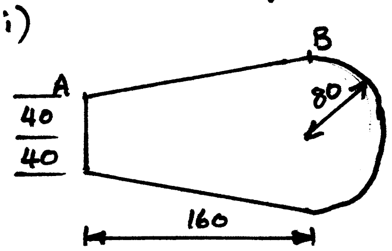
Part IB Paper 2. STRUCTURES June 1996.

Q1 a) Yield is associated with shear; Tresca assumes that the max. shear stress alone is significant; von Mises takes account of the max. shear of all three Mohr's circles representing the state of stress.

b) i)



Applied torque $= 2.F$ [Nm]

$$A_e = 160 \times 120 + \pi(80)^2/2 = 29{,}253 \text{ mm}^2$$

$$\tau = \frac{T}{2A_e t} = \frac{2.F}{2A_e t} \Rightarrow F = A_e t_{min} \tau_{max}$$

$$\therefore \quad F = \underset{[m^2]}{(29\,253 \times 10^{-6})} \underset{[m]}{(1 \times 10^{-3})} \underset{[N/m^2]}{(125 \times 10^6)} \underset{[m^{-1}]}{(1)} = \underline{\underline{3657 \text{ N}}}$$

ii) Torsional rigidity $= \dfrac{G\,4A_e^2}{\oint \frac{ds}{t}}$

Length $AB = \sqrt{40^2 + 160^2} = 164.9$ mm

$$\oint \frac{ds}{t} = 2 \times \left(\frac{40 + 164.9 \text{ mm}}{1 \text{ mm}}\right) + \frac{\pi(80) \text{ mm}}{2 \text{ mm}} = \underline{\underline{535.5}}$$

For aluminium alloy, $G = 26 \times 10^3$ N/mm² (Data book)

$$\therefore \text{ Torsional rigidity} = \frac{(26 \times 10^3)(4)(29\,253)^2}{535.5} = 166.2 \times 10^9 \text{ Nmm}^2$$

$$\phi = \frac{T}{\text{tors. rig.}} = \frac{2 \times 3.657}{166.2} \frac{\text{kNm}}{\text{kNm}^2} = 0.044 \text{ m}^{-1}$$

$$\phi = \frac{\theta}{L} \quad \therefore \quad \theta = (0.044)(4.2) = 0.185 \text{ rads} = \underline{\underline{10.6^\circ}}$$

iii) Effect of doubling the thickness t, with A_e and τ unchanged.

$$T = 2A_e t \tau \qquad \therefore \quad T(\text{new}) = 2T(\text{old})$$

$$\text{Tors. rig. } GJ = \frac{G\,4A_e^2}{\oint \frac{ds}{t}} \qquad \therefore \quad GJ(\text{new}) = 2GJ(\text{old})$$

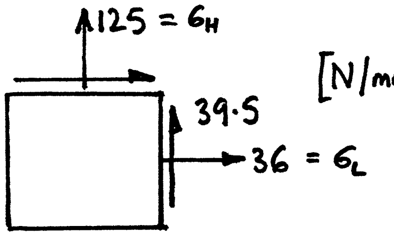
$$\phi(\text{new}) = \frac{T(\text{new})}{GJ(\text{new})} = \frac{2T(\text{old})}{2GJ(\text{old})} = \phi(\text{old})$$

$\therefore$ ϕ unchanged

$\therefore$ θ unchanged, $= \underline{\underline{10.6^\circ}}$

Q2

a)



$$\text{Torque} = (2\pi R^2 t)\,\tau = 2\pi(1000)^2(20)(39.5) \ \text{Nmm} = \underline{\underline{4964 \text{ kNm}}}$$

$$\text{Hoop stress} \Rightarrow p = \frac{\sigma_H t}{R} = \frac{125 \times 20}{1000} = 2.5 \text{ N/mm}^2 = \underline{\underline{2500 \text{ kPa}}}$$

(internal > external).

$$\text{Axial force} = \sigma_L(2\pi R t) = \underset{[\text{N/mm}^2]\ [\text{mm}^2]}{36(2.\pi.1000.20)} = \underline{\underline{4524 \text{ kN}}} \ \text{tension}$$

(Note: If the pipe is closed, there will be a longitudinal tensile stress σ_L^P associated with the internal pressure: $\sigma_L^P = \frac{pR}{2t} = \frac{\sigma_H}{2} = 62.5$ MPa tensile.

