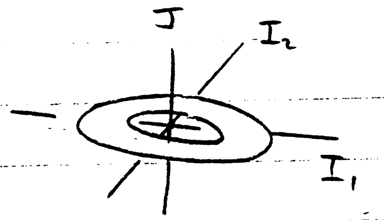
Engineering Tripos Part IB June 1997
Paper 1 Mechanics

(a) $J = \int_b^a r^2\,dm = \rho\int_b^a r^2\, 2\pi r\,dr = \rho\, \frac{2\pi r^4}{4}\Big|_b^a$



and use $m = \rho\pi(a^2-b^2)$

$\therefore\ J = m\,\frac{a^2+b^2}{2}$

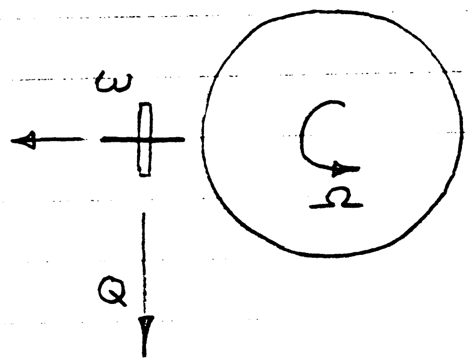
Perpendicular axis theorem : $J = I_1 + I_2$
for a lamina and $I_1 = I_2$ (axisymmetry)

$\therefore\ I = \frac{J}{2} = m\,\frac{a^2+b^2}{4}$

$\boxed{10 \text{ marks}}$

(b) Gyroscopic Couple $Q = J\omega\Omega$

$J = \frac{1}{2}mr^2 = \frac{1}{2}\times 96\times 1^2$
$= 48\ \text{kgm}^2$



$\Omega =$ 1 revolution/day
$= \frac{2\pi}{24\times 3600}$ rad/s

$\omega = 3600$ rad/s

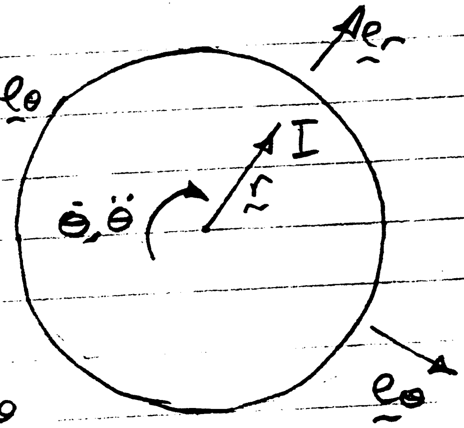
$\therefore\ Q = J\omega\Omega = \frac{2\pi\times 48\times 3600}{24\times 3600}$

$= 4\pi$ Nm
$= 12.5$ Nm in the direction shown

The Earth's rotation generates a centrifugal d'Alembert force in addition to the gyroscopic couple calculated above

$\boxed{10 \text{ marks}}$

2(a) $\ddot{\underset{\sim}{r}} = (\ddot{r} - r\dot{\theta}^2)\underset{\sim}{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underset{\sim}{e_\theta}$



is the acceleration of the insect at I. Assume that the Insect is moving at constant speed

$\therefore \ddot{\underset{\sim}{r}} = -r\dot{\theta}^2 \underset{\sim}{e_r} + 2\dot{r}\dot{\theta}\, \underset{\sim}{e_\theta}$

centripetal — coriolis

$\dot{\theta} = \dfrac{2\pi \times 33\frac{1}{3}}{60} = 3.49$ rad/s

FORCES

Centripetal : always acts towards the centre

Maximum magnitude at outside edge

$m r \dot{\theta}^2 = 0.001 \cdot \dfrac{0.3}{2} \cdot (3.49)^2 = \underline{\underline{1.83 \text{ mN}}}$

Coriolis : a constant force acting from "left to right" assuming the insect knows left from right !

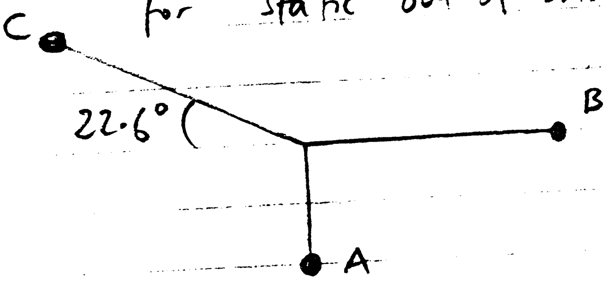
$m\, 2\dot{r}\dot{\theta} = 0.001 \times 2 \times \dfrac{0.3}{60} \times 3.49$

