ENGINEERING TRIPOS PART IB 1997

PAPER 7 MATHEMATICAL METHODS

ANSWERS AND HINTS

1. $a = 3$, $b = 3/2$, or *vice versa*, $\phi = \sin(x + 3y)$.

2. Identities required are $\nabla \times (\boldsymbol{u}_1 \times \boldsymbol{u}_2)$, $\nabla \times (\phi \boldsymbol{u})$.
   Also note $\nabla r^n = nr^{n-1}\nabla r = nr^{n-2}\boldsymbol{r}$
   $\nabla(fg) = f\nabla g + g\nabla f$ is useful for last part.

3. (ii) Note that, if $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{c}$, $\boldsymbol{b} \neq \boldsymbol{c}$ in general.
   (iii) Define $\bar{\boldsymbol{r}}$ rigorously in terms of $\boldsymbol{r}$ first.

4. $\phi = 8\ell n x + 2$
   Using central differencing for interior points and backward differencing for $d\phi/dx$ at $x = 4$, the numerical solution is $\phi_1 = 2 \cdot 00$, $\phi_2 = 7 \cdot 55$, $\phi_3 = 10 \cdot 79$, $\phi_4 = 13 \cdot 09$

5. a) $\alpha \Delta t / \Delta x^2 = 1/6$ (Note that $\partial^2 u / \partial t^2 = \alpha\, \partial^4 u / \partial x^4$).
   b) $A\, \Delta t / \Delta x < 1$ (Note that disturbance propagates downwind only).

6. For the discrete representation:
   $|F(o)| = 1 \cdot 0$, $|F(\pi)| = 0 \cdot 6533$, $|F(2\pi)| = 0$, $|F(3\pi)| = 0 \cdot 2706$, $|F(4\pi)| = 0$,
   (Other values are symmetrical).
   (Do not forget to multiply the F_k by the sampling time).

7. For the second part, direct integration is most straightforward, noting that $F(2\pi k/T)$ is not a function of t. For the last part, note that the equation is valid for <u>all</u> ω. Hence, if $F(\omega)$ is sampled at intervals of $\Delta\omega = 2\pi/T$, $F(2\pi k/T)$ can be calculated for all k and $F(\omega)$ reconstructed in its entirety

8. Construct a tree diagram to find the probability that the game ends with n bags and hence that

$$m(t) = \sum_{n=2}^{\infty} \frac{(n-1)}{n!} t^n$$

   Then show that this is the same as the given expression.
   Finally, mean $(N) = e$, var $(N) = 3e - e^2$.

PART IB PAPER 7 1997

MATHEMATICAL METHODS

SOLUTIONS BY J.B. YOUNG

1. First application of the chain rule gives:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} = a\frac{\partial\phi}{\partial u} + b\frac{\partial\phi}{\partial v}$$

Second application gives:

$$\frac{\partial^2\phi}{\partial x^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v}\right) = \frac{\partial^2\phi}{\partial u^2} + 2\frac{\partial^2\phi}{\partial u\partial v} + \frac{\partial^2\phi}{\partial v^2}$$

$$\frac{\partial^2\phi}{\partial x\partial y} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(a\frac{\partial\phi}{\partial u} + b\frac{\partial\phi}{\partial v}\right) = a\frac{\partial^2\phi}{\partial u^2} + (a+b)\frac{\partial^2\phi}{\partial u\partial v} + b\frac{\partial^2\phi}{\partial v^2}$$

$$\frac{\partial^2\phi}{\partial y^2} = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right)\left(a\frac{\partial\phi}{\partial u} + b\frac{\partial\phi}{\partial v}\right) = a^2\frac{\partial^2\phi}{\partial u^2} + 2ab\frac{\partial^2\phi}{\partial u\partial v} + b^2\frac{\partial^2\phi}{\partial v^2}$$

