ENGINEERING TRIPOS - PART IB - 1998

PAPER 7 : MATHEMATICAL METHODS

SOLUTIONS

(J.B.Young)

1. (a) Transformation equation $\eta = x(2\alpha t)^{-1/2}$

$$\frac{\partial}{\partial x} = \frac{\partial \eta}{\partial x}\frac{d}{d\eta} = (2\alpha t)^{-1/2}\frac{d}{d\eta}$$

$$\frac{\partial^2}{\partial x^2} = (2\alpha t)^{-1}\frac{d^2}{d\eta^2}$$

$$\frac{\partial}{\partial t} = -\frac{x}{2}(2\alpha)^{-1/2}t^{-3/2}\frac{d}{d\eta} = -\frac{\eta}{2t}\frac{d}{d\eta}$$

$$\left.\begin{aligned} \frac{\partial \phi}{\partial t} &= -\frac{\eta}{2t}\frac{d\phi}{d\eta} \\ \alpha\frac{\partial^2 \phi}{\partial x^2} &= \frac{\alpha}{2\alpha t}\frac{d^2\phi}{d\eta^2} \end{aligned}\right\} \quad \frac{d^2\phi}{d\eta^2} + \eta\frac{d\phi}{d\eta} = 0$$

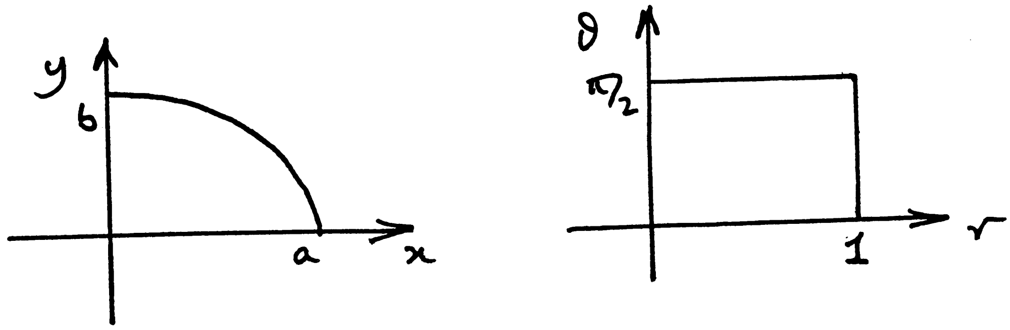
(b) The Jacobian of the transformation is :

$$J = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}$$

$$= (a\cos\theta)(br\cos\theta) - (-ar\sin\theta)(b\sin\theta)$$

$$= abr$$

The region of integration is transformed to a rectangle :



$$\therefore \quad I = \iint_R xy \, dx\,dy$$

$$= \iint abr^2 \sin\theta\cos\theta \, |J| \, dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 a^2b^2r^3 \sin\theta\cos\theta \, dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{a^2b^2}{4} \sin\theta\cos\theta \, d\theta$$

$$= \frac{a^2b^2}{8} \sin^2\theta \Big|_0^{\pi/2}$$

$$= \frac{a^2b^2}{8}$$

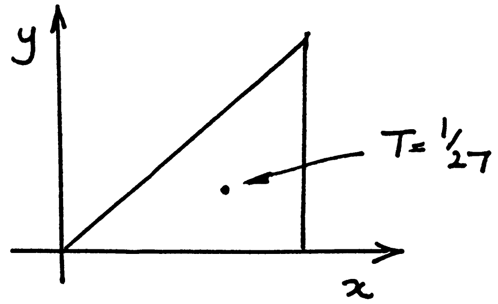
2(i) Max. T where $\partial T/\partial x = \partial T/\partial y = 0$

$$\frac{\partial T}{\partial x} = y(1-2x+y) = 0 \rightarrow y = 2x-1$$

$$\frac{\partial T}{\partial y} = (x-2y)(1-x) = 0 \rightarrow y = \frac{x}{2}$$

Together these give:

