IB Paper 7. Mathematical Methods (1996)

Q1

i) Volume, $V = \iint_{R'} z \, dx\,dy = \iint_{R} xy \, dx\,dy$

In 2D cartesian to polar transformation

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The Jacobian of the transformation, $J = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$

$(dx\,dy \rightarrow |J|\,dr\,d\theta)$

$$= r$$

$$V = \iint_{R'} xy \, dx\,dy = \iint_{R} r^2\cos\theta\sin\theta \, |J| \, dr\,d\theta$$

$$= \iint_{R} r^3\cos\theta\sin\theta \, dr\,d\theta$$

1 ii. Region of integration

R': $y=0$, $y=x$, $(x-a)^2+y^2=a^2$

R: $\theta=0$ to $\theta=\frac{\pi}{4}$

$(r\cos\theta-a)^2+(r\sin\theta)^2=a^2 \implies r^2-2ar\cos\theta=0$

$r(r-2a\cos\theta)=0$

$r=0$ to $r=2a\cos\theta$

$$\therefore \text{Volume, } V=\int_0^{\frac{\pi}{4}}\int_0^{2a\cos\theta} r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\cos\theta\sin\theta\int_0^{2a\cos\theta} r^3\,dr$$

$$=\int_0^{\frac{\pi}{4}}\sin\theta\cos\theta\,\frac{(2a\cos\theta)^4}{4}\,d\theta$$

$$=4a^4\int_0^{\frac{\pi}{4}}\sin\theta\cos^5\theta\,d\theta$$

$$=4a^4\left[-\frac{\cos^6\theta}{6}\right]_0^{\pi/4}$$

$$=\frac{4a^4}{6}\left[1-\left(\frac{1}{\sqrt{2}}\right)^6\right]$$

$$\underline{\underline{V=\frac{7a^4}{12}}}$$

Q2

a). A vector field $\underline{u}(\underline{r})$ is:

i) <u>solenoidal</u> if $\boxed{\nabla . \underline{u} = 0}$ everywhere.

i.e. $\exists\ \underline{A}$ such that $\underline{u} = \nabla \times \underline{A}$

ii). <u>irrotational</u> if $\boxed{\nabla \times \underline{u} = 0}$ everywhere.

i.e. $\exists\ \phi$ such that $\underline{u} = \nabla \phi$

iii) <u>conservative</u> if the line integral $\int_A^B \underline{u} . d\underline{r}$ is <u>independent of the path of integration</u>

i.e. $\exists\ \phi$ such that $\underline{u} = \nabla \phi$ since $\int_A^B \underline{u} . d\underline{r} = \int_A^B d\phi = \phi_B - \phi_A$

and $\oint_C \underline{u} . d\underline{r} = 0$ for all paths C.

Note that the condition for $\underline{u}$ to be conservative is the same as the condition for it to be irrotational since $\nabla \times \nabla \phi = 0$

Q2b.

$$\underline{v} = \underline{u} + \underline{\omega} \times \underline{r}$$

i) From data book Vector Calculus identities: (p12)

$$\nabla \times \underline{v} = \nabla \times \underline{u} + \nabla \times (\underline{\omega} \times \underline{r})$$

$$= 0 + \underline{\omega}(\nabla . \underline{r}) - \underline{r}(\nabla . \underline{\omega}) + (\underline{r}.\nabla)\underline{\omega} - (\underline{\omega}.\nabla)\underline{r}$$

$$\nabla \times \underline{v} = \underline{\omega}(\nabla . \underline{r}) - (\underline{\omega}.\nabla)\underline{r}$$

since $\underline{u}$ = constant and $\nabla . \underline{\omega}$ and $(\underline{r}.\nabla)\underline{\omega} = 0$

$$\nabla . \underline{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$(\underline{\omega}.\nabla)\underline{r} = \left[\omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z}\right][x\underline{i} + y\underline{j} + z\underline{k}]$$

$$= \underline{\omega}$$

$$\therefore \underline{\underline{\nabla \times \underline{v} = 3\underline{\omega} - \underline{\omega} = 2\underline{\omega}}}$$

ii). $\nabla . \underline{v} = \nabla . \underline{u} + \nabla . (\underline{\omega} \times \underline{r}) = \nabla . (\underline{\omega} \times \underline{r})$

$$\nabla . (\underline{\omega} \times \underline{r}) = \underline{r} . (\nabla \times \underline{\omega}) - \underline{\omega} . (\nabla \times \underline{r})$$

