ENGINEERING TRIPOS PART IB

Friday 5 June 1998 9 - 11

Paper 7

MATHEMATICAL METHODS

Answer not more than **four** *questions.*

Answer at least **one** *question from each section.*

The **approximate** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

(TURN OVER

SECTION A

Answer at least **one** *question from this section.*

1 (a) Show that the partial differential equation,

$$\alpha \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \phi}{\partial t}$$

where α is a constant, can be transformed into the ordinary differential equation

$$\frac{d^2 \phi}{d\eta^2} + \eta \frac{d\phi}{d\eta} = 0$$

by the change of variable $\eta = \eta(x,t) = \dfrac{x}{\sqrt{2\alpha t}}$. [8]

(b) Use the transformation,

$$x = a r \cos\theta$$
$$y = b r \sin\theta$$

where a and b are constants, to evaluate,

$$\iint_R x\,y \;dx\,dy$$

where R is the region $x \geq 0,\; y \geq 0,$ within the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ [12]

2 A long metal bar has a cross-section defined by the lines $y = 0$, $y = x$, $x = 1$. Ohmic heating produces a temperature distribution in the bar which is given (in dimensionless form) by

$$T = y(x-y)(1-x)$$

and which does not vary along the length of the bar. Heat flow in the bar obeys Fourier's law of heat conduction, $\boldsymbol{q} = -\lambda \nabla T$, with constant thermal conductivity λ.

(i) Find the position and magnitude of the maximum value of T and derive an expression for the heat flux vector $\boldsymbol{q}$. [5]

(ii) By evaluating a suitable integral over the surface of the bar, calculate the total heat loss per unit length from the bar. [8]

(iii) Verify this result by using Gauss's theorem. [7]

3 Consider the partial differential equation

$$c^2 \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2} + 2k \frac{\partial \phi}{\partial t} + k^2 \phi$$

where c and k are positive constants. Given (do not prove) a solution

$$\phi(x,t) = e^{-kt}[F(ct-x) + G(ct+x)]$$

where F and G are arbitrary functions, explain the significance of the characteristic lines $dx/dt = \pm c$ and the constant k. [5]

Using the method of separation of variables, and not otherwise, find the solution of the differential equation for $0 \le x \le 1$ and $t \ge 0$ which satisfies the boundary conditions,

$\phi = 0$ at $x = 0$ for $t \ge 0$

$\phi = 0$ at $x = 1$ for $t \ge 0$

$\phi = 5\sin(2\pi x)$ for $0 \le x \le 1$ at $t = 0$

$\dfrac{\partial \phi}{\partial t} + k\phi = 0$ for $0 \le x \le 1$ at $t = 0$. [15]

SECTION B

Answer at least **one** *question from this section.*

4 For an equation of the form $f(x) = 0$, derive the Newton-Raphson iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}$$

where $g(x_n) = df/dx$ at $x = x_n$. [5]

If $x = \alpha$ is a root of $f(x) = 0$ and ε_n is the error at the n^{th} iteration, then $x_n = \alpha + \varepsilon_n$. By expanding both $f(x_n)$ and $g(x_n)$ as Taylor series about the point $x = \alpha$, show that the Newton-Raphson method converges according to,

$$\varepsilon_{n+1} = C\varepsilon_n^2 + \dots$$

and find an expression for the coefficient C. [8]

Using the Newton-Raphson method, find the root of

$$f(x) = x^4 - 7x^3 + 11x^2 + 7x - 12$$

in the interval $0 \leq x \leq 2$ from a first guess of $x = 1.5$. [7]

(TURN OVER

5 A quadratic polynomial of the form

$$y = \sum_{k=0}^{2} a_k x^k$$

is to be fitted to N data points $(X_i, Y_i,\ i = 1, 2, \ldots, N)$ in such a way as to minimise the function

$$q = \sum_{i=1}^{N} (y_i - Y_i)^2$$

where $y_i = \sum_{k=0}^{2} a_k X_i^k$.