Now $\underset{\uparrow \text{ from applied external forces}}{\sigma_L^P + \sigma_L^F} = \sigma_L = 36$ MPa $\Rightarrow$ $\sigma_L^F = 36 - 62.5 = -26.5$ MPa

giving $F_{\substack{ext.\\appl.}} = \sigma_L^F(2\pi R t) = 3330$ kN compression .)

This was not really asked for, however.

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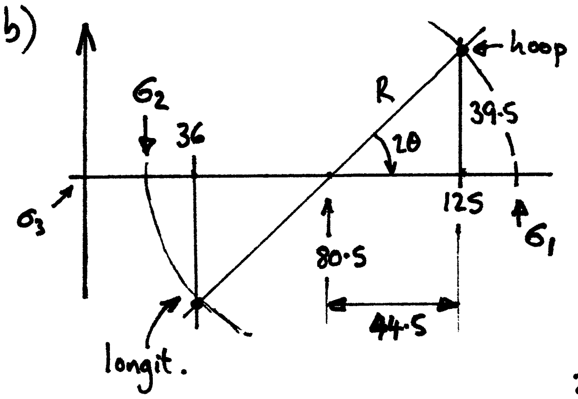
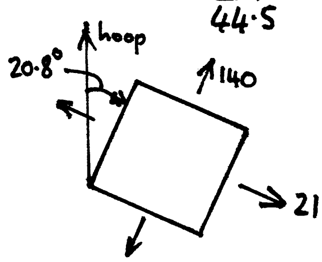
b)



$$R = \sqrt{44.5^2 + 39.5^2} = 59.5$$

$$\sigma_1 = 80.5 + 59.5 = 140 \text{ N/mm}^2$$
$$\sigma_2 = 80.5 - 59.5 = 21 \text{ N/mm}^2$$
$$\sigma_3 \approx 0$$

$$\tan 2\theta = \frac{39.5}{44.5} \Rightarrow 2\theta = 41.6^\circ \quad \therefore \underline{\underline{\theta = 20.8^\circ}}$$



Strains:

$$\varepsilon_1 = \frac{1}{E}\sigma_1 - \frac{\nu}{E}(\sigma_2+\sigma_3) = \frac{1}{210\times10^3}(140 - 0.3(21)) = \underline{\underline{637\,\mu\varepsilon}}$$ "extension" "microstrain"

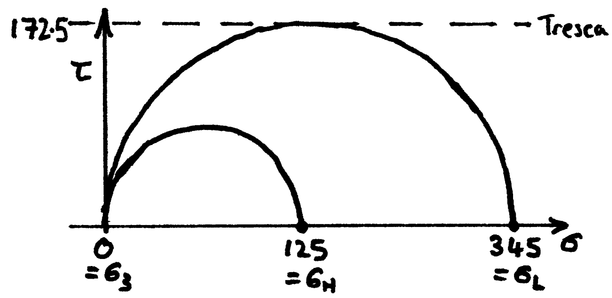
$$\varepsilon_2 = \frac{1}{E}\sigma_2 - \frac{\nu}{E}(\sigma_3+\sigma_1) = \frac{1}{210\times10^3}(21 - 0.3(140)) = \underline{\underline{-100\,\mu\varepsilon}}$$ "contraction"

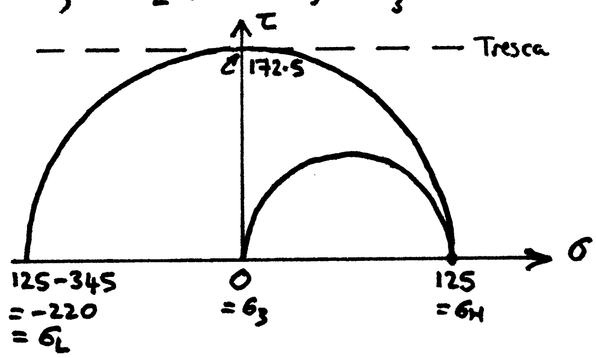
$$\varepsilon_3 = \frac{1}{E}\sigma_3 - \frac{\nu}{E}(\sigma_1+\sigma_2) = \frac{1}{210\times10^3}(-0.3(140+21)) = \underline{\underline{-230\,\mu\varepsilon}}$$ "contraction"

Q2 cont'd.

c) No torque $\Rightarrow$ σ_H, σ_L are principal stresses

$\sigma_H = 125\ N/mm^2$ due to pressure ; σ_L varies ; $\sigma_3 = 0$





$$\text{Area} = 2\pi R t = 2\pi(1000)(20) = 125\,664\ mm^2$$

$$\therefore \text{Axial force range} = \begin{cases} +345\ N/mm^2 \times 125664\ mm^2 = \underline{+43.3\ MN} & \text{tension} \\ -220\ N/mm^2 \times 125664\ mm^2 = \underline{-27.6\ MN} & \text{compression} \end{cases}$$

