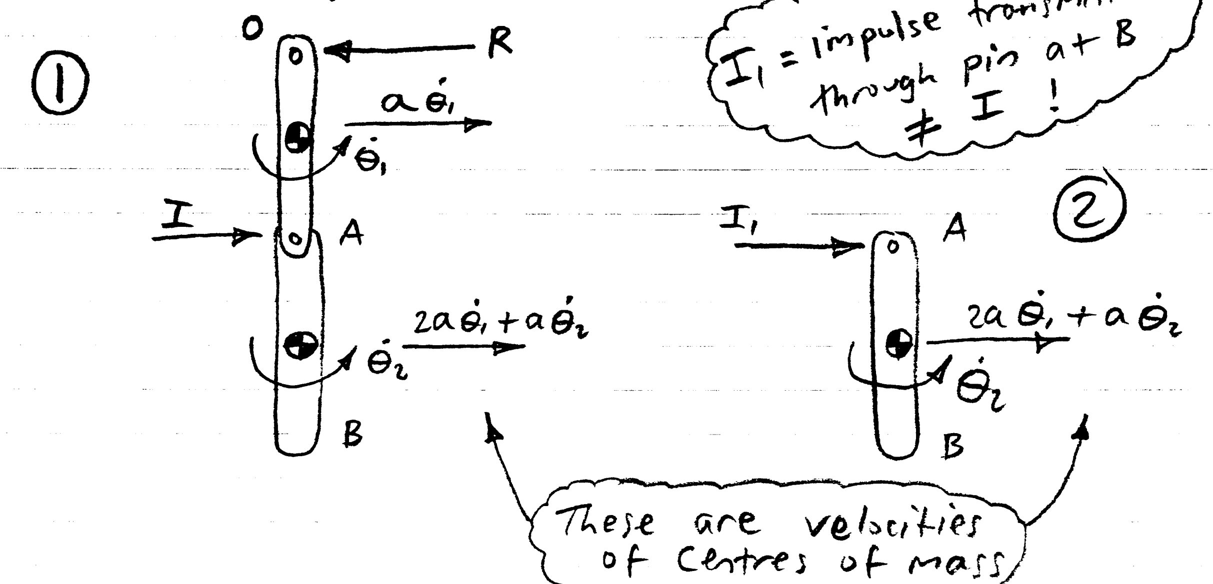
ENGINEERING TRIPOS PART IB 1998 Paper 1 Mechanics

1. In this question <u>many</u> candidates assumed gravity was important. Motion is in a horizontal plane (as stated) and gravity does <u>not</u> influence the motion (as stated).
It is really important that you <u>read the question</u>!

(a) Free body diagrams:



② Moment of Momentum about A:

$$0 = a\,m(2a\dot{\theta}_1 + a\dot{\theta}_2) + J\dot{\theta}_2$$

$$J = \tfrac{1}{12}m(2a)^2 = \tfrac{1}{3}ma^2$$

$$\therefore \quad 2\dot{\theta}_1 + \left(1 + \tfrac{1}{3}\right)\dot{\theta}_2 = 0 \qquad \therefore \quad \underline{\dot{\theta}_2 = -\tfrac{3}{2}\dot{\theta}_1}$$

① Moment of Momentum about O:

$$2aI = a\,ma\dot{\theta}_1 + 3a\,m(2a\dot{\theta}_1 + a\dot{\theta}_2) + J\dot{\theta}_1 + J\dot{\theta}_2$$

$$= ma^2\left[\left(7 + \tfrac{1}{3}\right)\dot{\theta}_1 + \left(3 + \tfrac{1}{3}\right)\dot{\theta}_2\right]$$

$$\therefore \frac{6I}{ma} = \left(22 + 10\left(-\tfrac{3}{2}\right)\right)\dot{\theta}_1 = 7\dot{\theta}_1$$

$$\therefore \quad \underline{\underline{\dot{\theta}_1 = \frac{6}{7}\frac{I}{ma}}}, \qquad \underline{\underline{\dot{\theta}_2 = -\frac{9}{7}\frac{I}{ma}}}$$