$\nabla \times \underline{\omega} = 0$ and $\nabla \times \underline{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$

$$\therefore \underline{\underline{\nabla . \underline{v} = 0}}$$

Q2 c). $\phi = \frac{k}{r}$ and potential increases with proximity to origin

i) $\underline{p} = -\nabla\phi$ since force is repulsive

Using databook relation for spherical symmetry:

$$\nabla\phi = \underline{e_r}\frac{d\phi}{dr} = -\underline{e_r}.\frac{k}{r^2}$$

$$= -\frac{k\underline{r}}{r^3}$$

$$\therefore \underline{p} = \frac{k\underline{r}}{r^3}$$

ii). $\nabla \times \underline{p} = \nabla \times (\nabla\phi) = 0$ $\therefore$ conservative field

$\nabla . \underline{p} = \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2 k}{r^2}\right) = 0$ $r>0$ $\therefore$ solenoidal field

$\nabla^2\phi = \nabla . \nabla\phi = 0$ for $r>0$

iii) Work done on particle $= -\int_A^B \underline{p}.\underline{dl} = \int_A^B + \underline{\nabla\phi}.\underline{dl} = +\int_A^B d\phi$

$= +(\phi_B - \phi_A)$ and is independent of path.

$= \frac{k}{6} - \frac{k}{5}$

$$W = -\frac{k}{30}$$

Q3. a) From databook (p12) Stokes' Theorem:

$$\iint_S (\nabla \times \underline{u}) \cdot d\underline{A} = \oint_C \underline{u} \cdot d\underline{l}$$

If we consider the surface S to be in the x-y plane with a closed boundary curve C in the planes and assume $\underline{u} = (u_x, u_y, u_z)$

$$d\underline{A} = dx\,dy\,\underline{k} \quad \text{and} \quad d\underline{L} = dx\,\underline{i} + dy\,\underline{j}$$

$$\therefore \quad \underline{k} \cdot (\nabla \times \underline{u}) = \underline{k} \cdot \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}$$

Thus Stokes' theorem becomes Green's Theorem in the plane:

$$\iint_S \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) dx\,dy = \oint_C u_x\,dx + u_y\,dy$$

Let $\underline{u} = (P, Q, R) \Rightarrow \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \oint_C P\,dx + Q\,dy$

3b) In general $P = P(x,y)$, $Q = Q(x,y)$

If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ then l.h.s becomes:

$$\iint_S 1 \, dx\, dy \equiv \text{area of } S \text{ enclosed by curve } C, A$$

i) ∴ let $P = -y/2$, $Q = \frac{x}{2}$ $\qquad A = \oint_C -\frac{y\,dx}{2} + \frac{x\,dy}{2}$

$$\underline{A = \frac{1}{2} \oint x\,dy - y\,dx}$$

ii). let $P = 0$, $Q = x$, $\qquad \underline{A = \oint_c x\,dy}$

iii) let $P = -y$, $Q = 0$, $\qquad A = -\oint_c y\,dx$

$$\therefore \underline{\text{Area of } S = \frac{1}{2}\oint_C x\,dy - y\,dx = \oint_C x\,dy = -\oint_C y\,dx}$$

iv). For an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x = a\cos\theta$
$y = b\sin\theta \quad dy = b\cos\theta\, d\theta$

$$\therefore \text{Area} = \oint_C x\,dy = \int_0^{2\pi} \underbrace{a\cos\theta}_{x}\ \underbrace{b\cos\theta\, d\theta}_{dy} = ab\int_0^{2\pi} \cos^2\theta\, d\theta$$

$$\underline{\underline{A = \pi ab}}$$

Q4 Assume functions have continuous derivatives and use Taylor series expansion:

$$u_{i+1}^n = u\left(i\Delta x + \Delta x,\ n\Delta t\right)$$

a). $$u_{i+1}^n = u_i^n + \Delta x \left.\frac{\partial u}{\partial x}\right|_{i,n} + \frac{\Delta x^2}{2!}\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,n} + \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + O\left(\Delta x^4\right).$$

$$u_{i-1}^n = u_i^n - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + O\left(\Delta x^4\right)$$

i). $$\frac{u_{i+1}^n - u_{i-1}^n}{\Delta x} = \frac{\partial u}{\partial x} + \frac{\Delta x^2}{3}\frac{\partial^3 u}{\partial x^3} + O\left(\Delta x^3\right)$$

$$= \underline{\frac{\partial u}{\partial x} + O\left(\Delta x^2\right)}$$

ie. the finite difference expression is consistent with $\frac{\partial u}{\partial x}$ and is second-order accurate since the leading truncation error term is $O\left(\Delta x^2\right)$. Centred finite difference.