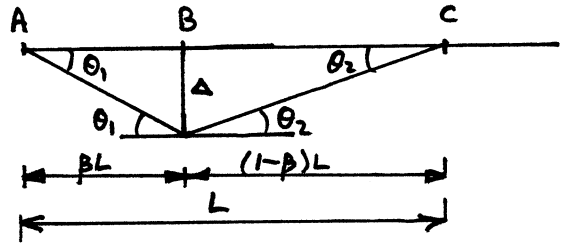
Note: again, if the pipe is closed, these can be converted to applied external axial forces if desired.
Longitudinal stress associated with internal pressure, $\sigma_L^P = \frac{\sigma_H}{2} = 62.5\ MPa$

$$\therefore \text{External force range} = \begin{cases} (345 - 62.5) \times 125664 \\ (-220 - 62.5) \times 125664 \end{cases}$$

$$= \pm 282.5 \times 125664 = \pm 35.5\ MN.$$

Again, however, this was not really asked for.

Q3.



Compatibility:

$$\theta_1 = \frac{\Delta}{\beta L} \Rightarrow \Delta = \theta_1 \beta L$$

$$\theta_2 = \frac{\Delta}{(1-\beta)L} \Rightarrow \Delta = \theta_2 (1-\beta) L$$

$$\theta_1 = \theta_2 \left(\frac{1-\beta}{\beta}\right)$$

$$\text{External work} = (\lambda w L)\frac{\Delta}{2} = \frac{\lambda w L^2}{2}(1-\beta)\theta_2$$

$$\text{Internal work} = \text{"}\sum M_p |\theta|\text{"} = \underbrace{M_p |\theta_1 + \theta_2|}_{\text{at } B} + \underbrace{(\alpha M_p)|\theta_2|}_{\text{at } C}$$

$$= M_p \theta_2 \left(\frac{1-\beta}{\beta} + 1\right) + \alpha M_p \theta_2$$

Q3 cont'd. $\quad$ I.W. $= M_p\theta_2\left[\frac{1}{\beta}+\alpha\right] = M_p\theta_2\left[\frac{1+\alpha\beta}{\beta}\right]$

Work equation:

$$\frac{\lambda w L^2}{2}(1-\beta)\theta_2 = M_p\theta_2\left(\frac{1+\alpha\beta}{\beta}\right)$$

$$\therefore \quad \underline{\underline{\lambda = \frac{2M_p}{wL^2}\,\frac{(1+\alpha\beta)}{\beta(1-\beta)}}}$$

Minimum when $\frac{\partial\lambda}{\partial\beta}=0 \Rightarrow$

$$\alpha\beta(1-\beta) - (1+\alpha\beta)(1-2\beta) = 0$$
$$\alpha\beta - \alpha\beta^2 - (1+\alpha\beta-2\beta-2\alpha\beta^2) = 0$$
$$\alpha\beta - \alpha\beta^2 - 1 - \alpha\beta + 2\beta + 2\alpha\beta^2 = 0$$

$$\therefore \quad \underline{\underline{\alpha\beta^2 + 2\beta - 1 = 0}} \quad \text{for minimum}$$

[$\therefore$ β_c satisfies $\quad \alpha\beta_c^2 = 1-2\beta_c$

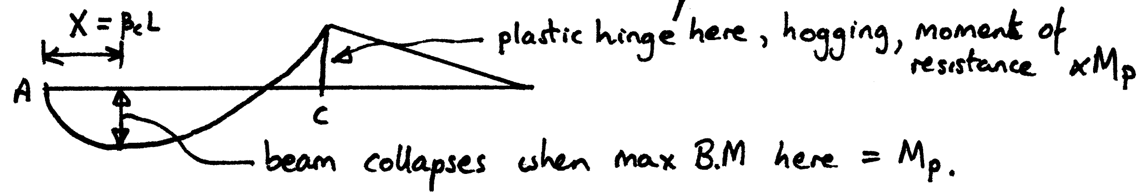
$$\alpha\beta_c = \frac{1}{\beta_c} - 2 \quad \Rightarrow \quad \underline{\underline{1+\alpha\beta_c = \frac{1}{\beta_c} - 1 = \frac{1-\beta_c}{\beta_c}}}$$

(Given)]