$x = \frac{2}{3}$, $y = \frac{1}{3}$, $T_{max} = \frac{1}{27}$



$$\underset{\sim}{q} = -\lambda \nabla T = -\lambda\left[y(1-2x+y)\underset{\sim}{i} + (x-2y)(1-x)\underset{\sim}{j}\right]$$

(ii) Total heat escape rate is given by $Q = \oint\!\!\oint \underset{\sim}{q}\cdot d\underset{\sim}{A}$

Surface $x=1$ (unit depth in z-direction):

$\underset{\sim}{q} = \lambda y(1-y)\underset{\sim}{i}$, $\quad d\underset{\sim}{A} = dy\,\underset{\sim}{i}$

$$Q_1 = \int_0^1 \lambda y(1-y)\,dy = \lambda\left(\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1 = \frac{\lambda}{6}$$

Surface $y=0$ (unit depth in z-direction):

$\underset{\sim}{q} = -\lambda x(1-x)\underset{\sim}{j}$, $\quad d\underset{\sim}{A} = -dx\,\underset{\sim}{j}$

$$Q_2 = \int_0^1 \lambda x(1-x)\,dx = \frac{\lambda}{6}$$

Surface $y=x$ (unit depth in z-direction):

$\underset{\sim}{q} = -\lambda y(1-y)\underset{\sim}{i} + \lambda x(1-x)\underset{\sim}{j}$, $\quad d\underset{\sim}{A} = -dy\,\underset{\sim}{i} + dx\,\underset{\sim}{j}$

$$Q_3 = \int_0^1 \lambda y(1-y)\,dy + \int_0^1 \lambda x(1-x)\,dx = \frac{\lambda}{6} + \frac{\lambda}{6} = \frac{\lambda}{3}$$

$$\therefore Q = Q_1 + Q_2 + Q_3 = \frac{2\lambda}{3}.$$

(iii) By Gauss's theorem:

$$\underset{\text{surface}}{\oiint} \underset{\sim}{q} \cdot d\underset{\sim}{A} = \underset{\text{volume}}{\iiint} (\nabla \cdot \underset{\sim}{q})\, dv$$

$$\nabla \cdot \underset{\sim}{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = \lambda\left[2y + 2(1-x)\right] = 2\lambda(1-x+y)$$

For unit depth in z-direction:

$$Q = \int_{x=0}^{1} \int_{y=0}^{x} 2\lambda(1-x+y)\, dy\, dx \qquad \left(\begin{array}{c} y \text{ integration first:} \\ \text{note limits} \end{array}\right)$$

$$= \int_0^1 2\lambda\left(y - xy + \frac{y^2}{2}\right)_0^x dx$$

$$= \int_0^1 2\lambda\left(x - \frac{x^2}{2}\right) dx$$

$$= 2\lambda\left(\frac{x^2}{2} - \frac{x^3}{6}\right)_0^1$$

$$= \frac{2\lambda}{3}$$

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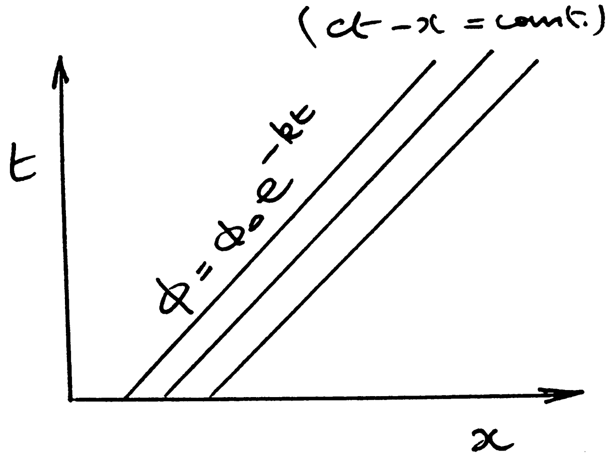
3. Along $dx/dt = c$, $ct - x =$ constant.

$$\phi = e^{-kt} F(ct - x)$$

represents waves convected with speed c in the positive x-direction.