ii). $$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t + \ldots$$

$$= \underline{\frac{\partial u}{\partial t} + O\left(\Delta t\right)}$$

ie. consistent with $\frac{\partial u}{\partial t}$ and is first-order accurate since leading truncation error term is $O(\Delta t)$. One-sided finite difference.

b) $$\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + O\left(\Delta x^4\right)$$

$$= \underline{\frac{\partial^2 u}{\partial x^2} + O\left(\Delta x^2\right)}$$

ie. consistent with $\frac{\partial^2 u}{\partial x^2}$ and second-order accurate in space increment Δx.

Q4 c) i) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} = 0$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 u}{\partial t^2} = 0$$

Since $\dfrac{\partial^2 u}{\partial x \partial t} = \dfrac{\partial^2 u}{\partial t \partial x}$, $\underline{\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} = \dfrac{-\partial^2 u}{\partial x \partial t}}$

ii). The Lax-Wendroff equation becomes:

$$\frac{\partial u}{\partial t} + \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right) + O(\Delta t^2) + \frac{\partial u}{\partial x} + O(\Delta x^2) = \frac{\Delta t}{2}\left[\frac{\partial^2 u}{\partial x^2} + O(\Delta x^2)\right]$$

Since $\dfrac{\Delta t}{2}\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\Delta t}{2}\dfrac{\partial^2 u}{\partial x^2}$ we can cancel the terms which are first-order accurate in time interval Δt to get an equation which is second-order accurate in both space and time.

$$\underline{\frac{\partial u}{\partial t} + O(\Delta t^2) + \frac{\partial u}{\partial x} + O(\Delta x^2) = 0}$$

This has been achieved <u>without</u> using centred differences in time.

Q5

Model of data: $\hat{y}_i = a + b x_i$

a) i) Assume all measurement error occurs in measurements of $\hat{y}_i$, y_i

$$\therefore \quad y_i = a + b x_i + \varepsilon_i$$

The least squares estimate minimizes the sum of the squares of the distances of data points to the straight line, E where:

$$E = \sum_{i=1}^{n} (y_i - \hat{y}_i) = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left(y_i - (a + b x_i)\right)^2$$

The minimum E occurs for values of a and b such that:

$$\frac{\partial E}{\partial a} = 0 \quad \text{and} \quad \frac{\partial E}{\partial b} = 0$$

$$\frac{\partial E}{\partial a} = 0 \implies -2 \sum_{i=1}^{n} \left(y_i - (a + b x_i)\right) = 0$$

$$\frac{\partial E}{\partial b} = 0 \implies -2 \sum_{i=1}^{n} x_i \left(y_i - (a + b x_i)\right) = 0$$

Minimum least squares error solution for a and b:

$$n a + b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

$$a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

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ii) If the error occurs in the measurement of x_i only then minimise the distance measured in the x-direction instead.