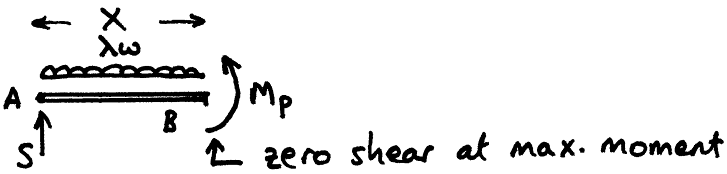
$$\therefore \quad \underline{\underline{\lambda_{cr}}} = \frac{2M_p}{wL^2}\,\frac{1}{\beta_c(1-\beta_c)}\left(\frac{1-\beta_c}{\beta_c}\right) = \underline{\underline{\frac{2M_p}{wX^2}}}$$

where $X = \beta_c L$

[Alternative derivation based on static equilibrium:—



<u>AB:</u>

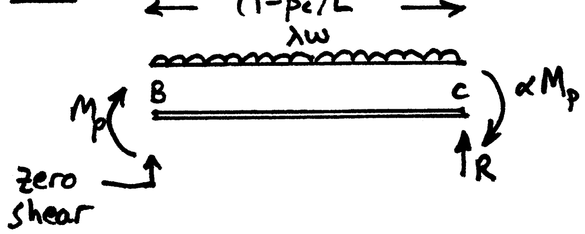


Moments about A:

$$M_p = (\lambda w X)\left(\frac{X}{2}\right)$$

$$\Rightarrow \quad \underline{\underline{\lambda = 2M_p/wX^2}}$$

<u>BC:</u>



Moments about C:

$$M_p + \alpha M_p = \lambda w(1-\beta_c)^2 L^2/2$$

$$\therefore \quad M_p(1+\alpha) = \left(\frac{2M_p}{\beta_c^2 L^2}\right)(1-\beta_c)^2\frac{L^2}{2}$$

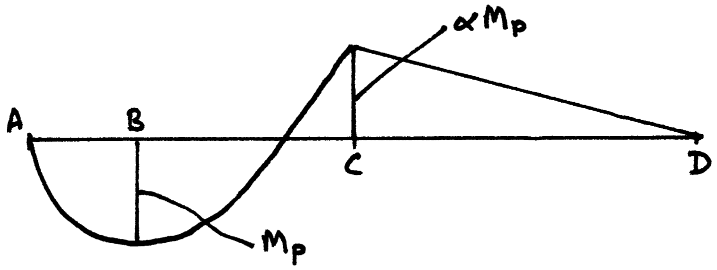
$$\therefore \quad (1+\alpha)\beta_c^2 = 1 - 2\beta_c + \beta_c^2$$

$$\Rightarrow \quad \underline{\underline{\alpha\beta_c^2 + 2\beta_c - 1 = 0}}$$]

Q3 cont'd.

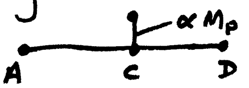
b)

BMD:



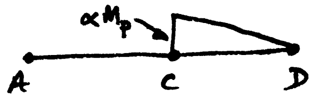
[Examiner's note: most students could not sketch this, so simple 'rules' for creating this diagram are given here.

CD: Moment at $C = \alpha M_p$, plastic hinge, hogging.
Tension on top side of beam at C, and using the convention that BMD goes on tension side of beam $\rightarrow$



No load on CD $\Rightarrow$ shear force constant $\Rightarrow$ BMD linear.
We also know moment at $D = 0$ (hinge support)

$\Rightarrow$

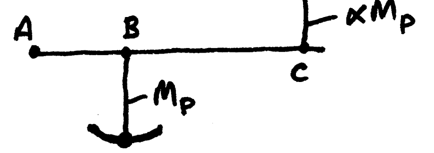


ABC: UDL on ABC $\Rightarrow$ linear shear force $\Rightarrow$ quadratic BMD (i.e. a parabola).

Also know $M_A = 0$ (hinge support)
$M_C = \alpha M_p$, hogging, (from above)
$M_B = M_p$, plastic hinge, sagging (tension on bottom of beam)

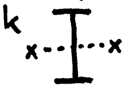
and max. sagging moment is at B (otherwise hinge would form somewhere other than B)

$\therefore$



connect with a parabola.]

c) i) 762 x 267 UB 173 $\rightarrow$ $Z_p = 6197 \times 10^3\ mm^3$
762 x 267 UB 147 $\rightarrow$ $Z_p = 5174 \times 10^3\ mm^3$ } Universal Beams Plastic moduli from Data Book