Hence $9\dfrac{\partial^2\phi}{\partial x^2} - 9\dfrac{\partial^2\phi}{\partial x\partial y} + 2\dfrac{\partial^2\phi}{\partial y^2} = 0$ becomes,

$$\left[9 - 9a + 2a^2\right]\frac{\partial^2\phi}{\partial u^2} + \left[18 - 9(a+b) + 4ab\right]\frac{\partial^2\phi}{\partial u\partial v} + \left[9 - 9b + 2b^2\right]\frac{\partial^2\phi}{\partial v^2} = 0$$

This reduces to $\dfrac{\partial^2\phi}{\partial u\partial v} = 0$ when:

$$(9 - 9a + 2a^2) = (3-a)(3-2a) = 0 \quad \rightarrow \quad a = 3 \text{ or } \tfrac{3}{2}$$

$$(9 - 9b + 2b^2) = (3-b)(3-2b) = 0 \quad \rightarrow \quad b = 3 \text{ or } \tfrac{3}{2}$$

$$\left[18 - 9(a+b) + 4ab\right] \neq 0$$

Hence $a = 3$, $b = 3/2$ or vice versa. [10]

By inspection, $\phi = f(u)$ and $\phi = g(v)$ both satisfy $\frac{\partial^2\phi}{\partial u \partial v} = 0$. As the equation is linear, the general solution is

$$\phi = f(u) + g(v) = f(x+3y) + g(x + \tfrac{3}{2}y)$$

where f and g are arbitrary functions. [3]

To find the particular solution, note that,

$$\phi(x,0) = f(x) + g(x) = \sin x \quad \ldots\ldots (i)$$

$$\frac{\partial\phi(x,0)}{\partial y} = 3f'(x) + \tfrac{3}{2}g'(x) = 3\cos x \quad \ldots\ldots (ii)$$

Differentiating (i) gives

$$f'(x) + g'(x) = \cos x$$

Combining with (ii) gives $g'(x) = 0$, $f'(x) = \cos x$.

Hence $f(x) = \sin x$, $g(x) = \text{constant}$.

But $g(x) = 0$ to satisfy (i). Hence the solution is

$$\phi(x,y) = f(x+3y) = \sin(x+3y)$$ [7]

2. Using the vector identity for $\nabla \times (\underset{\sim}{u}_1 \times \underset{\sim}{u}_2)$,

$$\nabla \times (\underset{\sim}{a} \times \underset{\sim}{r}) = \underset{\sim}{a}(\nabla \cdot \underset{\sim}{r}) - \underset{\sim}{r}(\nabla \cdot \underset{\sim}{a}) + (\underset{\sim}{r} \cdot \nabla)\underset{\sim}{a} - (\underset{\sim}{a} \cdot \nabla)\underset{\sim}{r}$$

$$= \underset{\sim}{a}(\nabla \cdot \underset{\sim}{r}) - (\underset{\sim}{a} \cdot \nabla)\underset{\sim}{r} \qquad [\underset{\sim}{a} = \text{constant}]$$

Now:

$$\nabla \cdot \underset{\sim}{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$(\underset{\sim}{a} \cdot \nabla)\underset{\sim}{r} = \left(a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}\right)(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k})$$

$$= a_x \underset{\sim}{i} + a_y \underset{\sim}{j} + a_z \underset{\sim}{k}$$

$$= \underset{\sim}{a}$$

Hence, [6]

$$\underline{\nabla \times (\underset{\sim}{a} \times \underset{\sim}{r}) = 3\underset{\sim}{a} - \underset{\sim}{a} = 2\underset{\sim}{a}}$$

Using the vector identity for $\nabla \times (\phi \underset{\sim}{u})$,

$$\nabla \times [r^n(\underset{\sim}{a} \times \underset{\sim}{r})] = r^n \nabla \times (\underset{\sim}{a} \times \underset{\sim}{r}) + \nabla r^n \times (\underset{\sim}{a} \times \underset{\sim}{r})$$

$$= 2r^n \underset{\sim}{a} + \nabla r^n \times (\underset{\sim}{a} \times \underset{\sim}{r})$$

