ENGINEERING TRIPOS PART IB

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Friday 6 June 1997 9 to 11

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Paper 7

MATHEMATICAL METHODS

*Answer not more than* **four** *questions.*

*Answer at least* **one** *question from each section.*

*The* **approximate** *number of marks allocated to each part of a question is indicated in the right margin.*

*Answers to questions in each section should be tied together and handed in separately.*

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## SECTION A

*Answer at least* **one** *question from this section.*
*Do not answer more than* **two** *questions from this section.*

1 Find the values of the numbers $a$ and $b$ such that the change of variables,

$$u = x + ay$$
$$v = x + by$$

transforms the partial differential equation,

$$9\frac{\partial^2 \phi}{\partial x^2} - 9\frac{\partial^2 \phi}{\partial x \partial y} + 2\frac{\partial^2 \phi}{\partial y^2} = 0$$

into,

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0$$ [10]

Hence deduce that the general solution of the equation is,

$$\phi(x,y) = f(x+3y) + g\left(x+\tfrac{3}{2}y\right)$$

where $f$ and $g$ are arbitrary functions. [3]

Find the particular solution which satisfies the conditions,

$$\left.\begin{aligned} \phi &= \sin x \\ \frac{\partial \phi}{\partial y} &= 3\cos x \end{aligned}\right\} \text{ for all } x \text{ on } y = 0.$$ [7]

2 If $\mathbf{a}$ is a fixed vector and $\mathbf{r}$ the position vector, show that,

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{r}) = 2\boldsymbol{a}$$ [6]

If $\boldsymbol{F} = r^n(\boldsymbol{a} \times \boldsymbol{r})$, where $r = |\boldsymbol{r}|$, show that,

$$\nabla \times \boldsymbol{F} = (2+n)r^n\boldsymbol{a} - nr^{n-2}(\boldsymbol{r} \cdot \boldsymbol{a})\boldsymbol{r}$$ [9]

Hence prove that,

$$\nabla\left(\frac{\boldsymbol{a} \cdot \boldsymbol{r}}{r^3}\right) = -\nabla \times \left(\frac{\boldsymbol{a} \times \boldsymbol{r}}{r^3}\right)$$ [5]

Make free use of the vector identities in the Mathematics Databook and state clearly which ones you are using.

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3 (i) Prove that the position vector $\boldsymbol{r}$ has a scalar potential and find it. [5]

(ii) Consider a volume $V$ of arbitrary shape completely enclosed by a surface $S$ and let $\boldsymbol{dA}$ be a vectorial element of surface area. By applying Gauss's theorem to the vector field $\phi\, \boldsymbol{e}$ where $\phi$ is a non-uniform scalar and $\boldsymbol{e}$ is an arbitrary unit vector, prove that,

$$\oiint_S \phi \; \boldsymbol{dA} = \iiint_V \nabla\phi \; dV$$ [8]

(iii) Use the results of (i) and (ii) to show that the position vector $\bar{\boldsymbol{r}}$ of the centre of mass of an arbitrarily shaped body of uniform density and volume $V$ can be written,

$$\bar{\boldsymbol{r}} = \frac{1}{V} \oiint_S \frac{r^2}{2} \; \boldsymbol{dA}$$ [7]

where $r = |\boldsymbol{r}|$.

SECTION B

*Answer at least* **one** *question from this section.*

4 Solve the ordinary differential equation,

$$x\frac{d^2\phi}{dx^2} + \frac{d\phi}{dx} = 0$$

analytically in the range $1 \leq x \leq 4$, subject to the boundary conditions $\phi = 2$ at $x = 1$ and $d\phi/dx = 2$ at $x = 4$. [4]

Now set up a finite difference scheme to solve the equation numerically, stating the order of accuracy of your discretization and commenting on the implementation of the boundary conditions. Dividing the range of integration into equal increments of magnitude $\Delta x = 1$, assemble the difference equations into matrix form and solve the matrix equations. Compare the numerical and analytical values of $\phi$ and comment on the results. [16]