$\therefore \quad \alpha = 5174/6197 = 0.8349$

β_{cr} from $\alpha\beta^2 + 2\beta - 1 = 0 \quad \Rightarrow \beta_{cr} = (-2 \pm \sqrt{4+4\alpha})/2\alpha$

Need β_{cr} +ve $\quad \therefore \quad \beta_{cr} = (-1 + \sqrt{1+\alpha})/\alpha = 0.4247$

$X = \beta_{cr} L = 0.4247(27.4) = 11.64\ m$

Put $\lambda = 1$ for collapse.

$$\frac{wX^2}{2} = M_p = (345\ N/mm^2) \times (6197 \times 10^3\ mm^3) \times 10^{-6}$$

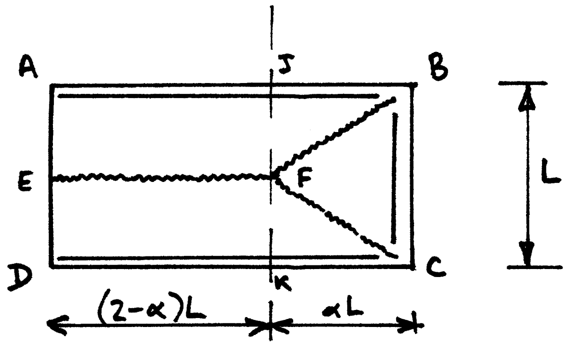
Nmm → kNm

$$= \underline{2138\ kNm}.$$

$$w = 2(2138)/(11.64)^2 = \underline{\underline{31.56\ kN/m}} \text{ for collapse}$$

ii) Collapse load unchanged by settlement.

Q4



Let EF deflect downwards by a distance Δ

External Work.

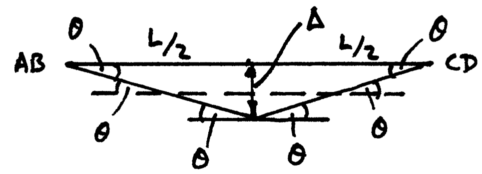
AJKD carries $(1-\alpha/2)$ of λW and deflects down $\Delta/2$ on average.

JBCK carries $\alpha/2$ of λW and deflects down $\Delta/3$ on average.

$$\therefore \quad WD = \left(1-\frac{\alpha}{2}\right)\lambda W.\frac{\Delta}{2} + \frac{\alpha}{2}\lambda W.\frac{\Delta}{3} = \frac{\lambda W\Delta}{2}\left(1-\frac{\alpha}{6}\right) = \frac{\lambda W\Delta}{12}(6-\alpha)$$

Internal work.

In span direction:



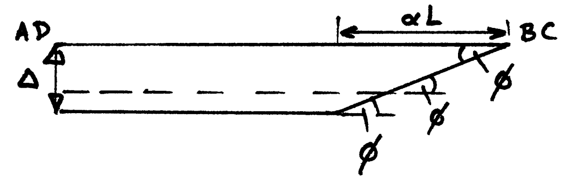
Compat: $\frac{\theta L}{2} = \Delta$

$\therefore \quad \theta = 2\Delta/L$

Total angle $= 4\theta$

$$\therefore \quad WD_{(span)} = (2L)\,m\,(4\theta) = 8mL\left(\frac{2\Delta}{L}\right) = 16m\Delta$$

In long direction:



Compat: $\phi(\alpha L) = \Delta$

$\therefore \quad \phi = \frac{\Delta}{\alpha L}$

Total angle $= 2\phi$

$$\therefore \quad WD_{(long)} = (L)\,m\,(2\phi) = 2mL\frac{\Delta}{\alpha L} = \frac{2m\Delta}{\alpha}$$

$$\therefore \text{ Total for all hinges} = 16m\Delta + \frac{2m\Delta}{\alpha} = 2m\Delta\left(\frac{8\alpha+1}{\alpha}\right)$$

WORK EQN:

$$\frac{\lambda W\Delta}{12}(6-\alpha) = 2m\Delta\left(\frac{8\alpha+1}{\alpha}\right)$$

$$\therefore \quad \lambda = 24\frac{m}{W}\left(\frac{8\alpha+1}{\alpha(6-\alpha)}\right)$$

b) Let $f = \frac{8\alpha+1}{\alpha(6-\alpha)}$

α	f
1	1.8
0.9	1.79
0.8	1.78
0.7	1.78

to 3 sig. figs.