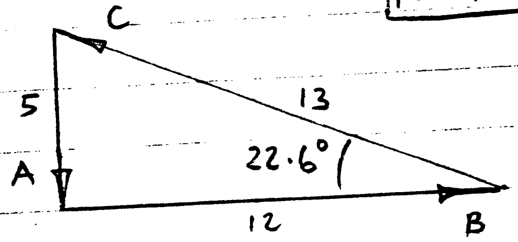
$= \underline{\underline{0.0349 \text{ mN}}}$

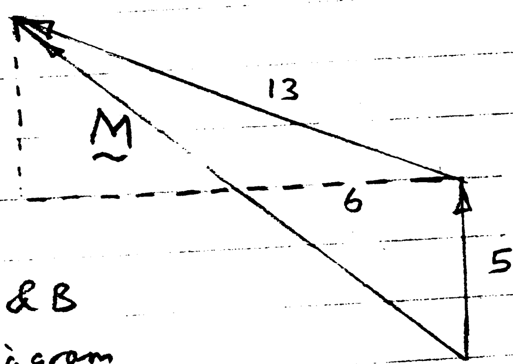
This is the only force that acts on the insect at the centre.

$\boxed{10 \text{ marks}}$

(b) A closed force polygon for static out of balance







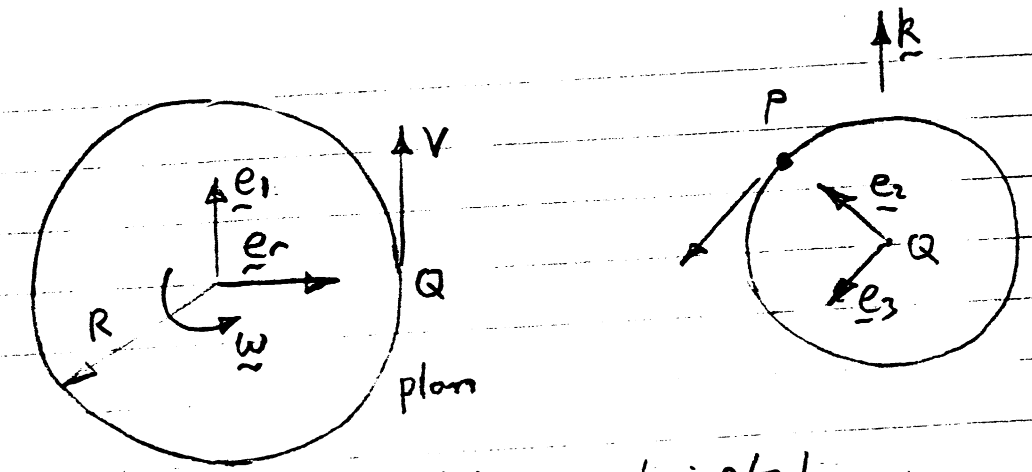
For the couple, consider moments about B caused by o.o.b from A & B being the length of $\underset{\sim}{M}$ in the diagram

$= 2\sqrt{5^2 + 6^2} \times 10^{-3} \times 0.1 \times 1000^2$

$= \underline{1562 \text{ Nm}}$

$\boxed{10 \text{ marks}}$

3



(i) Acceleration of Q is solely centripetal $\boxed{2}$

$$\underset{\sim}{a}_Q = -\frac{V^2}{R}\,\underset{\sim}{e}_r$$

(ii) Use the data book expressions for velocity and acceleration in an accelerating frame of reference :

$$\underset{\sim}{v}_P = \underset{\sim}{v}_Q + \dot{\underset{\sim}{r}}_P\big|_R + \underset{\sim}{\omega}\times\underset{\sim}{r}_P$$

with $\underset{\sim}{r}_P = a\underset{\sim}{e}_2$ and $\underset{\sim}{\omega} = \omega\underset{\sim}{k}$

$$= V\underset{\sim}{e}_1 + a\dot{\theta}\,\underset{\sim}{e}_3 + a\omega\,\underset{\sim}{k}\times\underset{\sim}{e}_2$$

and $$\underset{\sim}{a}_P = \underset{\sim}{a}_Q + \ddot{\underset{\sim}{r}}_P\big|_R + \dot{\underset{\sim}{\omega}}\times\underset{\sim}{r}_P + 2\underset{\sim}{\omega}\times\dot{\underset{\sim}{r}}_P\big|_R + \underset{\sim}{\omega}\times(\underset{\sim}{\omega}\times\underset{\sim}{r}_P)$$

$$= -\frac{V^2}{R}\underset{\sim}{e}_r - a\dot{\theta}^2\underset{\sim}{e}_2 + \underset{\sim}{0} + 2a\omega\dot{\theta}\,\underset{\sim}{k}\times\underset{\sim}{e}_3 + a\omega^2\,\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{e}_2)$$

$$= \left(-\frac{V^2}{R} - 2a\omega\dot{\theta}\cos\theta\right)\underset{\sim}{e}_r - a\omega^2\sin\theta\,\underset{\sim}{e}_1 - a\dot{\theta}^2\underset{\sim}{e}_2 \quad \boxed{12}$$