5 (a) The diffusion equation,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad [u = u(x,t),\ \alpha = \text{constant} > 0]$$

is to be integrated using the finite difference scheme,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{\alpha\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right)}{\Delta x^2}$$

where $U_i^n$ is the discrete approximation to $u$ at $x = i\Delta x,\ t = n\Delta t$. Use Taylor series expansions about the point $(i,n)$, and the original differential equation, to show that the scheme has, in general, a truncation error of order $\Delta x^2$. For what value of $\alpha\Delta t/\Delta x^2$ is the truncation error of order $\Delta x^4$ ? [10]

(b) The first order wave equation,

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \qquad [u = u(x,t),\ A = \text{constant} > 0]$$

is to be integrated using the finite difference scheme,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{A\left(U_i^n - U_{i-1}^n\right)}{\Delta x} = 0$$

where $U_i^n$ is the discrete approximation to $u$ at $x = i\Delta x,\ t = n\Delta t$. Starting from the condition $U_j^n = 0$ for all $j \neq i,\ U_i^n = \varepsilon$, march the equation forward in time for 4 steps and hence deduce the criterion for numerical stability. [10]

## SECTION C

*Answer at least* **one** *question from this section.*
*Do not answer more than* **two** *questions from this section.*

6 A signal $f(t)$ is defined by,

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

with $t$ in seconds. Show by direct integration that the Fourier transform of the signal is,

$$F(\omega) = e^{-j\omega/2}\text{sinc}\frac{\omega}{2}$$

where sinc $x = (\sin x)/x$. [5]

It is proposed to estimate $F(\omega)$ using 8 samples taken from $f(t)$ at intervals of 0·25s starting from $t = 0$. Obtain an expression for the discrete Fourier transform and calculate its magnitude at $\omega = 0, \pi, 2\pi, 3\pi$ and $4\pi$. [11]

Sketch the magnitudes of the actual and discrete Fourier transforms in the range $0 \le \omega \le 8\pi$ . [4]

Note that the DFT for $N$ samples $f_0, f_1, \ldots f_{N-1}$ is defined by

$$F_k = \sum_{n=0}^{N-1} f_n \; e^{-jkn2\pi/N}$$

7 Consider a signal $f(t)$ such that $f(t) = 0$ for $|t| > T/2$. The Fourier transform of $f(t)$ is $F(\omega)$. Assuming that $f(t)$ can be represented as a Fourier series in the range $|t| \leq T/2$, show that the complex Fourier coefficients are given by,

$$c_k = \frac{1}{T} F\left(\frac{2\pi k}{T}\right)$$ [5]

Hence show that

$$F(\omega) = \sum_{k=-\infty}^{\infty} F\left(\frac{2\pi k}{T}\right) \operatorname{sinc}\left[\left(\omega - \frac{2\pi k}{T}\right)\left(\frac{T}{2}\right)\right]$$

where $\operatorname{sinc} x = (\sin x)/x$. [12]

Explain how this implies that perfect reconstruction of $F(\omega)$ at all $\omega$ is possible by sampling at a particular discrete interval in the frequency domain. [3]

8 Explain what is meant by a *moment generating function* for a discrete random variable and explain why it is useful. [4]

A game is played whereby a ball is placed in one of two bags and a player has to guess which one does not contain the ball. If the player chooses the bag that contains the ball, the ball is replaced in the bag, a third (empty) bag is added and the bags are shuffled. The player then chooses one from amongst the three, again attempting to avoid the one which contains the ball. The process is repeated until the player picks an empty bag, with an empty bag being added each time an incorrect guess is made.

If $N$ is the number of bags present when the player eventually chooses an empty bag, show that a moment generating function $m(t)$ for $N$ is given by,

$$m(t) = te^t - e^t + 1$$ [12]

Find the mean and variance of $N$. [4]

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