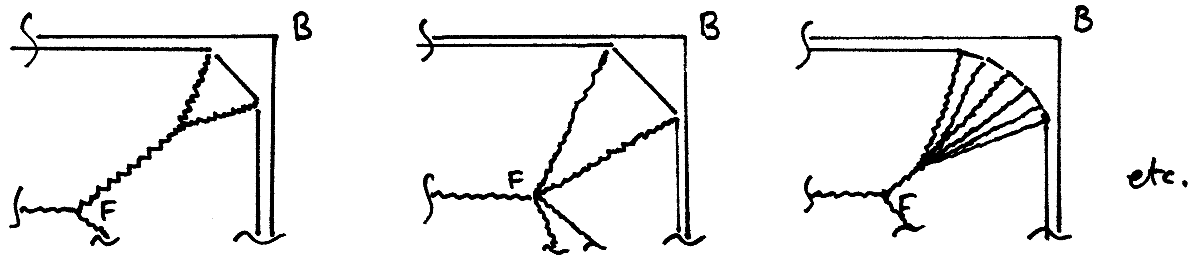
0.75 → 1.78

$$\therefore \lambda = 24\frac{m}{W}(1.78) = 42.7\frac{m}{W}$$

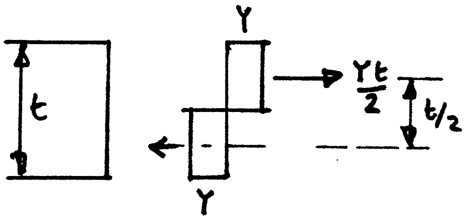
(or $43\frac{m}{W}$ to 2 sig. figs)

Q4 cont'd

c) Other mechanisms:



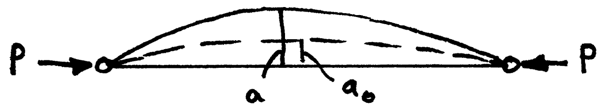
d) $\lambda = 40\frac{m}{W}$, with $\lambda = 1$ $\rightarrow$ $m = \frac{W}{40} = \frac{12\,kN}{40} = \underline{0.3\ kNm/m}$.



$m = \left(\frac{Yt}{2}\right)\left(\frac{t}{2}\right) = \underline{\underline{\frac{Yt^2}{4}}}$

$\frac{Yt^2}{4} = 0.3\ kNm/m$ $\Rightarrow$ $t = \sqrt{\frac{300\,N \times 4}{355\ N/mm^2}} = \underline{\underline{1.84\,mm}}$

Q5



a) $\sigma_{\substack{extreme\\fibre}} = \frac{P}{A} \pm \frac{M}{I} y_{\substack{ext.\\fib.}}$ $\therefore$ Max compressive stress $= \frac{P}{A} + \frac{M}{I} h$

(LINEAR ELASTIC THEORY)

h dist. from centroid to extreme fibre on concave side.

Now, $M_{max} = Pa$ and $I = Ar^2$

$$\therefore \sigma_{max} = \frac{P}{A} + \frac{Pah}{Ar^2} = \sigma\left(1 + \frac{ah}{r^2}\right)$$

where $\sigma = P/A$ (stress at centroid)

$$= \sigma\left(1 + \frac{a_0}{(1-\sigma/\sigma_E)}\frac{h}{r^2}\right) = \underline{\underline{\sigma\left(1 + \frac{\eta}{1-\sigma/\sigma_E}\right)}}$$

b) Perry-Robertson: $\sigma = \sigma_{cr}$ when $\sigma_{max} = \sigma_y$.

$$\therefore \sigma_y = \sigma_{cr} + \frac{\sigma_{cr}\eta}{(1-\sigma_{cr}/\sigma_E)}$$

$$\therefore (\sigma_y - \sigma_{cr}) = \frac{\sigma_E\sigma_{cr}\eta}{(\sigma_E - \sigma_{cr})}$$

$$\therefore \underline{\underline{(\sigma_y - \sigma_{cr})(\sigma_E - \sigma_{cr}) = \eta\,\sigma_{cr}\sigma_E}}$$

Q5 cont'd.

c) 305 x 305 UC 240 Universal Column section.
$\sigma_y = 345\ N/mm^2$, $L = 4.0\,m$

Data book: $A = 30560\ mm^2$
$r_{yy} = 81.4\,mm$ (Buckles about MINOR axis).