(iii) $\omega = \frac{V}{R} = \frac{5}{25}$ rad/s

$$\frac{V^2}{R} = \frac{25}{25} = 1 \text{ m/s}^2$$
$$2a\omega\dot{\theta} = 2\cdot(0.1)\cdot\frac{5}{25}\cdot 10 = 0.4 \text{ m/s}^2$$
$$a\omega^2 = (0.1)\cdot\left(\frac{5}{25}\right)^2 = 0.004 \text{ m/s}^2$$
$$a\dot{\theta}^2 = 0.1\cdot 100 = 10 \text{ m/s}^2$$

$\boxed{4}$

$\theta = 0$: $\underset{\sim}{a}_P = (-1 \underbrace{-0.4}_{\text{coriolis}})\,\underset{\sim}{e}_r - 10\,\underset{\sim}{e}_2$ m/s²

$\theta = \frac{\pi}{2}$: $\underset{\sim}{a}_P = -\underset{\sim}{e}_r - 0.004\,\underset{\sim}{e}_1 - 10\,\underset{\sim}{e}_2$ m/s²

(no coriolis)

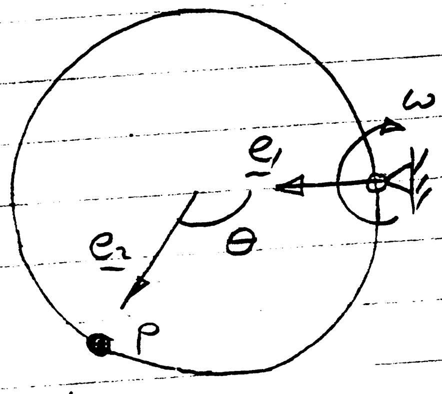
(iv) Inertia force/weight force = acceleration/g

∴ Coriolis is 4% of weight, etc.

... is 100% of weight

[20 Marks]

4 (i) $\underset{\sim}{r}_p = a\,\underset{\sim}{e}_1 + a\,\underset{\sim}{e}_2$



$\dot{\underset{\sim}{r}}_p = a\,\dot{\underset{\sim}{e}}_1 + a\,\dot{\underset{\sim}{e}}_2$

$= a\,\omega\,\underset{\sim}{e}_1^* + a(\omega+\dot\theta)\underset{\sim}{e}_2^*$

$\ddot{\underset{\sim}{r}}_p = a\,\dot\omega\,\underset{\sim}{e}_1^* + a\,\omega\,\dot{\underset{\sim}{e}}_1^*$
$+ a(\dot\omega+\ddot\theta)\underset{\sim}{e}_2^* + a(\omega+\dot\theta)\,\dot{\underset{\sim}{e}}_2^*$

$= a\,\dot\omega\,\underset{\sim}{e}_1^* - a\omega^2\underset{\sim}{e}_1 + a(\dot\omega+\ddot\theta)\underset{\sim}{e}_2^* - a(\omega+\dot\theta)^2\underset{\sim}{e}_2$ $\boxed{8}$

(ii) No friction on bead $\therefore$ $\ddot{\underset{\sim}{r}}_p \cdot \underset{\sim}{e}_2^* = 0$

$\therefore\; a\dot\omega\,\underset{\sim}{e}_1^*\cdot\underset{\sim}{e}_2^* - a\omega^2\,e_1\cdot e_2^* + a(\dot\omega+\ddot\theta) = 0$

$\therefore\; a\dot\omega\,(-\cos\theta) - a\omega^2(\sin\theta) + a(\dot\omega+\ddot\theta) = 0$ $\boxed{6}$

$\therefore\; \underline{\underline{\ddot\theta = \dot\omega(\cos\theta-1) + \omega^2\sin\theta}}$

(iii) $\theta = \frac{\pi}{2}$ $\therefore$ $\ddot\theta = \omega^2 - \dot\omega$