Now:

$$\nabla r^n = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right) r^n \qquad \left[r = (x^2+y^2+z^2)^{1/2}\right]$$

$$= n r^{n-1}\left(\underset{\sim}{i}\frac{x}{r} + \underset{\sim}{j}\frac{y}{r} + \underset{\sim}{k}\frac{z}{r}\right)$$

$$= n r^{n-2} \underset{\sim}{r}$$

$$\therefore \quad \nabla \times \underset{\sim}{F} = 2r^n \underset{\sim}{a} + n r^{n-2} \underset{\sim}{r} \times (\underset{\sim}{a} \times \underset{\sim}{r})$$

Using the triple vector-product formula,

$$\underset{\sim}{r} \times (\underset{\sim}{a} \times \underset{\sim}{r}) = (\underset{\sim}{r}\cdot\underset{\sim}{r})\underset{\sim}{a} - (\underset{\sim}{r}\cdot\underset{\sim}{a})\underset{\sim}{r}$$

$$\therefore \quad \nabla \times \underset{\sim}{F} = 2r^n \underset{\sim}{a} + n r^{n-2}\left[r^2 \underset{\sim}{a} - (\underset{\sim}{r}\cdot\underset{\sim}{a})\underset{\sim}{r}\right]$$

$$\underline{\nabla \times \underset{\sim}{F} = (2+n) r^n \underset{\sim}{a} - n r^{n-2}(\underset{\sim}{r}\cdot\underset{\sim}{a})\underset{\sim}{r}} \qquad [9]$$

Setting $n=-3$, we have

$$-\nabla \times \left(\frac{\underset{\sim}{a} \times \underset{\sim}{r}}{r^3}\right) = r^{-3}\underset{\sim}{a} - 3r^{-5}(\underset{\sim}{r}\cdot\underset{\sim}{a})\underset{\sim}{r}$$

Now :

$$\nabla\left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{r}}{r^3}\right) = \frac{1}{r^3}\nabla(\underset{\sim}{a}\cdot\underset{\sim}{r}) + (\underset{\sim}{a}\cdot\underset{\sim}{r})\nabla r^{-3}$$

$$\nabla(\underset{\sim}{a}\cdot\underset{\sim}{r}) = \left(\underset{\sim}{i}\frac{\partial}{\partial x} + \underset{\sim}{j}\frac{\partial}{\partial y} + \underset{\sim}{k}\frac{\partial}{\partial z}\right)(x a_x + y a_y + z a_z) = \underset{\sim}{a}$$

$$\nabla r^{-3} = -3r^{-5}\underset{\sim}{r}$$

$$\therefore \quad \nabla\left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{r}}{r^3}\right) = r^{-3}\underset{\sim}{a} - 3r^{-5}(\underset{\sim}{r}\cdot\underset{\sim}{a})\underset{\sim}{r}$$

Hence, $\quad \underline{\nabla\left(\frac{a\cdot r}{r^3}\right) = -\nabla \times \left(\frac{a \times r}{r^3}\right)} \qquad [5]$

3. (i) $\underset{\sim}{r}$ has a scaler potential if $\nabla \times \underset{\sim}{r} = 0$

$$\nabla \times \underset{\sim}{r} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0$$

$$\underset{\sim}{r} = \nabla\phi \; ; \quad \frac{\partial\phi}{\partial x} = x \, , \quad \frac{\partial\phi}{\partial y} = y \, , \quad \frac{\partial\phi}{\partial z} = z$$

$$\rightarrow \quad \phi = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} \quad \rightarrow \quad \phi = \frac{r^2}{2}$$ [5]

(ii) Apply Gauss's theorem to the vector field $\phi \underset{\sim}{e}$:

$$\oiint_S \phi \underset{\sim}{e} \cdot d\underset{\sim}{A} = \iiint_V \nabla \cdot (\phi \underset{\sim}{e}) \, dV$$

$$\nabla \cdot (\phi \underset{\sim}{e}) = \phi (\nabla \cdot \underset{\sim}{e}) + \underset{\sim}{e} \cdot \nabla\phi = \underset{\sim}{e} \cdot \nabla\phi \qquad (\nabla \cdot \underset{\sim}{e} = 0)$$