$$\frac{L}{r} = \frac{4000}{81.4} = 49.14$$

$$\sigma_E = \frac{\pi^2 E}{(L/r)^2} = \frac{\pi^2(210\times10^3\ N/mm^2)}{(49.14)^2} = \underline{858.3\ N/mm^2}$$

$$\eta = 0.0055\,\frac{L}{r} = 0.0055(49.14) = \underline{0.2703}$$

Perry-Robertson: $(345-\sigma)(858.3-\sigma) = 0.2703(858.3)\,\sigma$

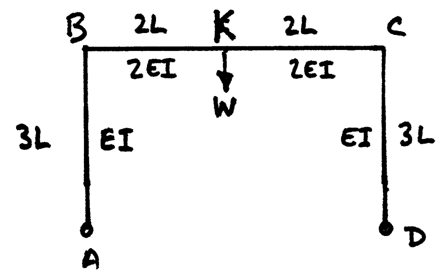
$$\therefore\quad \sigma^2 - 1435.3\,\sigma + 296{,}114 = 0$$

$$\therefore\quad \sigma_{cr} = (1435.3 \pm 935.8)/2 \qquad \text{(take -ve sign)}$$

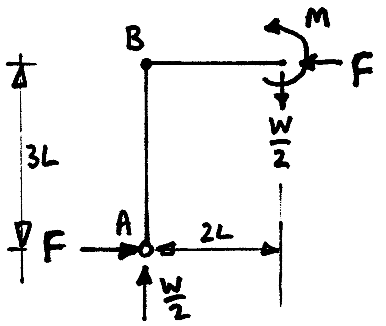
$$\underline{\underline{\sigma_{cr} = 250\ N/mm^2}}$$

$$P_{cr} = \sigma_{cr}A = 250\times30560 = \underline{\underline{7640\ kN}}$$

Q6



Consider half of structure, due to symmetry.



Moments about A:

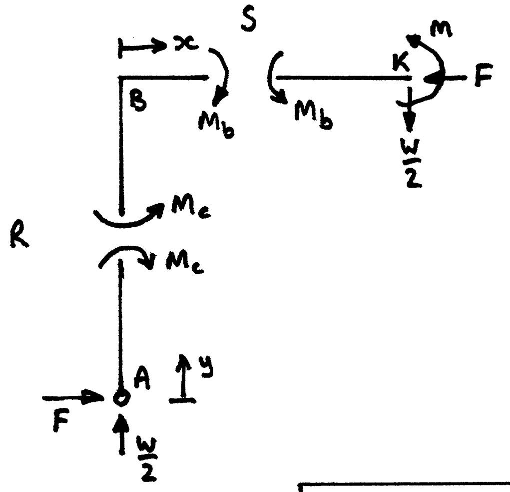
$$M = \left(\frac{W}{2}\right)(2L) - F(3L)$$

$$= \underline{\underline{WL - (3L)F}}$$

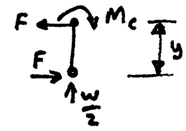
Q6 cont'd.

Use Virtual Work.

REAL SYSTEM:

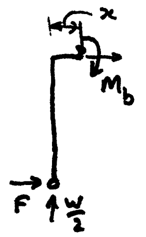


Moments about R:



$$\underline{\underline{M_c = Fy}}$$

Moments about S:



$$\underline{\underline{M_b = 3LF - \frac{W}{2}x}}$$

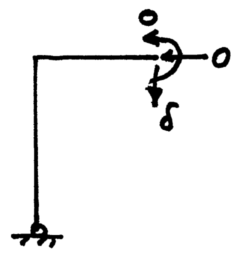
Real generalised strains:

$$\varepsilon_b = \alpha \Delta T$$

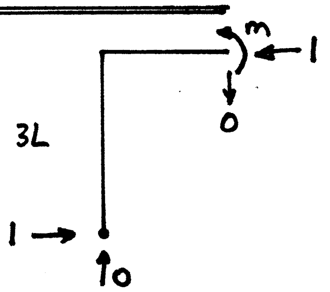
$$K_b = \frac{M_b}{2EI} = \frac{3LF}{2EI} - \frac{Wx}{4EI}$$

$$K_c = \frac{M_c}{EI} = \frac{Fy}{EI}$$

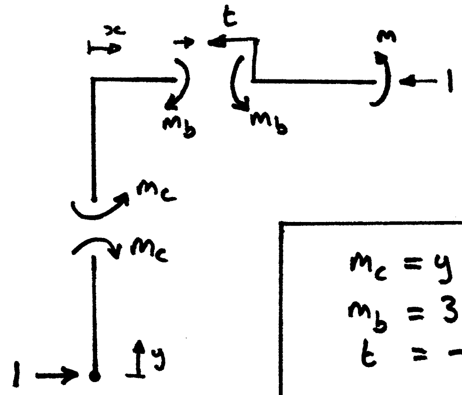
Real displacements:



VIRTUAL SYSTEM:



$m = -3L$



$m_c = y$
$m_b = 3L$
$t = -1$

Virtual Stress Resultants.