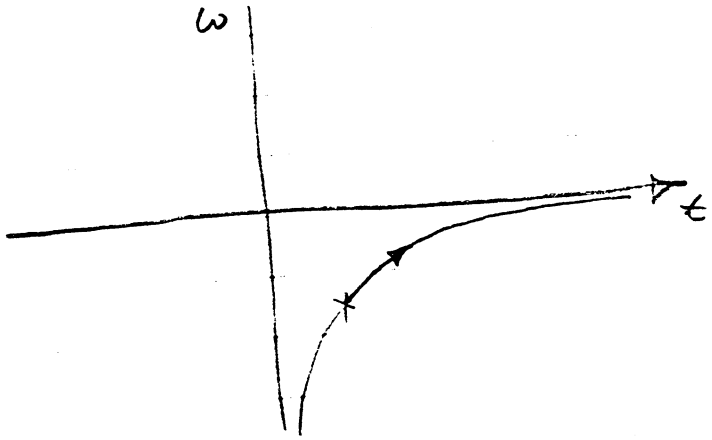
and if $\dot\omega = \omega^2$ $\therefore$ $\ddot\theta = 0$ $\therefore$ no slip $\boxed{2}$

(iv) $\frac{d\omega}{dt} = \omega^2$ $\therefore$ $\int\frac{d\omega}{\omega^2} = \int dt$

$\therefore\; -\frac{1}{\omega} = t \quad (+\text{ const})$

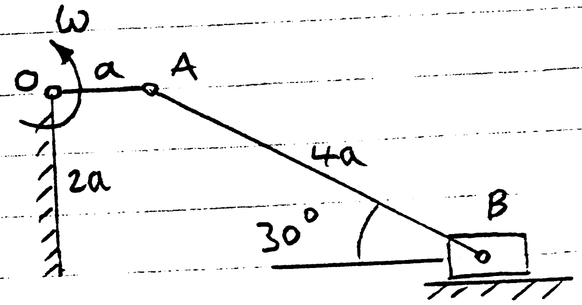
$\therefore\; \omega = -\frac{1}{t} \quad (+\text{ const})$

which is how ω must vary with time to bring the ring to rest without bead sliding $\boxed{4}$

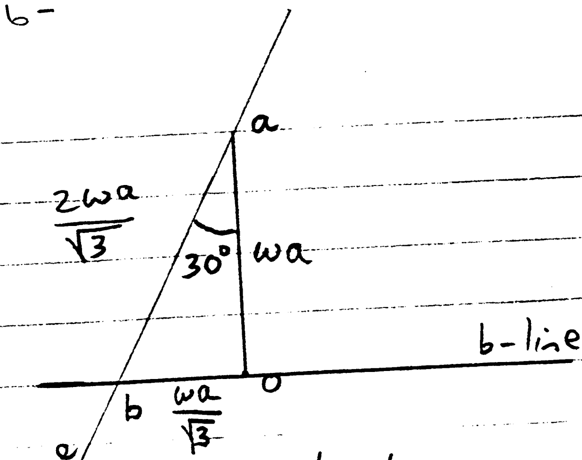


$\boxed{\text{20 MARKS}}$

5



space diagram

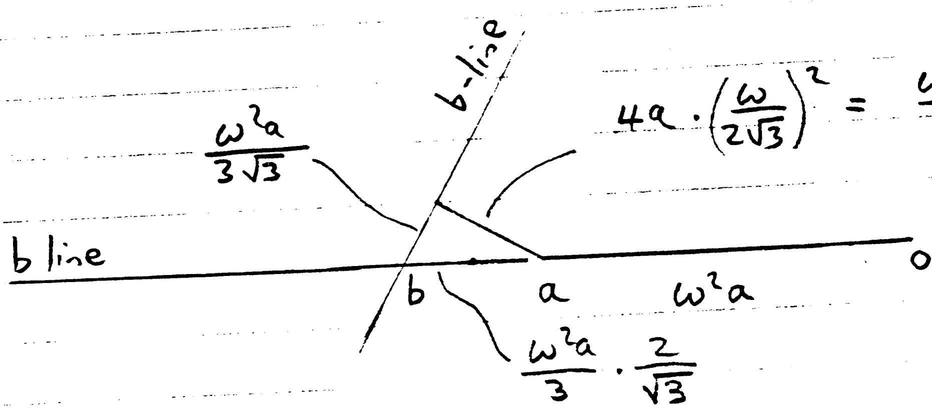


velocity diagram

Angular velocity of AB $= \dfrac{v_{AB}}{AB} = \dfrac{2\omega a}{\sqrt{3}\cdot 4a}$