Hence :

$$\oiint_S \underset{\sim}{e} \cdot (\phi \, d\underset{\sim}{A}) = \iiint_V \underset{\sim}{e} \cdot \nabla\phi \, dV$$

But $\underset{\sim}{e}$ is a constant vector so,

$$\underset{\sim}{e} \cdot \oiint_S \phi \, d\underset{\sim}{A} = \underset{\sim}{e} \cdot \iiint_V \nabla\phi \, dV$$

But $\underset{\sim}{e}$ is <u>any</u> unit vector so,

$$\oiint_S \phi \, d\underset{\sim}{A} = \iiint_V \nabla\phi \, dV \qquad \text{(Gradient theorem)}$$ [8]

(iii) Position vector of centre of mass is defined by

$$\bar{\underset{\sim}{r}} \iiint_V \rho \, dV = \iiint_V \underset{\sim}{r} \rho \, dV$$

Setting $\underset{\sim}{r} = \nabla\phi$ with $\phi = r^2/2$ and uniform ρ gives

$$\bar{\underset{\sim}{r}} \iiint_V dV = \iiint_V \nabla\left(\frac{r^2}{2}\right) dV$$

$$\therefore \quad \bar{\underset{\sim}{r}} = \frac{1}{V} \oiint_S \frac{r^2}{2} \, d\underset{\sim}{A} \qquad \text{(from gradient theorem)}$$ [7]

4. $$x\frac{d^2\phi}{dx^2} + \frac{d\phi}{dx} = \frac{d}{dx}\left(x\frac{d\phi}{dx}\right) = 0$$

$$\therefore \quad x\frac{d\phi}{dx} = c_1 \quad ; \quad \phi = c_1 \ln x + c_2$$

Applying the boundary conditions:

$$\left.\begin{aligned} 2 &= c_1 \ln 1 + c_2 \\ 2 &= c_1/4 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 8 \\ c_2 &= 2 \end{aligned}$$

[4]

A 2nd order accurate discretization for interior points is,

$$x_i\frac{(\phi_{i+1} - 2\phi_i + \phi_{i-1})}{\Delta x^2} + \frac{(\phi_{i+1} - \phi_{i-1})}{2\Delta x} = 0$$

With $\Delta x = 1$,

$$\phi_{i-1}(x_i - 0.5) + \phi_i(-2x_i) + \phi_{i+1}(x_i + 0.5) = 0$$

At $x = 1$, $\phi(1) = 2 \longrightarrow \phi_1 = 2$

At $x = 4$, $\frac{d\phi(4)}{dx} = 2 \longrightarrow \frac{\phi_4 - \phi_3}{\Delta x} = 2$

Second BC is only 1st order accurate. (This could be improved by fitting a quadratic through points (2,3,4) and calculating the slope at point 4.)

[5]

In matrix form the difference equations are:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1.5 & -4 & 2.5 & 0 \\ 0 & 2.5 & -6 & 3.5 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & -4 & 2.5 & 0 & -3 \\ 0 & 2.5 & -6 & 3.5 & 0 \\ 0 & 0 & -1 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & -4 & 2.5 & 0 & -3 \\ 0 & 0 & -4.4375 & 3.5 & -1.875 \\ 0 & 0 & -1 & 1 & 2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & -4 & 2.5 & 0 & -3 \\ 0 & 0 & -4.4375 & 3.5 & -1.875 \\ 0 & 0 & 0 & 0.21127 & 2.42254 \end{array}\right]$$

Back substitution gives :

$\phi_4 = 11.467$ $\qquad -4.4375\,\phi_3 + 3.5\,\phi_4 = -1.875$

$\phi_3 = 9.467$ $\qquad -4\,\phi_2 + 2.5\,\phi_3 = -3$

$\phi_2 = 6.667$ [8]