(b) (i) Moment of Momentum about O is equal to the Moment of external impulses $= \underline{\underline{2aI}}$

1. cont (b)(ii)

$$K.E. = \tfrac{1}{2} m\left[(a\dot{\theta}_1)^2 + (2a\dot{\theta}_1 + a\dot{\theta}_2)^2\right] + \tfrac{1}{2} J\left[\dot{\theta}_1^2 + \dot{\theta}_2^2\right]$$

$$= \tfrac{1}{2} ma^2\left[\dot{\theta}_1^2 + 4\dot{\theta}_1^2 + 4\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2 + \tfrac{1}{3}\dot{\theta}_1^2 + \tfrac{1}{3}\dot{\theta}_2^2\right] \quad \text{(eq. 1)}$$

$$= \frac{1}{2}\frac{I^2}{m}\left[\frac{16}{3}\cdot\left(\frac{6}{7}\right)^2 - 4\cdot\frac{6}{7}\cdot\frac{9}{7} + \frac{4}{3}\cdot\left(\frac{9}{7}\right)^2\right]$$

$$\underline{\underline{K.E. = \frac{6}{7}\frac{I^2}{m}}}$$

(c) In position (b), moment of momentum about O must be $2aI$, as always (conserved) and K.E. must be $\frac{6}{7}\frac{I^2}{m}$ (conserved).

$$\underline{K.E.}$$

$$\frac{6}{7}\frac{I^2}{m} = \tfrac{1}{2} ma^2\left[\frac{16}{3}\dot{\theta}_1^2 + 4\dot{\theta}_1\dot{\theta}_2 + \frac{4}{3}\dot{\theta}_2^2\right] \quad \text{(from eq. 1)}$$

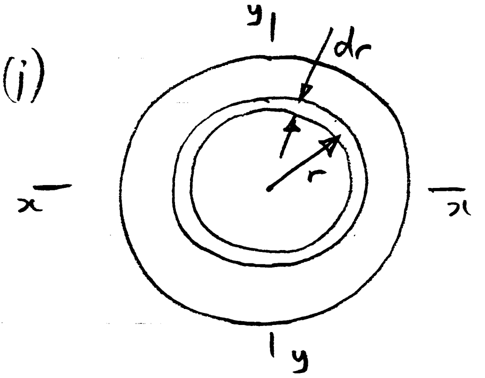
<u>Mom of Mom</u>

$$2aI = ma^2\left[\frac{22}{3}\dot{\theta}_1 + \frac{10}{3}\dot{\theta}_2\right]$$

and the resulting quadratic has <u>two</u> solutions, one of which we already have.

This question was easy to get full marks on (and many did) but most didn't draw free body diagrams and so cocked it all up.

2(a)(i)



Polar moment of inertia

$$I_{zz} = \int_0^a r^2\, dm$$
$$= \int_0^a (x^2 + y^2)\, dm$$
$$= \int_0^a x^2\, dm + \int_0^a y^2\, dm$$
$$= I_{xx} + I_{yy}$$

This is the perpendicular axis theorem

$$I_{zz} = \int_0^a r^2\, dm \qquad dm = 2\pi r\, dr \frac{m}{\pi a^2} = \frac{2m}{a^2} r\, dr$$
$$= \frac{2m}{a^2} \int_0^a r^3\, dr$$
$$= \frac{2m}{a^2} \left[\tfrac{1}{4} r^4\right]_0^a = \tfrac{1}{2} m a^2$$