$= \dfrac{\omega}{2\sqrt{3}}$

4

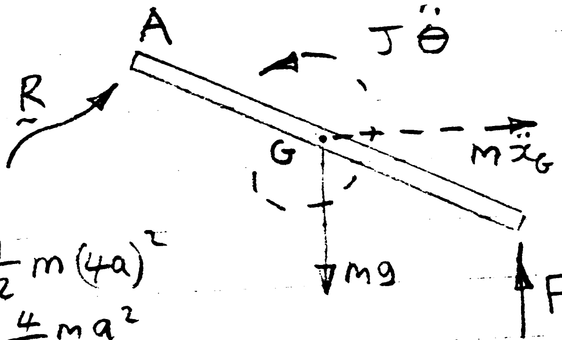


acceleration diagram

6

Angular acceleration of AB $= \dfrac{a_{AB}}{AB}$

$= \dfrac{\omega^2 a}{3\sqrt{3}} \Big/ 4a = \dfrac{\omega^2}{12\sqrt{3}}$



$J = \frac{1}{12}m(4a)^2$
$= \frac{4}{3}ma^2$

Acceleration of G
$= \omega^2 a\left(1 + \frac{1}{3\sqrt{3}}\right)$ ←

Take moments of d'Alembe[rt] forces about A

$\therefore F4a\frac{\sqrt{3}}{2} - mg2a\frac{\sqrt{3}}{2} + m\omega^2 a\left(1+\frac{1}{3\sqrt{3}}\right)a$
$+ \frac{4}{3}ma^2\frac{\omega^2}{12\sqrt{3}} = 0$

8

$\therefore F = \dfrac{mg}{2} - m\omega^2 a\left(\dfrac{9\sqrt{3}+4}{54}\right)$

$\therefore \omega^2 = \dfrac{g}{a}\,\dfrac{27}{9\sqrt{3}+4}$ $\quad\therefore$ $\omega = 3.68$ rad/s

2

20

6 (i) $I_1 = \int F\,dt = \left[\frac{1}{2}\cdot 4 + (12-4)\right]\cdot 1$

$= 10 \text{ Ns}$ $\boxed{4}$

(ii) I doesn't affect angular velocity

$\therefore I_1 a = J\,\Delta\omega$

$J = \frac{1}{2} m r^2$

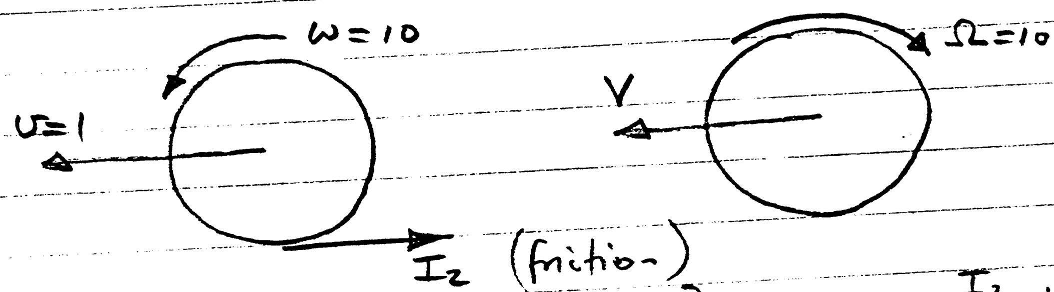
$= \frac{1}{2}\times 20 \times \cdot 1^2$

$= 0{\cdot}1 \text{ kgm}^2$

$\therefore \Delta\omega = \frac{I_1 a}{J} = \frac{10\times 0{\cdot}1}{0{\cdot}1} = 10 \text{ rad/s}$ $\boxed{4}$

(iii) After skidding, $v = 1\text{m/s}$

$\omega = \frac{1}{0{\cdot}1} = 10 \text{ rad/s}$



$\left.\begin{aligned} I_2 &= m\,(V - v) \\ \text{and } aI_2 &= J\,(\Omega + \omega) \end{aligned}\right\}$ $\therefore V = \frac{I_2}{m} + v$

$= \frac{J}{ma}(\Omega + \omega) + v$

$= \frac{0{\cdot}1}{20\cdot 0{\cdot}1}(10+10) + 1$

$= \underline{\underline{2 \text{ m/s}}}$ $\boxed{6}$

(iv) $I - I_1 = mV = 20 \times 2$

$= 40 \text{ Ns}$

$\boxed{4}$

$\therefore \underline{\underline{I = 50 \text{ Ns}}}$

(v) Friction has to be a small effect during the impulse I_1. With $\mu = 0{\cdot}1$ then friction force $= 20\text{N} \ll 1\text{kN}$ $\therefore$ ok

$\boxed{2}$

$\boxed{20 \text{ Marks}}$