$\phi_1 = 2.0$

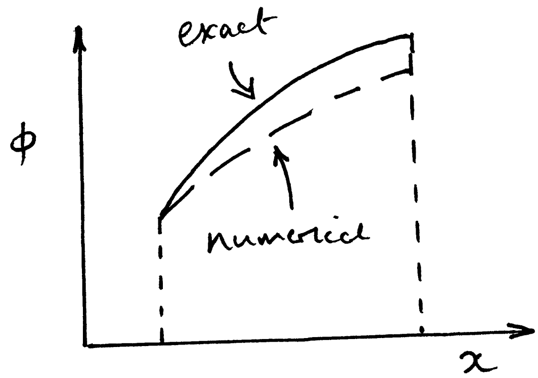
Exact values from solution of differential equation :

$\phi_4 = 13.090$

$\phi_3 = 10.789$

$\phi_2 = 7.545$

$\phi_1 = 2.0$



1st order discretization of BC at $x = 4$ affects solution badly. More points would improve matters.

[3]

5. (a)

$$u_i^{n+1} = u_i^n + \frac{\partial u}{\partial t}\Delta t + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t^2}{2} + \frac{\partial^3 u}{\partial t^3}\frac{\Delta t^3}{6} + \dots$$

$$u_{i+1}^n = u_i^n + \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{\Delta x^2}{2} + \frac{\partial^3 u}{\partial x^3}\frac{\Delta x^3}{6} + \frac{\partial^4 u}{\partial x^4}\frac{\Delta x^4}{24} + \frac{\partial^5 u}{\partial x^5}\frac{\Delta x^5}{120} + \frac{\partial^6 u}{\partial x^4}\frac{\Delta x^6}{720} + \dots$$

$$u_{i-1}^n = u_i^n - \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{\Delta x^2}{2} - \frac{\partial^3 u}{\partial x^3}\frac{\Delta x^3}{6} + \frac{\partial^4 u}{\partial x^4}\frac{\Delta x^4}{24} - \frac{\partial^5 u}{\partial x^5}\frac{\Delta x^5}{120} + \frac{\partial^6 u}{\partial x^4}\frac{\Delta x^6}{720} + \dots$$

Substituting into the finite difference approximation:

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t}{2} + \frac{\partial^3 u}{\partial t^3}\frac{\Delta t^2}{6} + \dots = \alpha\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4}\frac{\Delta x^2}{12} + \frac{\partial^6 u}{\partial x^6}\frac{\Delta x^4}{360} + \dots\right]$$

From the differential equation:

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^4 u}{\partial x^4} \quad ; \quad \frac{\partial^3 u}{\partial t^3} = \alpha^3 \frac{\partial^6 u}{\partial x^6}$$

Hence:

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} + \frac{\alpha}{12}\frac{\partial^4 u}{\partial x^4}\left[1 - \frac{6\alpha\Delta T}{\Delta x^2}\right]\Delta x^2 + \frac{\alpha}{360}\frac{\partial^6 u}{\partial x^4}\left[1 - \frac{60\alpha^2\Delta T^2}{\Delta x^4}\right]\Delta x^4 + \dots$$

$\therefore$ The truncation error is of order Δx^2 unless Δt is chosen such that $\frac{\alpha\Delta t}{\Delta x^2} = \frac{1}{6}$, when the error is of order Δx^4.