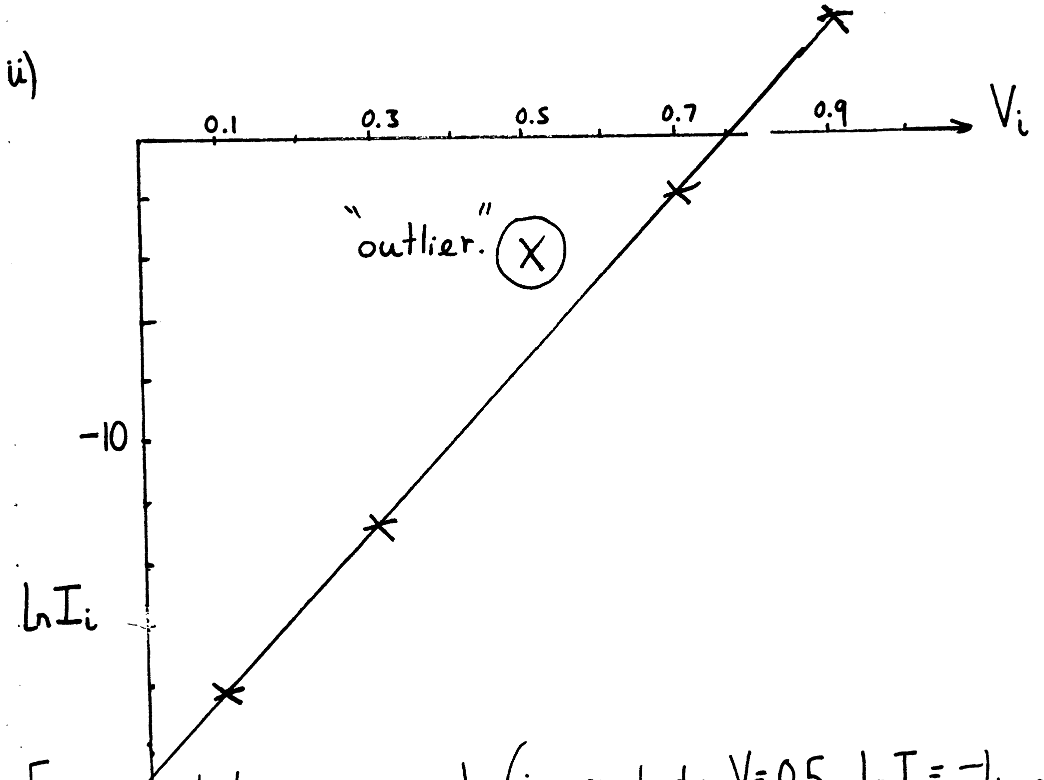
Q5b.

i) $I_i = I_o e^{\frac{V_i}{V_s}}$

$\therefore \ln I_i = \ln I_o + \left(\frac{1}{V_s}\right) V_i$

cf $y_i = a + b x_i$ where $a = \ln I_o$ and $b = 1/V_s$

ii)



For correct 4 measurements (ie. exclude $V = 0.5$, $\ln I = -4$ measurements)

$n = 4$
$\sum x_i = 2.0$
$\sum x_i^2 = 1.4$
$\sum x_i y_i = -3.5$
$\sum y_i = -29$

Least squares estimates for a and b satisfy
$$4a + 2b = -29$$
$$2a + 1.4b = -3.5$$

$$\Longrightarrow a = -21, \quad I_o = 7.58 \times 10^{-10}$$
$$b = 27.5, \quad \underline{V_s = 0.036\,V}$$

iii). $V = 0.9$ measurement is twice as accurate and the distance to line should be weighted by $\sqrt{2}$. Add this measurement twice so that $n = 5$ and $V = 0.9$ appears twice.

Q6 a). For a signal $x(t)$ with Fourier Transform $X(\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

let $y(t) = x(t)\, e^{j\omega_0 t}$ have Fourier Transform $Y(\omega)$.

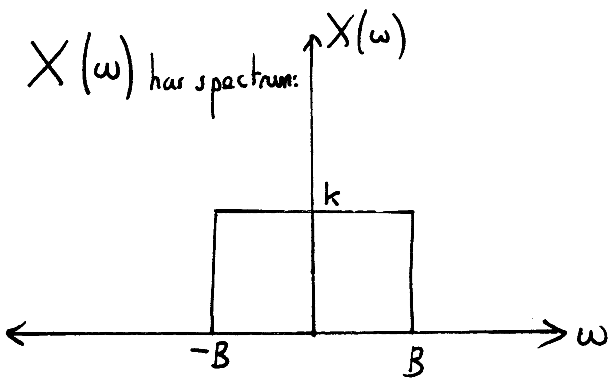
$$Y(\omega) = \int_{-\infty}^{\infty} \left(x(t)\, e^{j\omega_0 t}\right) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

$$= X(\omega') \quad \text{where } \omega' = \omega - \omega_0$$

$$\underline{Y(\omega) = X(\omega - \omega_0)} \qquad [\text{Frequency shift}]$$

b) $y(t) = x(t).s(t)$, and $X(\omega)$ has spectrum:



i) $s(t) = \cos \omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t} \right]$

$$\therefore\; y(t) = \frac{x(t)}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t} \right]$$