$\therefore I_{xx} = \frac{1}{4} m a^2$ by $\perp$ axis theorem

Radius of gyration k : $I_{xx} = m k^2$

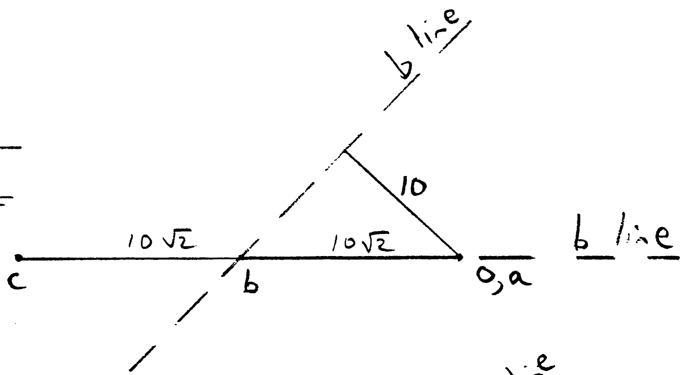
$$\therefore \underline{\underline{k = \frac{a}{2}}}$$

2(b) Velocity diagram

$\underline{\underline{\underset{\sim}{v}_c = 20\sqrt{2}\ \text{mm/s} \leftarrow}}$

$\omega_{OB} = \frac{10\sqrt{2}}{1} \times 10^{-3}$ rad/s

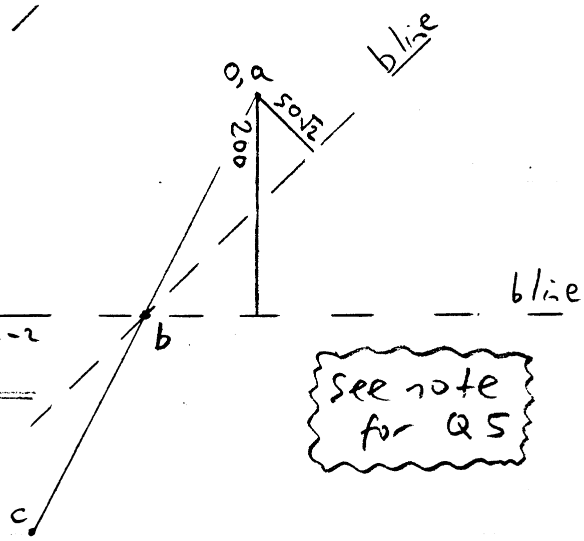
$\omega_{AB} = \frac{10}{\sqrt{2}} \times 10^{-3}$ rad/s



Acceleration diagram

$\overline{OB}\, \omega_{OB}^2 = 200 \times 10^{-6}\ \text{ms}^{-2}$

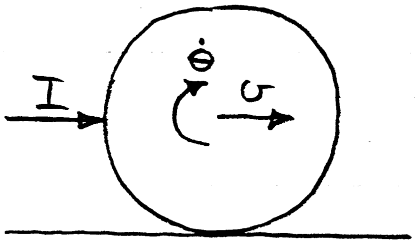
$AB\, \omega_{AB}^2 = 50\sqrt{2} \times 10^{-6}\ \text{ms}^{-2}$



$\underline{\underline{\underset{\sim}{a}_c = (400 \downarrow + 200 \leftarrow) \times 10^{-6}\ \text{ms}^{-2}}}$

see note for Q5

3 (a) (i)



ignore friction during impulse

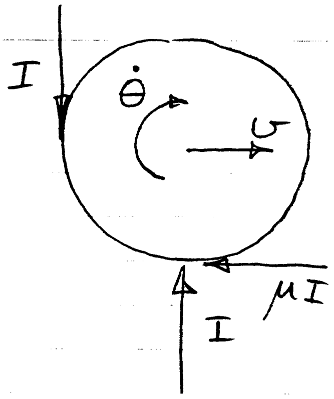
before : $v = 0$ $\quad \dot\theta = 0$

after : $v = v_1$ $\quad \dot\theta = \dot\theta_1$

$I = m v_1 \quad \therefore \quad \underline{v_1 = \frac{I}{m}}$

$\underline{\underline{\dot\theta_1 = 0}}$

(ii)



moment of inertia $= J$
$= \frac{2}{3} m a^2$

(expecting $\dot\theta$ to be negative)

