Answers to IB Mechanics (1999)

Q1 POA = 90^0: $\omega_{AP} = 0$; $a_P = 17.7\omega^2 \Uparrow$; Q+50F = - $441.9\omega^2$.
OAP = 90^0: $\omega_{AP} = \omega / 9$; ; Q+52.7 F = + $188.6\,\omega^2$.

Q2. $M_{max} = \frac{2}{27} mg\ell \sin\theta$.
Loses contact when $\cos\theta = \frac{2}{3}\cos\alpha \rightarrow M_{max} = \frac{2}{27} mg\ell\sqrt{1 - \frac{4}{9}\cos^2\alpha}$

Q3. Force on ring $-2m\Omega\dot{\theta}\cos\theta\,\mathbf{e}^*$. $\dot{\theta}^2 = \Omega^2\left(\sin^2\theta - \sin^2\alpha\right) + \frac{2g}{R}(\cos\theta - \cos\alpha)$
$\cos\theta = \cos\alpha$ or $\cos\theta = \dfrac{2g}{\Omega^2 R} - \cos\alpha$

Q4 (a) $\omega_2 = -\frac{1}{2}\omega_1$: (b) $\omega_2 = \frac{13}{7}\omega_1$, $\omega_2 = -\frac{11}{7}\omega_1$

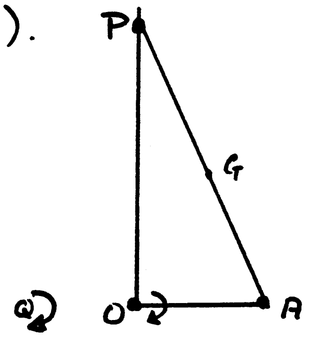
Q5 $\dfrac{m}{7}\left(v_0 - a\omega_0\right)^2$ where $v_0 = \dfrac{P}{m}(\cos\theta - \mu\sin\theta)$, $a\omega_0 = -\dfrac{5P}{2m}(1-\mu)\sin\theta$

Q6(i) AOC = 118^0, COB = 152^0, BOA=90^0. Q=350.9 Nm.
Reduce to 272Nm by putting C in middle: to zero by balancing:
eg ± 11 Kg mm on A, C at 43^0 to OA.

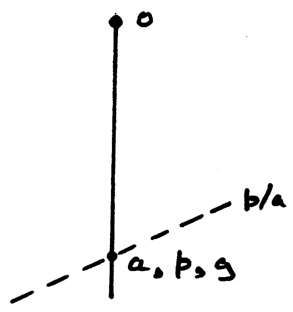
Q6(ii) $\dfrac{dh}{dt} = \Omega \times J\omega$:
Precession at $\Omega / 2$: spin $\omega = \Omega R / 2r$: $Q = 0.1 m R r \Omega^2$ pointing backwards along track.

IB Mechanics 1999 Q1.

(i).



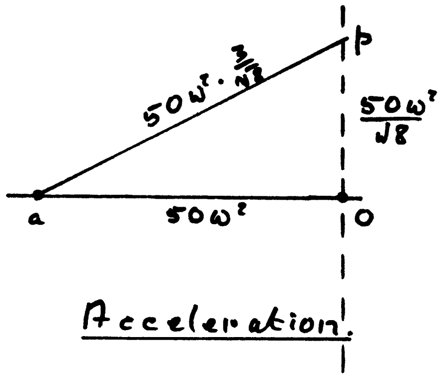
Real.
3 : 1 : √8 triangle



Velocity

$v_A = v_P = 50\omega \downarrow$

$\omega_{AP} = 0$



Acceleration.

$a_p = \frac{50\omega^2}{\sqrt{8}} = 17.68\omega^2 \uparrow$

[P wrt A represents the tangential acceleration $\dot{\omega}_{AP}$:

so $150\,\dot{\omega}_{AP} = 50\omega^2 \cdot \frac{3}{\sqrt{8}}$

and $\dot{\omega}_{AP} = \frac{\omega^2}{\sqrt{8}} = \cdot 354\,\omega^2$ ↻

but this will turn out to be irrelevant.]