At $t=0$, $\phi = \phi_0 = F(-x_0)$



If $ct - x =$ constant, then $F =$ constant along $dx/dt = c$.
Thus, along the characteristic, $\phi = \phi_0 e^{-kt}$, and each wave point decays in amplitude exponentially with time.
Similarly, $\phi = e^{-kt} G(ct + x)$ represent decaying waves convected with speed c in the negative x-direction.

Assume $\phi = X(x) T(t)$ and substitute in the PDE :

$$c^2 X'' T = T'' X + 2k X T' + k^2 X T$$

$$c^2 \frac{X''}{X} = \frac{T''}{T} + 2k \frac{T'}{T} + k^2 = -\alpha^2 c^2$$

(Negative separation constant to give trig. solution for X)

$$X'' + \alpha^2 X = 0 \quad \rightarrow \quad X = A \sin \alpha x + B \cos \alpha x$$

$\phi(0,t) = 0$ implies $B = 0$

$\phi(1,t) = 0$ implies $\alpha = n\pi$, $n = 1, 2, 3, \ldots$

$$T'' + 2kT' + (k^2+\alpha^2c^2)T = 0$$

Auxiliary equation is $m^2 + 2km + (k^2+\alpha^2c^2) = 0$

$$m = -k \pm \sqrt{k^2-(k^2+\alpha^2c^2)} = -k \pm \alpha c i$$

$$T = [A'\sin(\alpha ct) + B'\cos(\alpha ct)]e^{-kt}$$

Thus :

$$\phi = e^{-kt}[C\sin(\alpha ct) + D\cos(\alpha ct)]\sin(\alpha x)$$

$$\frac{\partial\phi}{\partial t} = -k\phi + e^{-kt}[\alpha c C\cos(\alpha ct) - \alpha c D\sin(\alpha ct)]\sin(\alpha x)$$

From $\frac{\partial\phi}{\partial t} + k\phi = 0$ at $t=0$ $\rightarrow$ $C = 0$

from $\phi = 5\sin(2\pi x)$ at $t=0$, $\alpha = 2\pi$, $D = 5$.

Solution is :

$$\phi(x,t) = 5e^{-kt}\cos(2\pi ct)\sin(2\pi x)$$

4. At iteration n, approximate solution is $x = x_n$.
$f(x_n) \neq 0$ but we require $f(x_{n+1}) = 0$.
Taylor expansion about x_n gives:

$$f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \tfrac{1}{2} f''(x_n)(x_{n+1} - x_n)^2 + \cdots$$

Truncating after the first derivative:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the true solution is $x = \alpha$, then $f(\alpha) = 0$.

$$(\alpha + \varepsilon_{n+1}) = (\alpha + \varepsilon_n) - \frac{f(\alpha + \varepsilon_n)}{g(\alpha + \varepsilon_n)} \qquad [g = f']$$

$$\therefore \quad \varepsilon_{n+1} = \varepsilon_n - \left\{ \frac{f(\alpha) + f'(\alpha)\varepsilon_n + \frac{1}{2} f''(\alpha)\varepsilon_n^2 + \cdots}{g(\alpha) + g'(\alpha)\varepsilon_n + \frac{1}{2} g''(\alpha)\varepsilon_n^2 + \cdots} \right\}$$

$$= \varepsilon_n - \frac{f'(\alpha)\varepsilon_n + \frac{1}{2} f''(\alpha)\varepsilon_n^2 + \cdots}{g(\alpha)\left[1 + \frac{g'(\alpha)}{g(\alpha)}\varepsilon_n + \cdots\right]}$$

$$= \varepsilon_n - \left[f'(\alpha)\varepsilon_n + \tfrac{1}{2} f''(\alpha)\varepsilon_n^2 + \cdots\right]\left[1 - \frac{g'(\alpha)}{g(\alpha)}\varepsilon_n + \cdots\right] / g(\alpha)$$

