ENGINEERING TRIPOS PART IB

Friday 7 June 1996 9 to 11

Paper 7

MATHEMATICAL METHODS

Answer not more than **four** *questions.*

Answer at least **one** *question from each section.*

Answers to questions in each section should be tied together and handed in separately.

(TURN OVER

SECTION A

Answer at least **one** *question from this section.*

1 A machined metal part has a base which is bounded by the positive x-axis, the circle $(x-a)^2+y^2=a^2$ and the line $y=x$ as shown in Fig. 1. The height, z, is given by xy.

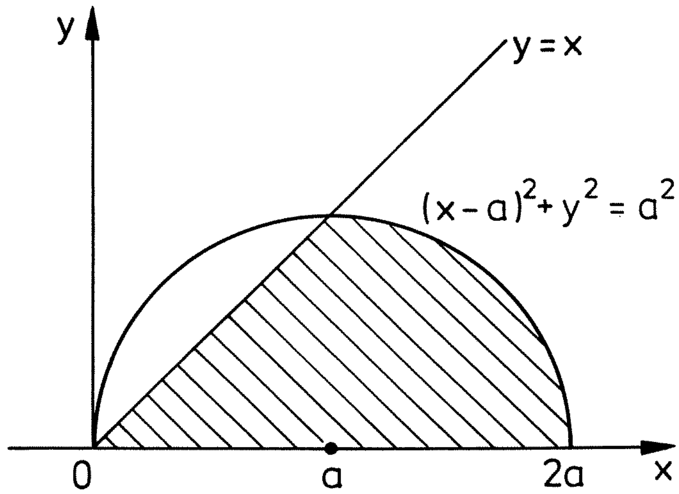

Fig. 1

(i) Show, by using the 2D cartesian to polar transformation, that the volume, V, can be evaluated as a double integral of the form

$$V=\iint_R f(r,\theta)\,dr\,d\theta$$

(ii) Find the region of integration and evaluate the integral.

2 (a) What are meant by solenoidal, irrotational and conservative vector fields?

(b) A vector field is defined by

$$\mathbf{v} = \mathbf{u} + \boldsymbol{\omega} \times \mathbf{r}$$

where $\mathbf{u}$ and $\boldsymbol{\omega}$ are constant vectors and $\mathbf{r}$ is the position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $\boldsymbol{\nabla} \times \mathbf{v} = 2\boldsymbol{\omega}$ and $\nabla.\mathbf{v} = 0$.

(c) A vector force field $\mathbf{p}$ has spherical symmetry and a scalar potential $\phi(x,y,z) = \frac{k}{r}$, where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

(i) Find the force $\mathbf{p}$ at position $\mathbf{r}$.

(ii) Find the curl and divergence of the vector force field and show that the scalar potential satisfies $\nabla^2\phi = 0$ for $r > 0$.

(iii) What work is done on a particle in moving it from position $(5,0,0)$ to $(0,0,6)$ in a straight line?

3 (a) A curve C is the boundary of a flat, closed region S in the xy-plane. Using Stokes' Theorem or otherwise, show that if P and Q are functions of x and y only, then

$$\iint_S \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dxdy = \oint_C Pdx + Qdy$$

(b) Use the result derived in part (a) to show that the area of S is given by

$$\frac{1}{2}\oint_C xdy - ydx = \oint_C xdy = -\oint_C ydx$$

and hence find the area of the ellipse parameterised by $x = a\cos\theta,\ y = b\sin\theta$.

SECTION B

Answer at least **one** *question from this section.*

4 (a) The partial differential equation $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ can be integrated numerically by the Lax-Wendroff finite-difference scheme:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \frac{\Delta t}{2}\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right]$$

where the discrete value of the solution $u(x,t)$ at time $t = n\Delta t$ and position $x = i\Delta x$ is represented by $u_i^n = u(i\Delta x, n\Delta t)$.

Show that the terms on the left hand-side are finite difference approximations for $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$ which are first order accurate in time increment Δt and second order accurate in space increment Δx respectively.

(b) Show that the finite difference expression

$$\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

is an approximation to the spatial derivative $\frac{\partial^2 u}{\partial x^2}$ and find the order of its accuracy.

By showing that the general solution $u(x,t)$ to $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ also satisfies

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

show how the Lax-Wendroff scheme achieves second order accuracy in both space and time.

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5 (a) The straight line $y = a + bx$ is to be fitted through a discrete set of data points $(x_1, y_1) \ldots (x_n, y_n)$ where $n > 2$. Show that the values of a and b which minimize the sum of the squares of the distances (measured in the y-direction) of the data points to the straight line satisfy the following equations:

$$an + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

$$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

When might it be appropriate to minimize the errors measured in the x-direction instead?