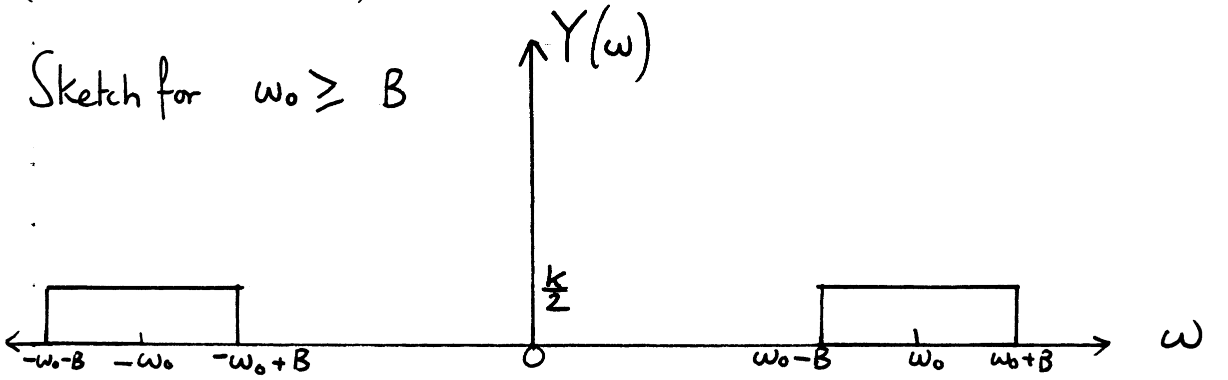
Q6b (cont)

Using result of part (a).

$$\therefore \; Y(\omega) = \frac{X(\omega-\omega_0)}{2} + \frac{X(\omega+\omega_0)}{2}$$

(i.e. modulation theorem)

Sketch for $\omega_0 \geq B$



ie. frequency shift in modulation.

Note alternative method

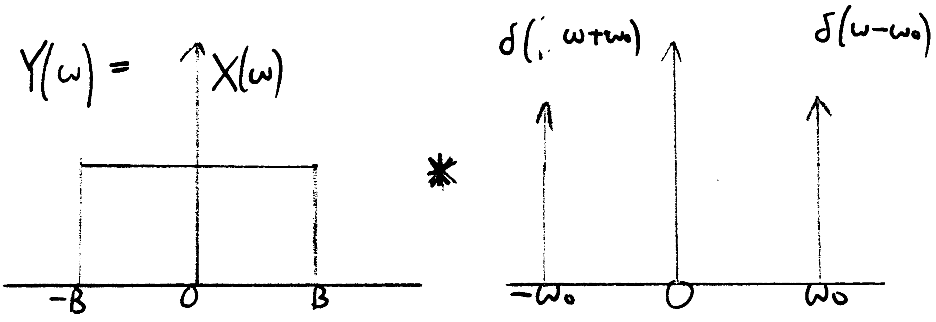
From convolution theorem, multiplication in time domain leads to convolution in the frequency dom

If $\quad y(t) = x(t).\,s(t)$

then $\quad Y(\omega) = \dfrac{1}{2\pi}\, X(\omega) * S(\omega)$

where $\quad S(\omega) = \pi\left[\,\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\,\right]$

i.e. $Y(\omega) =$



$$\therefore \; Y(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} X(\Omega)\,\delta(\omega-\omega_0-\Omega)\,d\Omega + \frac{1}{2}\int X(\Omega)\,\delta(\omega+\omega_0-\Omega).$$