But $g(\alpha) = f'(\alpha)$, $g'(\alpha) = f''(\alpha)$, etc. Hence,

$$\varepsilon_{n+1} = \varepsilon_n - \left[\varepsilon_n + \varepsilon_n^2 \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} - \varepsilon_n^2 \frac{f''(\alpha)}{f'(\alpha)} + O(\varepsilon_n^3)\right]$$

$$\therefore \quad \varepsilon_{n+1} = C\varepsilon_n^2 + \cdots \qquad \text{with} \quad C = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

i.e. <u>Second-order convergence</u>

$$f(x) = x^4 - 7x^3 + 11x^2 + 7x - 12$$

$$f'(x) = 4x^3 - 21x^2 + 22x + 7$$

n	x_n	$f(x_n)$	$f'(x_n)$
1	1.5000	4.6875	6.2500
2	0.7500	−3.1992	13.3750
3	0.9892	−0.1301	12.0854
4	0.9996	−0.0004	12.0003
5	1.0000		

Solution in range $0 \leq x \leq 2$ is $x = 1$.

(other roots are $x = -1, 3, 4$).

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(other roots are $x = -1, 3, 4$).

5. The condition for minimising $q = \sum_{i=1}^{N} (y_i - Y_i)^2$ with respect to the coefficients a_k is,

$$\frac{\partial q}{\partial a_k} = 0 \quad , \quad k = 0, 1, 2.$$

Hence,

$$\frac{\partial q}{\partial a_k} = \sum_{i=1}^{N} 2(y_i - Y_i)\frac{\partial y_i}{\partial a_k} = 0$$

From $y_i = a_0 + a_1 X_i + a_2 X_i^2$, $\quad \frac{\partial y_i}{\partial a_k} = X_i^k$

$$\therefore \quad \sum_{i=1}^{N} (y_i - Y_i) X_i^k = 0 \quad , \quad k = 0, 1, 2$$

$$\sum_{i=1}^{N} (a_0 + a_1 X_i + a_2 X_i^2 - Y_i) X_i^k = 0 \quad , \quad k = 0, 1, 2$$

$$\underline{\sum_{i=1}^{N} (a_0 X_i^k + a_1 X_i^{k+1} + a_2 X_i^{k+2}) = \sum_{i=1}^{N} Y_i X_i^k \quad k = 0, 1, 2.}$$

For the 5 points :

$\sum X_i^0 = 5$, $\quad \sum X_i^1 = 1.9$, $\quad \sum X_i^2 = 1.23$,

$\sum X_i^3 = 0.919$, $\quad \sum X_i^4 = 0.7443$.

$\sum Y_i = 4.1$, $\quad \sum Y_i X_i = 2.0$, $\quad \sum Y_i X_i^2 = 1.388$

Thus:

$$\begin{pmatrix} 5.0 & 1.9 & 1.23 \\ 1.9 & 1.23 & 0.919 \\ 1.23 & 0.919 & 0.7443 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 4.1 \\ 2.0 \\ 1.388 \end{pmatrix}$$

Gaussian elimination :

5.0	1.9	1.23	4.1
	0.508	0.4516	0.442
	0.4516	0.44172	0.3794

5.0	1.9	1.23	4.1
	0.508	0.4516	0.442
		0.040258	−0.13528

Back substitution gives :

$a_2 = -0.3360$, $a_1 = 1.1688$, $a_0 = 0.4585$

6. By definition $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

$$\therefore \quad I = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t') e^{-j\omega t'} dt'\right] e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t') e^{j\omega(t-t')} dt'\right] d\omega \qquad \left(\begin{array}{l} e^{j\omega t} \text{ does not} \\ \text{depend on } t' \end{array}\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t') e^{j\omega(t-t')} d\omega\, dt' \qquad \left(\begin{array}{l} \text{Change order} \\ \text{of integration} \end{array}\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t')\, 2\pi\, \delta(t-t')\, dt' \qquad (\text{From given result})$$

$$= f(t) \qquad (\delta \text{ function picks out value of } f(t') \text{ at } t'=t)$$