Q6 cont'd.

Virtual Work:

External. Internal.

$$m.0 + 1.0 + 0.\delta = \int_0^{2L} (-1)(\alpha \Delta T)\, dx + \int_0^{3L} (y)\left(\frac{Fy}{EI}\right) dy + \int_0^{2L} (3L)\left(\frac{3LF}{2EI} - \frac{Wx}{4EI}\right) dx$$

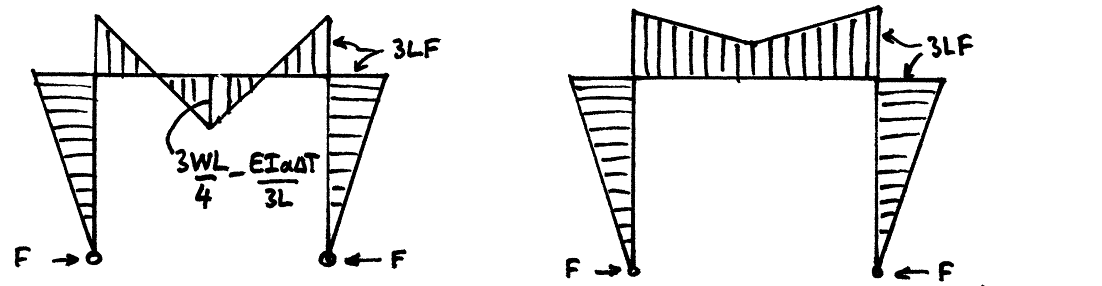
$$\therefore \quad 0 = -2\alpha \Delta T L + \frac{F}{EI}\frac{27L^3}{3} + \frac{9FL^2}{2EI}[2L] - \frac{3WL}{4EI}\frac{(2L)^2}{2}$$

$$\therefore \quad 0 = -2\alpha \Delta T L + \frac{9FL^3}{EI} + \frac{9FL^3}{EI} - \frac{3}{2}\frac{WL^3}{EI}$$

$$\therefore \quad 18\frac{FL^3}{EI} = 2\alpha \Delta T L + \frac{3}{2}\frac{WL^3}{EI} \qquad \therefore \quad \boxed{F = \frac{W}{12} + \frac{EI\alpha \Delta T}{9L^2}}$$

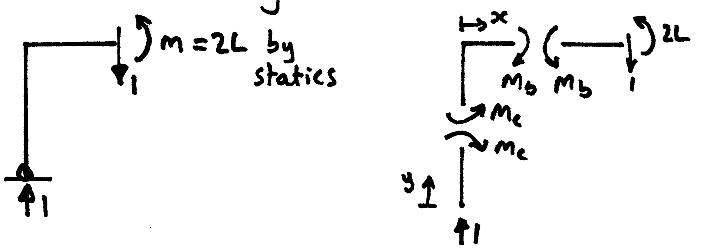
$$\therefore \quad M = WL - 3LF = WL - \left(\frac{EI\alpha\Delta T}{3L} + \frac{WL}{4}\right) = \frac{3WL}{4} - \frac{EI\alpha\Delta T}{3L}$$

BMD:



(for large W, small ΔT) (for large ΔT, small W)

ii) Create virtual system with unit load in direction of displacement δ.



Virtual Internal Stress Resultants

$t_b = 0$

$m_c = 0$

$m_b = -x$

$$V.W: \quad m \times 0 + 1 \times \delta + 0 \times 0 = \int_0^{2L} \underset{\text{real}}{K_b}\, \underset{\text{virtual}}{m_b}\, dx \qquad \text{and } K_b = \frac{3FL}{2EI} - \frac{Wx}{4EI}$$

$$\therefore \delta = \int_0^{2L}\left(-\frac{3}{2}\frac{FLx}{EI} + \frac{Wx^2}{4EI}\right) dx = -\frac{3FL^3}{EI} + \frac{2}{3}\frac{WL^3}{EI} = \frac{L^3}{EI}\left[-3F + \frac{2}{3}W\right]$$

$$= \frac{L^3}{EI}\left[-\frac{EI\alpha\Delta T}{3L^2} - \frac{W}{4} + \frac{2}{3}W\right] = \frac{5}{12}\frac{WL^3}{EI} - \frac{\alpha\Delta T L}{3}$$