{The relation between P and Q cannot be found – or not easily – by resolving or moments, as there may be a sideways force on the piston, and bearing forces at the crank. Many candidates did not realise this.}

Power $\quad Q.\omega + F.v_P = \frac{d}{dt}(KE)$

$= \frac{d}{dt}\left(\frac{1}{2}m_1 v_1^2 + \frac{1}{2}I_1\omega^2 + \frac{1}{2}m_2 v_2^2 \ldots\right)$

ie $Q.\omega + F.v_P = m_1 \underline{v}_1 . \underline{a}_1 + I_1\omega_1\dot{\omega}_1 + m_2 \underline{v}_2 . \underline{a}_2 + I_2\omega_2\dot{\omega}_2$

{which can be derived by virtual work + D'Alembert, but why bother?}

Since the crank is steadily rotating, it makes no contribution. Also $\omega_{AP} = 0$, leaving only

$Q.\omega + F v_P = m \underline{v}_G . \underline{a}_G$

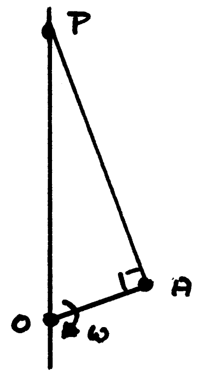
$\underline{v}_G = 50\omega \downarrow \; : \; \underline{a}_G = (25\omega^2 \leftarrow, \; 25\omega^2/\sqrt{8} \uparrow)$

so $\quad Q + 50F = -41.9\,\omega^2$

{Many candidates seemed not to be aware that the purpose of an internal combustion engine is to provide a torque by using a force derived from expansion of a gas – so hoped to find both Q and F}

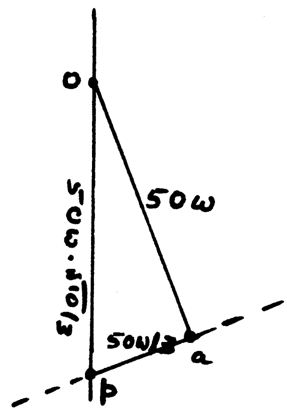
Careful drawing was acceptable, but much harder!

(iii)



Real.

$3:1:\sqrt{10}$ triangle.



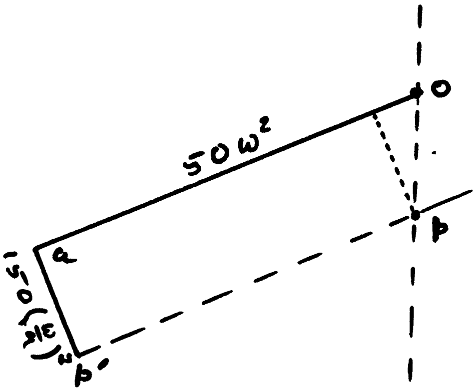
Velocity.

$v_P = 50\omega \cdot \frac{\sqrt{10}}{3}$

$= 52.70\omega \downarrow$

$150\,\omega_{AP} = \frac{50\omega}{3}$

$\omega_{AP} = \frac{\omega}{9}$ ↻



Acceleration

NOT TO SCALE!

{The centripetal acceleration of P/A is $150(\omega/9)^2 = 1.85\omega^2$ is barely visible on a scaled diagram.}

From similar triangles,

$a_P = op = 1.85\omega^2 \times \sqrt{10}/3$

$= \underline{1.952\,\omega^2}$

while the tangential acc^n

$l\dot{\omega}_{AP} = p'p$

$= 50\omega^2 - 1.85\omega^2 \times 1/3$

ie $\dot{\omega}_{AP} = 0.329\,\omega^2$ ↺

Use components of $\underline{v}_G$ and $\underline{a}_G$ along and normal to PA (again crank has constant KE so ignore it).

$\underline{v}_P = \left(\frac{50\omega}{3}\swarrow,\ 50\omega\searrow\right)$

$\underline{v}_A = \left(0\swarrow,\ 50\omega\searrow\right)$ $\quad \therefore\ \underline{v}_G = \left(\frac{25\omega}{3}\swarrow,\ 50\omega\searrow\right)$

$\underline{a}_P = \left(\frac{50\omega^2}{81}\swarrow,\ \frac{50\omega^2}{27}\searrow\right)$

$\underline{a}_A = \left(50\omega^2\swarrow,\ 0\right)$ $\quad \therefore\ \underline{a}_G = \left(50\omega^2\cdot\frac{41}{81}\swarrow,\ \frac{25\omega^2}{27}\searrow\right)$

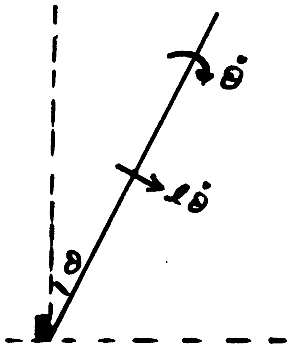
$I = \frac{1}{12}ml^2 = 150^2 m/12$

$\Rightarrow P(52.70\omega) + Q\omega = m\left\{\frac{25\omega}{3}\cdot 50\omega^2\cdot\frac{41}{81} + 50\omega\cdot\frac{25\omega^2}{27} - \frac{150^2}{12}\cdot\frac{\omega}{9}\cdot\frac{80\omega^2}{243}\right\}$

$= m\omega^3\{210.9 + 46.3 - 68.6\}$

ie $\underline{Q + 52.7P = 188.6\,m\omega^2}$

Q2.