$$Y(\omega) = \frac{X(\omega-\omega_0)}{2} + \frac{X(\omega+\omega_0)}{2}$$

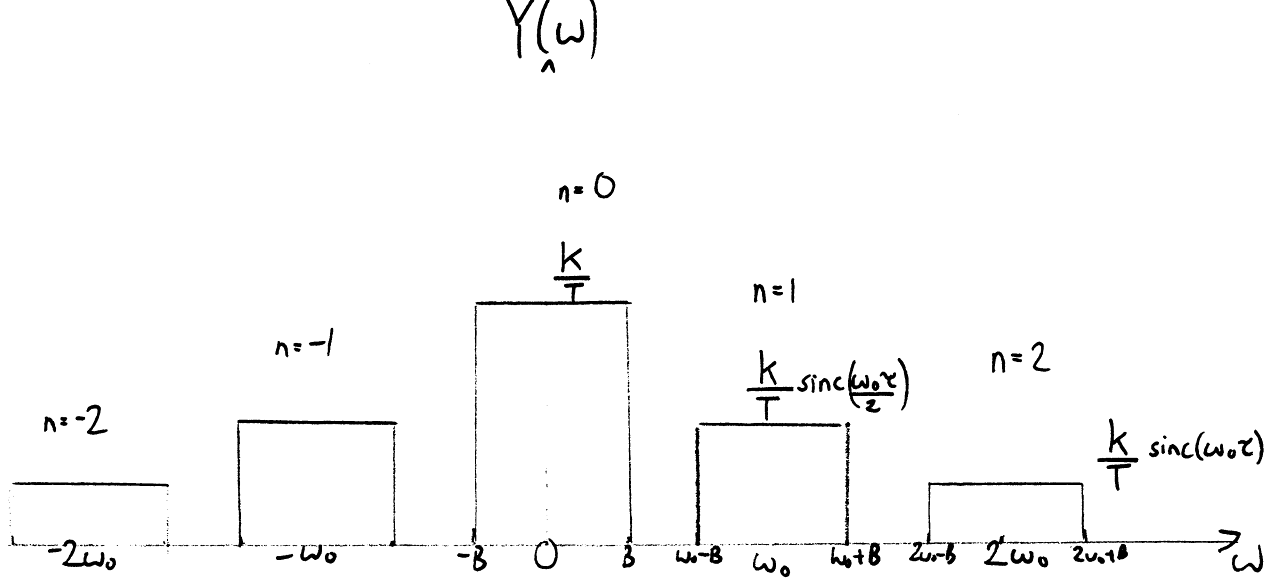
Q6 ii). $y(t) = \frac{x(t)}{T} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n\omega_0\tau}{2}\right) e^{jn\omega_0 t}$

$$y(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n\omega_0\tau}{2}\right) x(t) e^{jn\omega_0 t}$$

Using frequency shift result of (a):

$$\therefore Y(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n\omega_0\tau}{2}\right) X(\omega - n\omega_0)$$

Note: $s(t)$ is a pulse-train.



iii). $\tau \to 0$ $\quad s(t) \longrightarrow \delta_p(t)$ i.e. impulse train

$$s(t) \longrightarrow \frac{1}{T} \sum_{n=-\infty}^{\infty} 1 . e^{jn\omega_0 t}$$ (i.e. Impulse train fourier series)

since $\text{sinc}\left(\frac{\omega\tau}{2}\right) \longrightarrow 1$ as $\tau \to 0$

$$\therefore \text{ spectrum } Y(\omega) \longrightarrow \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)$$

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as in ideal sampling by impulse train.

6iii). As $\tau \longrightarrow T$ $\quad \sin n\frac{\omega_0 \tau}{2} \rightarrow \operatorname{sinc} \frac{n\omega_0 T}{2} = \operatorname{sinc}(n\pi)$

$$= 0 \quad n \neq 0$$
$$= 1 \quad n = 0$$

$$\therefore \; \underline{Y(\omega) \longrightarrow \frac{1}{T} X(\omega)}$$

i.e. no sampling but scaling by $1/T$. Obvious since $s(t) \rightarrow \frac{1}{T}$ and $y(t) = \frac{1}{T} x(t)$.

Q7)

a) Sampling frequency $= f_s = 20\text{ kHz}$ Sample period $T = 50\mu s$
no. of samples, $N = 1024$

i). For band limited signal $f_s \geq 2 f_b$
$\therefore$ maximum frequency after anti-aliasing filter, $f_b = \underline{10\text{ kHz}}$

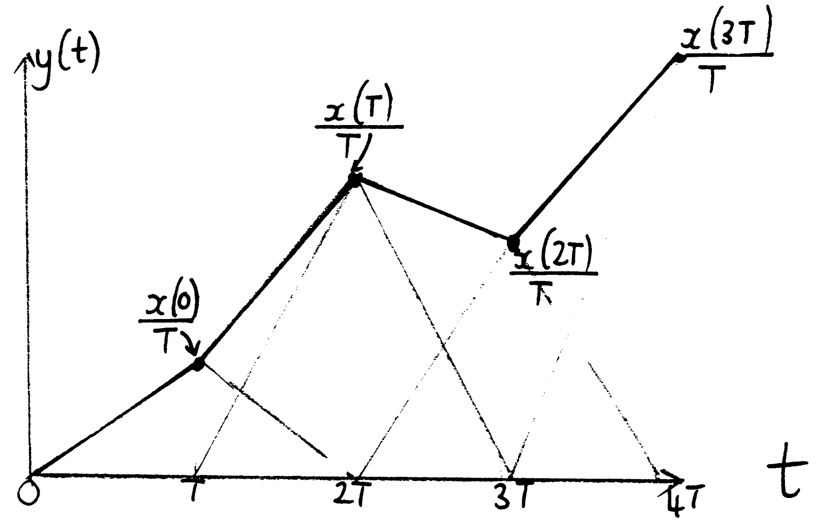
ii) frequency spacing $\Delta f = \frac{f_s}{N} = \frac{20000}{1024} = \underline{19.5\text{ Hz}}$