(b) An experiment is carried out to determine the relationship between current I and corresponding voltage V for a semiconductor diode. The relationship between I and V is expected to be of the form $I = I_0 \exp(V/V_S)$.

The table below gives the values of $\ln I$ measured for different voltages V.

V	0.1	0.3	0.5	0.7	0.9
$\ln I$	−18	−13	−4	−2	4

(i) Show that a linear relationship is expected between $\ln I$ and V.

(ii) One of the measurements was incorrectly recorded. Plot the data to find the incorrect measurement. Assuming that any measurement error occurs in the measurement of $\ln I$ estimate the parameters I_0 and V_S by the method of least squares using the 4 valid measurements.

(iii) The measurement error in the $V = 0.9$ reading is known to have an error distribution with variance σ^2. The other measurements have error distributions with variance $2\sigma^2$. How would you modify the method of least squares to exploit this information?

SECTION C

Answer at least **one** *question from this section.*

6 (a) A signal $x(t)$ has Fourier transform $X(\omega)$. Show that the Fourier transform of the signal $x(t)e^{j\omega_0 t}$ is $X(\omega - \omega_0)$.

(b) A band limited signal $x(t)$ has spectrum $X(\omega)$, where

$$\begin{aligned} X(\omega) &= k \qquad -B \leq \omega \leq B \\ X(\omega) &= 0 \qquad \text{otherwise.} \end{aligned}$$

The signal, $x(t)$, is modulated by being multiplied by a periodic waveform $s(t)$ with period $T = 2\pi/\omega_0$, so that

$$y(t) = x(t)s(t).$$

Find and sketch the spectrum of $y(t)$ for the following periodic signals $s(t)$:

(i) $s(t) = \cos \omega_0 t$;

(ii) $s(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n\omega_0 \tau}{2}\right) e^{jn\omega_0 t}$ where sinc (x) is defined as $\dfrac{\sin x}{x}$.

What happens to the spectrum of $y(t)$ in the limit when $\tau \to 0$? What happens when $\tau \to T$?

7 (a) An analogue signal is to be spectrum analysed. It is fed through an anti-aliasing filter, then sampled at 20kHz and the DFT of 1024 samples computed. What should be the highest frequency present after the anti-aliasing filter? Determine the frequency spacing of the spectral samples.

(b) A sampled signal $x_s(t) = \sum_{-\infty}^{\infty} x(t)\delta(t - nT)$ is passed through a filter with impulse response $h(t)$ shown in Fig. 2.

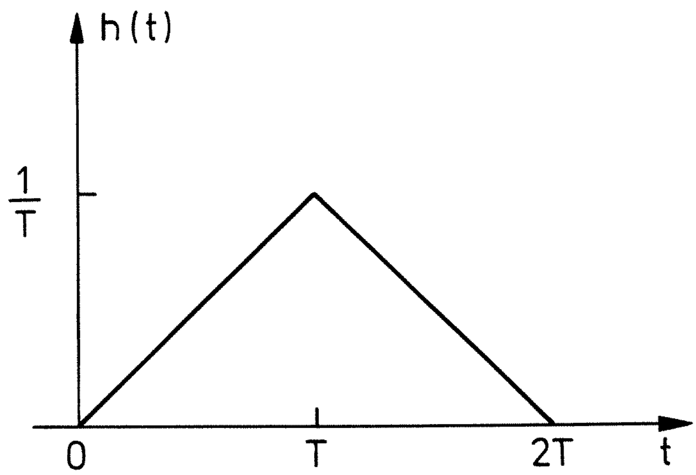

Fig.2

(i) Sketch the output signal $y(t)$ and give a practical use for this filter.

(ii) Find the frequency response $|H(\omega)|$ of the filter. Compare this response to the frequency response of the ideal filter required to recover the original signal $x(t)$ from its samples.

8 (a) What is meant by the moment generating function of a random variable? How is it used in the theory of probability?

(b) If X and Y are independent random variables with Normal distributions

$$X \sim N(\mu_1, \sigma_1)$$
$$Y \sim N(\mu_2, \sigma_2)$$

show that the random variable $Z = X - Y$ has a probability density function which is also Normal with distribution $N\left(\mu_1 - \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right)$.

(c) A manufactured product is sold in cans. The cans have a weight which is normally distributed with mean 200g and standard deviation 9g. The filling machine is set to give a total weight (can and contents) with mean W and standard deviation 12g.

What should be the least value of W to ensure that less than 0.25% of the filled cans have contents weighing less than 1000g?

END OF PAPER