By conservation of energy (assuming stationary when the axle breaks at $\theta = \alpha$)

$$mgl(\cos\alpha - \cos\theta) = \tfrac{1}{2}m(l\dot\theta)^2 + \tfrac{1}{2}I\dot\theta^2$$

with $I = \frac{1}{12}m(2l)^2$

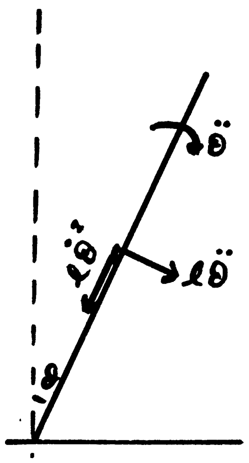
$$\therefore \frac{g}{l}(\cos\alpha - \cos\theta) = \frac{1}{2}\left[\dot\theta^2 + \frac{1}{3}\dot\theta^2\right]$$

$$\text{so } \dot\theta^2 = \frac{3g}{2l}(\cos\alpha - \cos\theta)$$

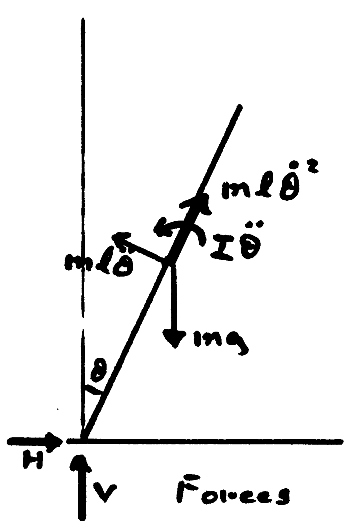
Differentiating wrt time:

$$2\dot\theta\ddot\theta = \frac{3g}{2l}(\sin\theta)\dot\theta \quad \text{ie } \ddot\theta = \frac{3g}{4l}\sin\theta$$

{Many candidates preferred to derive this ab initio by D'Alembert forces and taking moments, ignoring any connection between $\dot\theta^2$ and $\ddot\theta$ – indeed, often finding completely incompatible results.}



Accelerations.



Forces

The horizontal force at the foot must come from the reaction at the stop since frictionless ground, so $H > 0$. For the whole column,

$$H = ml\dot\theta^2\sin\theta - ml\ddot\theta\cos\theta$$

$$\text{so } H = 0 \text{ when } \dot\theta^2\sin\theta = \ddot\theta\cos\theta$$

124

Substituting,

2 contd.

$$\frac{3g}{2l}(\cos\alpha - \cos\theta)\sin\theta = \frac{3g}{4l}\sin\theta \, . \, \cos\theta$$

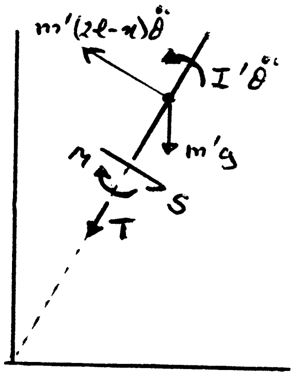
$\therefore \quad \sin\theta = 0 \quad$ or $\quad \cos\alpha - \cos\theta = \frac{1}{2}\cos\theta$

ie $\quad \cos\theta = \frac{2}{3}\cos\alpha$

so at this angle, column loses contact with stop.

eg $\alpha = 45^\circ \Rightarrow \theta = 62^\circ$ and clearly there is always an angle between $\theta = \alpha$ and $\theta = \pi/2$ when this occurs. Note that simply proving that contact is lost before $\theta = \pi/2$ is easier: for at $\theta = \frac{\pi}{2}$ we have $H = -ml\dot{\theta}^2$ and $\dot{\theta}^2$ is certainly not zero ($KE = PE$) so H has already changed sign before this point.

To examine the bending moment, cut off a length ($2x$) of column from the top.



mass $\quad m' = mx/l$

$I'_G = \frac{1}{3}m'x^2 = \frac{1}{3}mx^3/l$

Maximum BM is when $S = 0$ since $S = \frac{dM}{dx}$. But

$$S + m'g\sin\theta = m'(2l - x)\ddot{\theta}$$

so $S = 0$ when $\quad g\sin\theta = (2l - x)\ddot{\theta}$

$$= (2l - x)\frac{3g}{4l}\sin\theta$$

so, independent of θ, $\quad 4l = 3(2l - x)$

so $x = 2l/3$

ie $4l/3$ from top: $\frac{2l}{3}$ from foot.