or $\Delta f = \frac{1}{T_L} = \frac{1}{NT} = \frac{1}{1024 \times 50 \times 10^{-6}} = 19.5\text{Hz}$

b). $x_s(t) = \sum_{-\infty}^{\infty} x(nT)\,\delta(t-nT)$

$$y(t) = h(t) * x_s(t)$$
$$= h(t) * \sum_{-\infty}^{\infty} x(nT)\,\delta(t-nT)$$
$$y(t) = \sum_{-\infty}^{\infty} x(nT)\, h(t-nT)$$

$\therefore$



i.e. linear interpolation of samples, delayed by T, scaled by 1/T. Used in recovery of

Q7b.

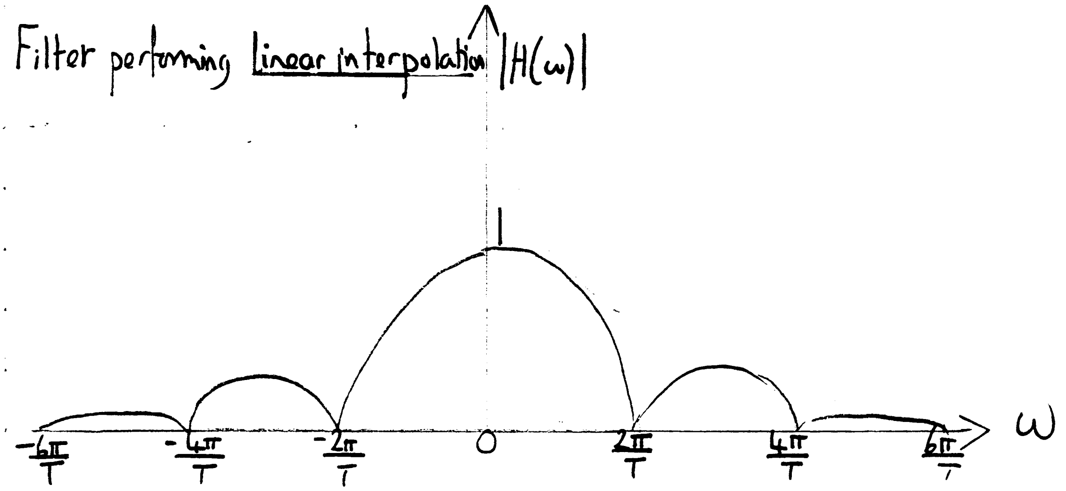
From databook for triangle pulse shifted (delayed) by T.

$$H(\omega) = e^{-j\omega T} \cdot \operatorname{sinc}^2\left(\frac{\omega T}{2}\right)$$

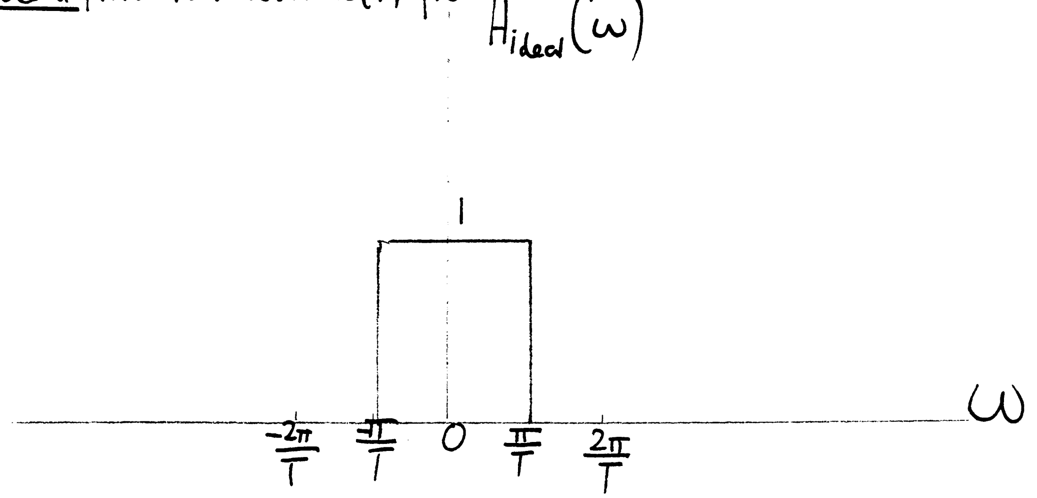
$$= \left[\frac{4}{\omega^2 T^2} \sin^2 \frac{\omega T}{2}\right] e^{-j\omega T}$$

$$\therefore \; |H(\omega)| = \operatorname{sinc}^2\left(\frac{\omega T}{2}\right)$$