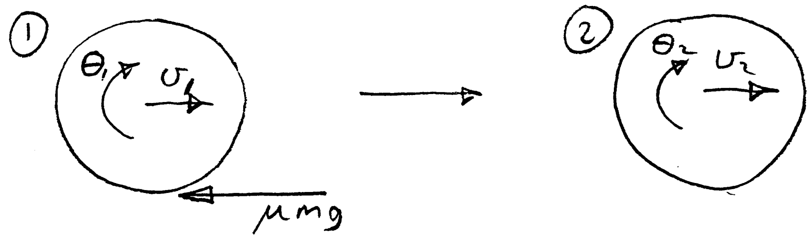
before : $v = 0$ $\quad \dot\theta = 0$

after : $v = v_1$ $\quad \dot\theta = \dot\theta_1$

$-\mu I = m v_1 \quad \therefore \quad \underline{\underline{v_1 = -\mu \frac{I}{m}}}$

$(\mu I - I) a = J \dot\theta_1 \quad \therefore \quad \dot\theta_1 = \frac{I a (\mu - 1)}{J}$ (-ve)

(b)



$v_2 = a \dot\theta_2$

Impulsive moment about centre :

$\mu m g a T = J(\dot\theta_2 - \dot\theta_1) = J\left(\frac{v_2}{a} - \dot\theta_1\right)$

Impulse $\leftrightarrow$ $\therefore \; -\mu m g T = m(v_2 - v_1)$

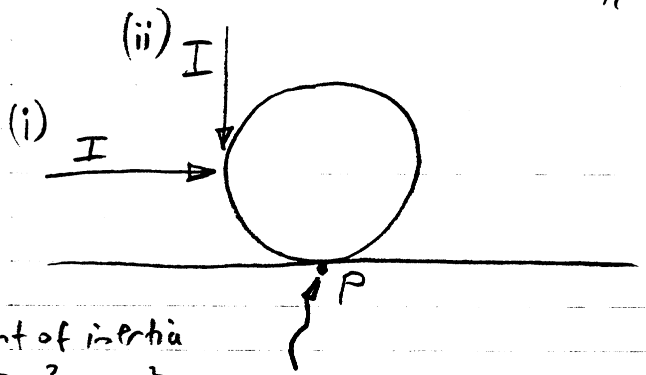
$\therefore \; -ma(v_2 - v_1) = \frac{2}{3} m a^2 \left(\frac{v_2}{a} - \dot\theta_1\right)$

$\therefore \; \frac{5}{3} v_2 = \frac{2}{3} a \dot\theta_1 + v_1$

$\therefore \; v_2 = \frac{2}{5} a \dot\theta_1 + \frac{3}{5} v_1 = \underline{\underline{\frac{3}{5}\frac{I}{m}}}$ (i)

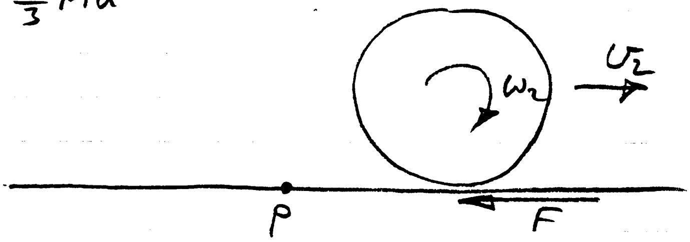
and for (ii) $\; v_2 = \frac{2}{5} a^2 (\mu - 1) \frac{I}{\frac{2}{3} m a^2} - \frac{3}{5} \mu \frac{I}{m} = \underline{\underline{-\frac{3}{5}\frac{I}{m}}}$

3 (b) (alt) The answer below was produced by a candidate in the exam. The examiner and most successful candidates weren't so clever, obtaining the correct answer but taking more time over it.



During either impulse there is an impulsive reaction through P
We needn't know its magnitude or direction

Moment of inertia
$J = \frac{2}{3} m a^2$



During subsequent slipping a friction force acts, but through P.