[10]

(b) The finite difference approximation is rearranged to give,

$$u_i^{n+1} = u_i^n - c\,(u_i^n - u_{i-1}^n) \qquad c = \frac{A\,\Delta t}{\Delta x}$$

i.e., $u_i^{n+1} = c\,u_{i-1}^n + (1-c)\,u_i^n$

	n	$n+1$	$n+2$	$n+3$	$n+4$
$i-1$	0	0	0	0	0
i	ε	$(1-c)\varepsilon$	$(1-c)^2\varepsilon$	$(1-c)^3\varepsilon$	$(1-c)^4\varepsilon$
$i+1$	0	$c\varepsilon$	$2c(1-c)\varepsilon$	$3c(1-c)^2\varepsilon$	$4c(1-c)^3\varepsilon$
$i+2$	0	0	$c^2\varepsilon$	$3c^2(1-c)\varepsilon$	$6c^2(1-c)^2\varepsilon$
$i+3$	0	0	0	$c^3\varepsilon$	$4c^3(1-c)\varepsilon$
$i+4$	0	0	0	0	$c^4\varepsilon$
$i+5$	0	0	0	0	0

Errors will only decay if $c < 1$ and so the stability limit is $c = \frac{A\,\Delta t}{\Delta x} = 1$.

{ For $c<1$, accuracy is lost because numerical diffusion causes the solution to "spread" in space.} [10]

6. $$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_0^1 e^{-j\omega t} dt$$

$$= \frac{e^{-j\omega t}}{-j\omega}\Big|_0^1 = \frac{e^{-j\omega} - 1}{-j\omega}$$

$$= e^{-j\omega/2} \frac{(e^{j\omega/2} - e^{-j\omega/2})}{j\omega} = e^{-j\omega/2} \frac{\sin(\omega/2)}{(\omega/2)}$$

[5]

$$= e^{-j\omega/2} \operatorname{sinc}\left(\frac{\omega}{2}\right)$$

Sampling generates the series :

$$\{f_n\}_{n=0}^{7} = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

The DFT is,

$$F_k = \sum_{n=0}^{7} f_n e^{-jkn 2\pi/N} \quad \text{with } N = 8$$

$$= \sum_{n=0}^{3} f_n e^{-jkn(\pi/4)} \quad \text{as } f_4 = f_5 = f_6 = f_7 = 0$$

The estimate of the FT is then the sequence

$$\{TF_k\}_{k=0}^{7} = \{0.25\ F_k\}_{k=0}^{7}$$

i.e., 8 values representing the FT at $\omega = k\Delta\omega = k\pi$, $k = 0 \to 7$.

$$0.25\,F_0 = 0.25\left[1+1+1+1\right] = 1$$

$$0.25\,F_1 = 0.25\left[1+e^{-j\pi/4}+e^{-j\pi/2}+e^{-j3\pi/4}\right] = 0.25 - j0.6036$$

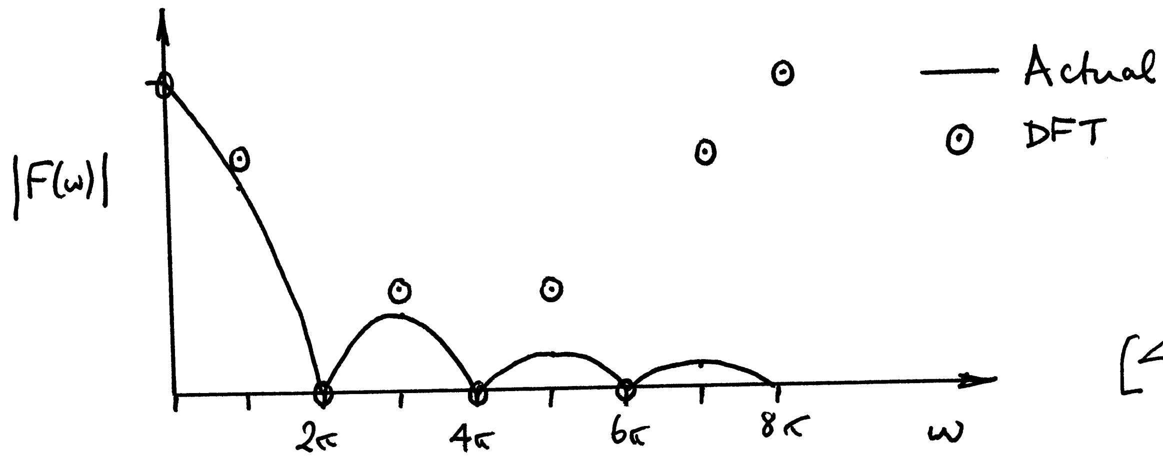
$$0.25\,F_2 = 0.25\left[1+e^{-j\pi/2}+e^{-j\pi}+e^{-j3\pi/2}\right] = 0$$