(i) Show that the polynomial coefficients can be determined by solving the simultaneous equations,

$$\sum_{i=1}^{N} \left(a_0 X_i^k + a_1 X_i^{k+1} + a_2 X_i^{k+2} \right) = \sum_{i=1}^{N} Y_i X_i^k , \qquad k = 0, 1, 2.$$ [6]

(ii) Such a quadratic polynomial is to be fitted to the following data points:

X	0.00	0.10	0.40	0.50	0.90
Y	0.40	0.65	0.90	0.90	1.25

Set up the equations for the coefficients in matrix form and solve them using Gaussian elimination. [14]

SECTION C

Answer at least **one** *question from this section.*

6 Using the result

$$\lim_{a \to \infty} \int_{-a}^{+a} e^{j\omega t} d\omega = 2\pi\delta(t)$$

show that, if $F(\omega)$ is the Fourier transform of $f(t)$, then the *inverse Fourier transform* is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega .$$ [7]

A signal $f(t)$ has Fourier transform $F(\omega)$, where,

$$F(\omega) = e^{a\omega} \qquad \omega < 0$$

$$F(\omega) = e^{-a\omega} \qquad \omega \geq 0$$

with $a > 0$. Using the above result, show that,

$$f(t) = \frac{a}{\pi(a^2 + t^2)} .$$ [6]

The *cross-correlation,* $R(\tau)$, of two real functions $g_1(t)$ and $g_2(t)$, is given by,

$$R(\tau) = \int_{-\infty}^{\infty} g_1(t)\, g_2(t+\tau)\, dt .$$

If g_1 and g_2 are even functions of time, show that $R(\tau)$ is equivalent to the convolution of g_1 and g_2. Hence find $R(\tau)$ for the functions:

$$g_1(t) = \frac{a}{\pi(a^2+t^2)} , \qquad g_2(t) = \frac{b}{\pi(b^2+t^2)} .$$ [7]

(TURN OVER

7 The sampling function,

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

can be written as a Fourier series in the form,

$$s(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

where $\omega_0 = 2\pi/T$. If a signal $f(t)$ is sampled every T seconds using this sampling function, the sampled signal $f_s(t)$ is represented by $f_s(t) = s(t)f(t)$.

Using the Fourier series expression for $s(t)$, show that the Fourier transform of the sampled signal is given by,

$$F_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_0).$$ [6]

With reference to this result, explain the meaning of the terms *aliasing* and *Nyquist frequency* when applied to a sampled signal. [5]

The *discrete Fourier transform* (DFT) for a set of N samples $\{f_0, f_1, \ldots, f_{N-1}\}$ is defined by,

$$F_k = \sum_{n=0}^{N-1} f_n e^{-jkn2\pi/N}.$$

Form the DFT of the sequence {0, 1, 0, -1} and verify the discrete version of Parseval's theorem,

$$\sum_{n=0}^{N-1} |f_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |F_k|^2.$$ [9]

8 X_1 and X_2 are independent random variables with means μ_1 and μ_2, and standard deviations σ_1 and σ_2 respectively. If Y is a random variable defined by $Y = aX_1 + bX_2$ (where a and b are constants), show that its mean μ_Y and standard deviation σ_Y are given by,

$$\mu_Y = a\mu_1 + b\mu_2$$

$$\sigma_Y = \sqrt{a^2\sigma_1^2 + b^2\sigma_2^2}$$ [8]

The manager of a fleet of 20 taxicabs knows that the annual mileage and service cost per car are both random variables with normal distributions $N(\mu,\sigma)$:

Mileage ~ N(25000 miles, 4000 miles)

Service cost ~ N(£1200, £300)

The cost of fuel is 10p per mile. The most expensive taxi in the fleet has an annual mileage of 31400 miles and a service cost of £1590. Adopting a suitable criterion for significance, establish whether or not these figures are individually statistically significant. [6]

Stating any further assumptions, comment on the significance of the combined cost of mileage and servicing. [6]

END OF PAPER