Finally, $U_2 = a\omega_2$ for rolling

Since P is a fixed point, take moments of momentum about it

$$\left.\begin{aligned} &\text{Case(i)} \quad +Ia \\ &\text{Case(ii)} \quad -Ia \end{aligned}\right\} = mU_2 a + J\omega_2$$

$$= mU_2 a + \frac{2}{3} m a^2 \frac{U_2}{a}$$

$$\therefore \quad \underline{\underline{U_2 = \pm \frac{3}{5} \frac{I}{m}}}$$

This solution is very elegant because:

i/ it explains why the final speed is the same for both
ii/ it explains why the answer is independent of μ
iii/ it cheered up the examiner!

(c) In part (ii), slip occurs in the correct direction only if $a\dot{\theta}_1 < U_1$

$$\therefore \quad \frac{Ia^2(\mu-1)}{\frac{2}{3} m a^2} < -\mu \frac{I}{m}$$

$$\therefore \quad \frac{3}{2}(\mu-1) < -\mu \qquad \therefore \quad \underline{\underline{\mu < \frac{3}{5}}}$$

Further elegance: If $U_1 = U_2$ then the sphere is rolling already so $\mu = \frac{3}{5}$ drops out neatly. A couple of students did it this way – full marks.

4(a) (i) $\underset{\sim}{r} = \underset{\sim}{r}_e + \underset{\sim}{r}_p + \underset{\sim}{r}_t$

(ii) $\dot{\underset{\sim}{r}} = \dot{\underset{\sim}{r}}_e + \dot{\underset{\sim}{r}}_p + \underset{\sim}{\omega} \times \underset{\sim}{r}_p + \dot{r}_t + \underset{\sim}{\omega} \times \underset{\sim}{r}_t$

but $\dot{\underset{\sim}{r}}_p = 0$

$\therefore \dot{\underset{\sim}{r}} = \dot{\underset{\sim}{r}}_e + \dot{\underset{\sim}{r}}_t + \underset{\sim}{\omega} \times (r_p + r_t)$

(iii) $\ddot{\underset{\sim}{r}} = \ddot{\underset{\sim}{r}}_e + \ddot{\underset{\sim}{r}}_t + \underset{\sim}{\omega} \times \dot{\underset{\sim}{r}}_t + \underset{\sim}{\omega} \times \dot{\underset{\sim}{r}}_t$
$+ \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times (\underset{\sim}{r}_p + \underset{\sim}{r}_t))$

$= \ddot{\underset{\sim}{r}}_e + \ddot{\underset{\sim}{r}}_t + 2\underset{\sim}{\omega} \times \dot{\underset{\sim}{r}}_t + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times (\underset{\sim}{r}_p + \underset{\sim}{r}_t))$

(b) for $\ddot{\underset{\sim}{r}} = \ddot{\underset{\sim}{r}}_e + \ddot{\underset{\sim}{r}}_t$

then $2\underset{\sim}{\omega} \times \dot{\underset{\sim}{r}}_t + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times (\underset{\sim}{r}_p + \underset{\sim}{r}_t)) = 0$

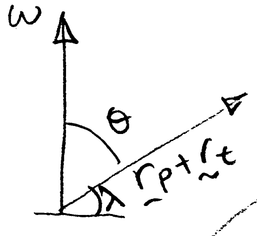
$\therefore \underset{\sim}{\omega} \times (2\dot{\underset{\sim}{r}}_t + \underset{\sim}{\omega} \times (\underset{\sim}{r}_p + \underset{\sim}{r}_t)) = 0$

So either $2\dot{\underset{\sim}{r}}_t + \underset{\sim}{\omega} \times (\underset{\sim}{r}_p + \underset{\sim}{r}_t) = 0$ ①

or $2\dot{\underset{\sim}{r}}_t + \underset{\sim}{\omega} \times (\underset{\sim}{r}_p + \underset{\sim}{r}_t) \;//^{l}\; \underset{\sim}{\omega}$ ②

Note that $\underset{\sim}{\omega} \times (\underset{\sim}{r}_p + \underset{\sim}{r}_t)$ is always in an Easterly direction so it has no component $//^{l}$ $\underset{\sim}{\omega}$.
So ② cannot hold unless ① holds