$$0.25\,F_3 = 0.25\left[1+e^{-j3\pi/4}+e^{-j3\pi/2}+e^{-j9\pi/4}\right] = 0.25 - j0.6036$$

$$0.25\,F_4 = 0.25\left[1+e^{-j\pi}+e^{-j2\pi}+e^{-j3\pi}\right] = 0$$

[11]

ω	Actual $\lvert F(\omega)\rvert$	Discrete $\lvert F(\omega)\rvert$
0	1.0	1.0
π	0.6366	0.6533
2π	0	0
3π	0.2122	0.2706
4π	0	0 Symmetrical
5π	0.1273	0.2706
6π	0	0
7π	0.0909	0.6533
8π	0	1.0



[4]

7. The Fourier transform of $f(t)$ is

$$F(\omega) = \int_{-T/2}^{T/2} f(t)\, e^{-j\omega t}\, dt$$

If $f(t)$ can be represented as a Fourier series, then

$$f(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{j\frac{2\pi k}{T}t}$$

where the complex coefficients are given by

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-j\frac{2\pi k}{T}t}\, dt$$

Thus, [5]

$$\underline{c_k = \frac{1}{T} F\left(\frac{2\pi k}{T}\right)}$$

Now,

$$f(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{j\frac{2\pi k}{T}t} = \sum_{k=-\infty}^{\infty} \frac{1}{T} F\left(\frac{2\pi k}{T}\right) e^{j\frac{2\pi k}{T}t}$$

The Fourier transform is thus,

$$F(\omega) = \int_{-T/2}^{T/2} \left\{ \sum_{k=-\infty}^{\infty} \frac{1}{T} F\left(\frac{2\pi k}{T}\right) e^{j\frac{2\pi k}{T}t} \right\} e^{-j\omega t}\, dt$$

As $F\left(\frac{2\pi k}{T}\right)$ is not a function of t, this may be written

$$F(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} F\left(\frac{2\pi k}{T}\right) \int_{-T/2}^{T/2} e^{-j(\omega - \frac{2\pi k}{T})t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} F\left(\frac{2\pi k}{T}\right) \left. \frac{e^{-j(\omega - \frac{2\pi k}{T})t}}{-j(\omega - \frac{2\pi k}{T})} \right|_{-T/2}^{T/2}$$

$$F(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} F\left(\frac{2\pi k}{T}\right) \left\{ \frac{e^{j\left(\omega - \frac{2\pi k}{T}\right)\frac{T}{2}} - e^{-j\left(\omega - \frac{2\pi k}{T}\right)\frac{T}{2}}}{j\left(\omega - \frac{2\pi k}{T}\right)} \right\}$$

$$F(\omega) = \sum_{k=-\infty}^{\infty} F\left(\frac{2\pi k}{T}\right) \operatorname{sinc}\left[\left(\omega - \frac{2\pi k}{T}\right)\left(\frac{T}{2}\right)\right] \qquad [12]$$

This equation is valid for all ω. Hence, if $F(\omega)$ is sampled at discrete intervals $\Delta\omega = 2\pi/T$, $F(2\pi k/T)$ can be calculated for all k and so $F(\omega)$ can be reconstructed in its entirety, not just at values of $\omega = 2\pi k/T$. [3]