Filter performing linear interpolation



Ideal filter to recover $x(t)$ from $x_s(t)$:



Note linear interpolator introduces high-frequency distortion. Ideal filter is impractical.

Q8 a). For a probability density function $f(x)$ of a random variable X the moment-generating function is defined by:

$$M_X(s) = \int_{-\infty}^{\infty} f(x)\, e^{-sx}\, dx = E\left(e^{-sX}\right)$$

(Note alternative definitions exist: $M_X(t) = E\left(e^{tX}\right)$).

i) Moments of the distribution can be computed from $M_X(s)$ by differentiation (if possible):

$$\left.\frac{d^n M_X(s)}{ds^n}\right|_{s=0} = (-1)^n E\left(X^n\right)$$

eg. $$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = -\left.\frac{d M_X(s)}{ds}\right|_{s=0}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \left.\frac{d^2 M_X(s)}{ds^2}\right|_{s=0}$$

ii) Moment-generating functions can be used to derive the probability density functions of sums of independent random variables (see (b)). (reproductive properties).

eg. If $Z = X + Y$

$$M_Z(s) = M_X(s)\, M_Y(s)$$

If X and Y are normally distributed random variables, Z will also have a normal distribution.

8b).

$$X \sim N(\mu_1, \sigma_1)$$
$$Y \sim N(\mu_2, \sigma_2)$$
$$(-Y) \sim N(-\mu_2, \sigma_2)$$

If $Z = X - Y$ and X and Y are independent

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(x-z)\, dx = \int_{-\infty}^{\infty} f_X(x)\, f_{-Y}(z-x)\, dx$$

i.e. convolution of pdf of X and $-Y$.

By using moment-generating functions (similar to Laplace transforms) the moment generating function can be obtaned by multiplication.

$$\therefore\ M_Z(s) = M_X(s)\, M_{-Y}(s)$$
$$= e^{\left(-s\mu_1 - \frac{\sigma_1^2 s^2}{2}\right)} \cdot e^{s\mu_2 - \frac{\sigma_2^2 s^2}{2}}$$
$$= e^{-s(\mu_1-\mu_2) - \frac{s^2}{2}(\sigma_1^2+\sigma_2^2)}$$

This is the moment-generating function of a gaussian distribution:

$$\therefore\ f_Z(z) = \frac{1}{\sigma'\sqrt{2\pi}}\, e^{-\frac{(x-\mu')^2}{2\sigma_1^2}} \qquad \text{where } \mu' = \mu_1 - \mu_2,\ \sigma^{2\prime} = \sigma_1^2 + \sigma_2^2$$

$$\therefore\ \underline{Z \sim N\left(\mu_1 - \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right)}$$

8c). X = total weight $\sim N(W, 12)$

Y = can weight $\sim N(200, 9)$

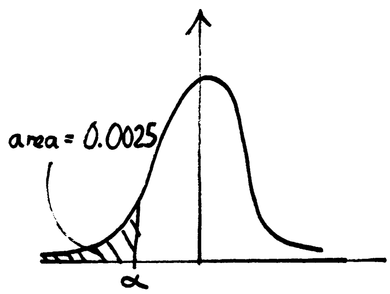
Z = contents = $X - Y \sim N\left(W-200, \sqrt{9^2+12^2}\right)$

$$\underline{Z \sim N(W-200, 15)}$$

$$\therefore \quad \frac{Z-(W-200)}{15} \sim N(0, 1)$$

We must ensure that $\dfrac{1000-(W-200)}{15} = \alpha$

where $\displaystyle \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{u^2}{2}}\, du = 0.0025$



From datasheet $\underline{\alpha = -2.81}$

$$\therefore \quad \frac{1000-(W-200)}{15} \leqslant -2.81$$

$$\therefore \quad W \geqslant 1200 + 2.81 \times 15$$
$$\underline{W \geqslant 1242g.}$$

R. Cipolla 19/6/96