① requires that $\dot{\underset{\sim}{r}}_t$ be in a westerly direction



$$|\dot{r}_t| = \frac{|\omega| R \sin\theta}{2} = \frac{|\omega| R \cos\lambda}{2}$$

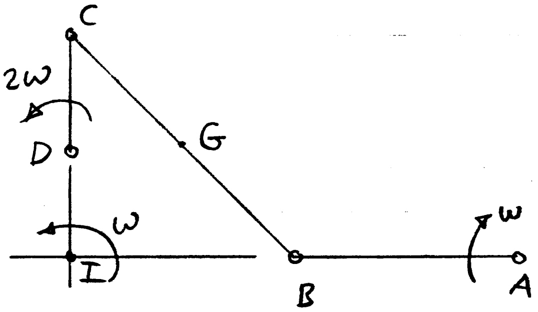
[Note that a train on the equator can have any Northerly component in addition to this since this is $//^{l}$ to $\underset{\sim}{\omega}$]

5 (i) By instantaneous centres

$\omega_{AB} = \curvearrowleft \omega$

$\omega_{BC} = \omega \curvearrowright$

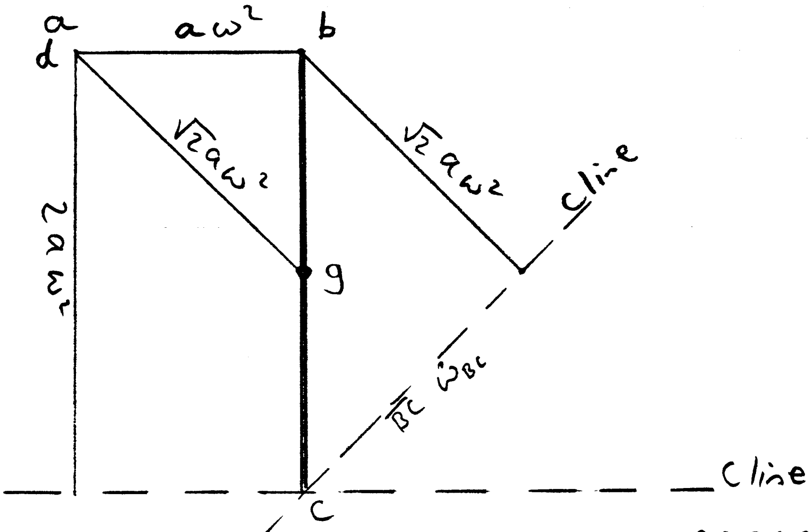
$\omega_{DC} = 2\omega \curvearrowright$



(ii) acceleration diagram

acceleration of G (mid point of BC) =

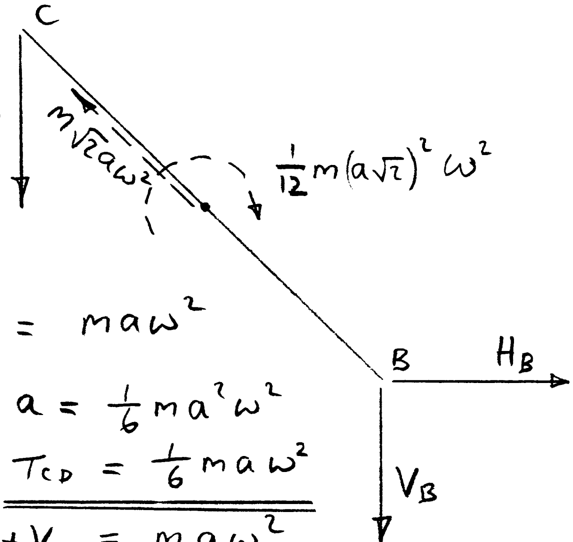
$\sqrt{2} a \omega^2 \searrow$



(iii) $\overline{BC}\,\dot{\omega}_{BC} = \sqrt{2}\, a \omega^2 \quad \therefore \quad \dot{\omega}_{BC} = \omega^2 \curvearrowright$