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8. If a discrete random variable N takes integer values n, then $m(t)$ is a moment generating function for N if

$$m(t) = \sum_{\substack{\text{relevant}\\ n}} p_n t^n$$

where $p_n =$ probability $(N=n)$. It is useful for generating means, variances and other moments:

$$m(t) = \sum p_n t^n \quad \rightarrow \quad m(1) = \sum p_n = 1$$

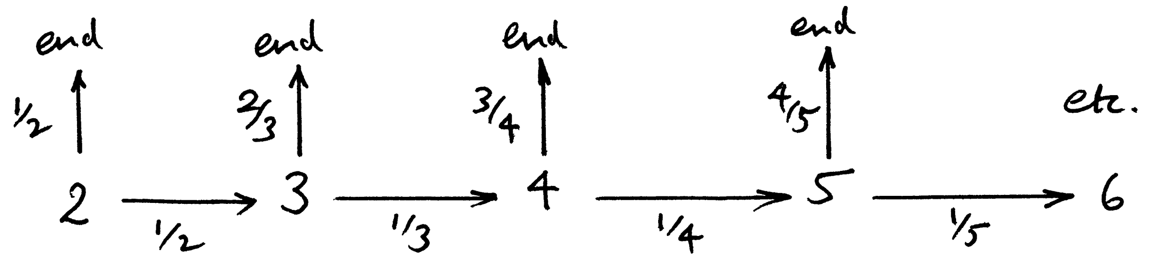
$$m'(t) = \sum n p_n t^{n-1} \quad \rightarrow \quad m'(1) = \sum n p_n = \text{Mean}(N)$$

$$m''(t) = \sum n(n-1) p_n t^{n-2} \rightarrow \quad m''(1) = \sum n^2 p_n - \sum n p_n$$

$$\text{Var}(N) = \sum n^2 p_n - \left(\sum n p_n\right)^2 = m''(1) + m'(1) - [m'(1)]^2$$

[4]

etc.



$$p(\text{reaches } 5) = \frac{1}{4} P(\text{reaches } 4) = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2}$$

$$p(\text{reaches } n) = \frac{1}{(n-1)!}$$

$$p(\text{ends at } 5) = \frac{4}{5} P(\text{reaches } 5)$$

$$p(\text{ends at } n) = \left(\frac{n-1}{n}\right) \frac{1}{(n-1)!} = \frac{(n-1)}{n!}$$

Moment generating function is

$$m(t) = \sum_{n=2}^{\infty} \frac{(n-1)}{n!} t^n$$

Now $$\sum_{n=2}^{\infty} \frac{(n-1)}{n!} t^n = \sum_{2}^{\infty} \frac{nt^n}{n!} - \sum_{2}^{\infty} \frac{t^n}{n!}$$

$$= t \sum_{2}^{\infty} \frac{t^{n-1}}{(n-1)!} - \sum_{2}^{\infty} \frac{t^n}{n!}$$

$$= t \sum_{1}^{\infty} \frac{t^m}{m!} - \sum_{2}^{\infty} \frac{t^n}{n!}$$

$$= t(e^t - 1) - (e^t - 1 - t)$$

$$= te^t - e^t + 1$$

Alternatively, work backwards :

$$te^t - e^t + 1 = t\left(1 + t + \frac{t^2}{2!} + \cdots\right) - \left(1 + t + \frac{t^2}{2!} + \cdots\right) + 1$$

$$= t^2\left[1 - \frac{1}{2!}\right] + t^3\left[\frac{1}{2!} - \frac{1}{3!}\right] + \cdots + t^n\left[\frac{1}{(n-1)!} - \frac{1}{n!}\right] + \cdots$$

But $\left[\frac{1}{(n-1)!} - \frac{1}{n!}\right] = \frac{n-1}{n!}$ [12]

$m(t) = te^t - e^t + 1 \longrightarrow m(1) = 1$

$m'(t) = e^t + te^t - e^t = te^t \longrightarrow m'(1) = e$

$m''(t) = e^t + te^t \longrightarrow m''(1) = 2e$

$\therefore$ Mean$(N) = e$;

Var$(N) = 2e + e - e^2 = 3e - e^2$ [4]