$$f(t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{0} e^{a\omega} e^{j\omega t} d\omega + \int_{0}^{\infty} e^{-a\omega} e^{j\omega t} d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\int_{-\infty}^{0} e^{\omega(a+jt)} d\omega + \int_{0}^{\infty} e^{-\omega(a-jt)} d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\left.\frac{e^{\omega(a+jt)}}{a+jt}\right|_{-\infty}^{0} + \left.\frac{e^{-\omega(a-jt)}}{-(a-jt)}\right|_{0}^{\infty}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{1}{a+jt} + \frac{1}{a-jt}\right\}$$

$$= \frac{1}{2\pi}\frac{2a}{(a^2+t^2)}$$

$$= \frac{a}{\pi(a^2+t^2)}$$

Cross-correlation : $R(\tau) = \int_{-\infty}^{\infty} g_1(t)\, g_2(t+\tau)\, dt$

Change variables : $u = (t+\tau)$; $du = dt$; $u \to \pm\infty$ as $t \to \pm\infty$

$$\therefore \quad R(\tau) = \int_{-\infty}^{\infty} g_1(u-\tau)\, g_2(u)\, du$$

But g_1 is even so $g_1(u-\tau) = g_1[-(u-\tau)] = g_1(\tau-u)$

$$\therefore \quad R(\tau) = \int_{-\infty}^{\infty} g_1(\tau-u)\, g_2(u)\, du = g_1 * g_2$$

i.e, convolution of g_1 and g_2

Now $\quad g_1 * g_2 \xleftrightarrow{F_T} G_1 \times G_2$

(convolution) (multiplication)

$$g_1(t) = \frac{a}{\pi(a^2+t^2)} \rightarrow G_1(\omega) = \begin{cases} e^{a\omega} & \omega < 0 \\ e^{-a\omega} & \omega \geqslant 0 \end{cases}$$

$$g_2(t) = \frac{b}{\pi(b^2+t^2)} \rightarrow G_2(\omega) = \begin{cases} e^{b\omega} & \omega < 0 \\ e^{-b\omega} & \omega \geqslant 0 \end{cases}$$

$$\therefore \quad G_1 \times G_2 = \begin{cases} e^{(a+b)\omega} & \omega < 0 \\ e^{-(a+b)\omega} & \omega \geqslant 0 \end{cases}$$

Hence,

$$R(\tau) = g_1 * g_2 = \frac{a+b}{\pi[(a+b)^2 + \tau^2]}$$

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7. The FT of the sampled signal $f_s(t)$ is

$$F_s(\omega) = \int_{-\infty}^{\infty} f_s(t)\, e^{-j\omega t}\, dt$$

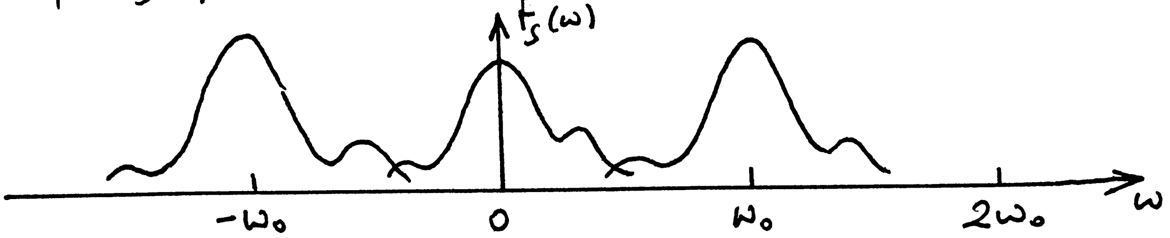
$$= \int_{-\infty}^{\infty} f(t) \left(\frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \right) e^{-j\omega t}\, dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} f(t) \sum_{n=-\infty}^{\infty} e^{-jt(\omega - n\omega_0)}\, dt$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{-j(\omega - n\omega_0)t}\, dt \quad \left(\text{Summation over } n \text{ and integration over } t \text{ are independent}\right)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_0)$$

Thus, the FT of the sampled signal is the FT of the original signal repeated at every integer multiple of the sampling frequency :



For a bandlimited signal $f(t)$ with FT $F(\omega)$, if the highest frequency present in $F(\omega)$ is ω_{max}, we require $\omega_0 \geq 2\omega_{max}$ to avoid overlap in the repeated signals. $2\omega_{max}$ is called the Nyquist frequency. If $\omega_0 < 2\omega_{max}$ the repeated spectra overlap resulting in aliasing. When aliasing occurs, the original signal cannot be perfectly reconstructed from its samples.