(iv)

Direction of T_{CD} is known because CD is light & pin at D is frictionless. CD could be replaced by a string in tension



$\Sigma F \leftrightarrow \quad \therefore \quad H_B = m a \omega^2$

$\Sigma M_c \quad \therefore \quad T_{CD}\, a = \frac{1}{6} m a^2 \omega^2$

$\therefore \quad \underline{\underline{T_{CD} = \frac{1}{6} m a \omega^2}}$

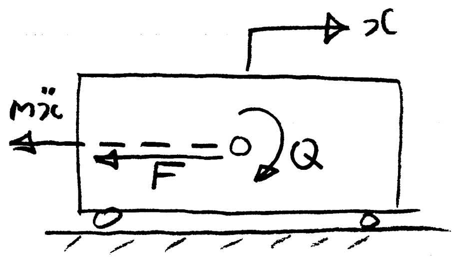
$\Sigma F \updownarrow \quad \therefore \quad T_{CD} + V_B = m a \omega^2$

$\therefore \quad V_B = \frac{5}{6} m a \omega^2$

Net force at $B = m a \omega^2 \sqrt{1 + \left(\frac{5}{6}\right)^2} = \underline{\underline{m a \omega^2 \frac{\sqrt{61}}{6}}}$

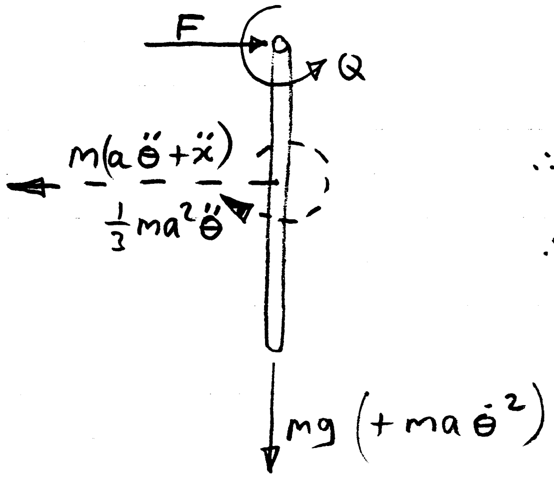
Dozens of people used the vector algebra method ($\underset{\sim}{e}, \underset{\sim}{e}^*$ etc)
There were only a couple of correct answers by this method.
Moreover, the time taken to obtain a solution (even the wrong one) was much greater.
In simple cases like this use inst. centres & accel diagrams.

6 (a)



$$F = -m\ddot{x}$$

Many did not draw free body diagrams. The problem is quite easy once you do.



$$F = m(a\ddot{\theta} + \ddot{x})$$

$$\therefore\ ma(\ddot{\theta} + \ddot{x}) = -m\ddot{x}$$

$$\therefore\ \ddot{x} = \frac{-a\ddot{\theta}}{2} = \frac{-a\pi}{4}$$

since $\ddot{\theta} = \frac{\pi}{2}$

(b)

$$Q = ma(a\ddot{\theta} + \ddot{x}) + \tfrac{1}{3}ma^2\ddot{\theta}$$

$$= ma^2\ddot{\theta}\left(1 - \tfrac{1}{2} + \tfrac{1}{3}\right)$$

$$= \tfrac{5}{6}ma^2\ddot{\theta}$$

$$Q = \tfrac{5}{12}ma^2\pi$$

(c) $\ddot{\theta} = \frac{\pi}{2}$ $\quad\therefore\ \dot{\theta} = \frac{\pi}{2}t$

$\therefore\ \theta = \frac{1}{2}\frac{\pi}{2}t^2 = \pi$ at $t = 2$

Same diagram as above (except upwards) and with a radial $ma\dot{\theta}^2$ term shown in brackets

This does not affect the equilibrium in (i) & (ii) so the answers are unchanged

(d) The centre of mass of the whole system does not move left or right so the amplitude of motion is $\pm\frac{a}{2}$