DFT : $F_k = \sum_{n=0}^{N-1} f_n e^{-jkn\frac{2\pi}{N}}$

and $f_n : \{0, 1, 0, -1\}$; $N = 4$

$F_0 = 0 + 1 + 0 - 1 = 0$

$F_1 = 0 + 1.e^{-j2\pi/4} + 0 - 1.e^{-j3\,2\pi/4}$

$= 0 - j + 0 - j$

$= -2j$

$F_2 = 0 + 1.e^{-j2\,2\pi/4} + 0 - 1.e^{-j6\,2\pi/4}$

$= 0 - 1 + 0 + 1$

$= 0$

$F_3 = 0 + 1.e^{-j3\,2\pi/4} + 0 - 1e^{-j9\,2\pi/4}$

$= 0 + j + 0 + j$

$= 2j$

$\therefore F_k : \{0, -2j, 0, 2j\}$

$\sum_{n=0}^{N-1} |f_n|^2 = 0 + 1 + 0 + 1 = 2$

$\frac{1}{N}\sum_{k=0}^{N-1} |F_k|^2 = \frac{1}{4}[0 + 4 + 0 + 4] = 2$

and the discrete form of Parseval's theorem is satisfied.

8. $$\mu_Y = E(aX_1 + bX_2)$$
$$= E(aX_1) + E(bX_2)$$
$$= aE(X_1) + bE(X_2)$$
$$= a\mu_1 + b\mu_2$$

$$\sigma_Y^2 = E((Y-\mu_Y)^2)$$
$$= E(((aX_1+bX_2) - (a\mu_1+b\mu_2))^2)$$
$$= E((a(X_1-\mu_1) + b(X_2-\mu_2))^2)$$
$$= E(a^2(X_1-\mu_1)^2 + 2ab(X_1-\mu_1)(X_2-\mu_2) + b^2(X_2-\mu_2)^2)$$
$$= a^2E((X_1-\mu_1)^2) + 2abE((X_1-\mu_1)(X_2-\mu_2)) + b^2E((X_2-\mu_2)^2)$$

As X_1 and X_2 are independent,

$$E((X_1-\mu_1)(X_2-\mu_2)) = E(X_1-\mu_1)E(X_2-\mu_2) = 0$$

$$\therefore \quad \sigma_Y^2 = a^2\sigma_1^2 + b^2\sigma_2^2$$
$$\sigma_Y = \sqrt{a^2\sigma_1^2 + b^2\sigma_2^2}$$

Mileage cost	N(£2500, £400)	Actual = £3140
Service cost	N(£1200, £300)	Actual = £1590

Mileage $\dfrac{M-\mu_M}{\sigma_M} = \dfrac{3140-2500}{400} = 1.6$; erf(1.6) = 0.9452

Service $\dfrac{S-\mu_S}{\sigma_S} = \dfrac{1590-1200}{300} = 1.3$; erf(1.3) = 0.9032

For 20 cabs take the 5% level (1/20) as being statistically significant. Hence, neither mileage nor servicing are significant.

For the combined cost, assume M and S are independent. (This is probably not true because a high mileage is likely to lead to a high service cost.) Because M and S are distributed normally, then M+S is also.

$$\mu_{M+S} = 2500 + 1200 = £3700$$

$$\sigma_{M+S} = \sqrt{400^2 + 300^2} = £500$$

$$\frac{(M+S) - \mu_{M+S}}{\sigma_{M+S}} = \frac{4730 - 3700}{500} = 2.06 \; ; \; \mathrm{erf}(2.06) = 0.9803$$

Hence the combined cost is significant at the 5% level (and also at the 2% level).