For this section, since $S = 0$, can take moments about centroid to get $\quad M = I'\ddot{\theta}$

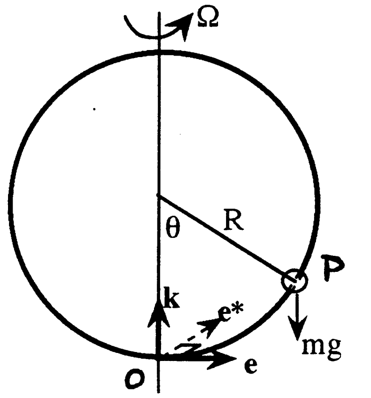
so $M_{max} = \frac{m}{3}\left(\frac{2l}{3}\right)^3/l \cdot \frac{3g}{4l}\sin\theta$

$$= \frac{2}{27} \cdot mgl\sin\theta$$

$\therefore M_{max}\big|_{\theta_{crit}} = \frac{2}{27} mgl \cdot \sqrt{1 - \frac{4}{9}\cos^2\alpha}$

p5

Q3.



OP is

$$\underline{r} = R\underline{k} - R\cos\theta\,\underline{k} + R\sin\theta\,\underline{e}$$

where $\underline{k}$ is a fixed direction

$$(\underline{\Omega} = \Omega\underline{k})$$

while $\underline{e}$ and $\underline{e}^*$ rotate at $\underline{\Omega}$, so that

$$\dot{\underline{e}} = \Omega\,\underline{e}^* : \dot{\underline{e}}^* = -\Omega\,\underline{e}$$

Then $$\underline{v} = R\sin\theta\,\dot\theta\,\underline{k} + R\cos\theta\,\dot\theta\,\underline{e} + R\sin\theta\,\Omega\,\underline{e}^*$$

and
$$\begin{aligned}\underline{a} &= R\cos\theta\,\dot\theta^2\,\underline{k} + R\sin\theta\,\ddot\theta\,\underline{k} - R\sin\theta\,\dot\theta^2\,\underline{e} \\ &\quad + R\cos\theta\,\ddot\theta\,\underline{e} + R\cos\theta\,\dot\theta\,\Omega\,\underline{e}^* + R\cos\theta\,\dot\theta\,\Omega\,\underline{e}^* \\ &\quad + R\sin\theta\,\Omega\,(-\Omega\,\underline{e}) \\ &= (R\cos\theta\,\dot\theta^2 + R\sin\theta\,\ddot\theta)\,\underline{k} \\ &\quad + (R\cos\theta\,\ddot\theta - R\sin\theta\,\dot\theta^2 - R\sin\theta\,\Omega^2)\,\underline{e} \\ &\quad + 2\Omega\dot\theta\cos\theta\,\underline{e}^*\end{aligned}$$

Can equally be found from Data Book: bead acceleration ignoring the ring rotation is $R\ddot\theta$ along the ring, ie $R\ddot\theta\cos\theta\,\underline{e} + R\ddot\theta\sin\theta\,\underline{k}$

Point on ring has acceleration $\Omega^2 R\sin\theta$ towards the axis, ie $-\Omega^2 R\sin\theta\,\underline{e}$ (and so a component $-\Omega^2 R\sin\theta\cos\theta$ along the hoop)

Relative velocity of bead wrt ring is $R\dot\theta$ along ring, so Coriolis acceleration $2\underline{\Omega}\times\underline{v}$ is $2\Omega R\dot\theta\cos\theta$ normal to the hoop.

Ring must exert a force $2m\Omega\dot\theta\cos\theta\,\underline{e}^*$ on the bead, so force exerted by bead on ring is

$$-2m\Omega\dot\theta\cos\theta\,\underline{e}^*$$

Tangent to ring at P is $\hat{\underline{t}} = \underline{e}\cos\theta + \underline{k}\sin\theta$, so tangential acceleration is

$\underline{a}\cdot\hat{\underline{t}} = R\ddot\theta - \Omega^2 R\sin\theta\cos\theta$ (by algebra, or directly from Data Book approach).

B6. (3 cont'd)

Since ring is frictionless, only force on bead in tangential direction is due to gravity, so

$$-mg\sin\theta = m\left[R\ddot{\theta} - \Omega^2 R\sin\theta\cos\theta\right]$$

ie $$\ddot{\theta} = \Omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta$$

Multiply by $2\dot{\theta}$ and integrate wrt time:

$$\dot{\theta}^2 = 2\Omega^2\int\sin\theta\cos\theta\,d\theta - \frac{2g}{R}\int\sin\theta\,d\theta$$

Since $\frac{d}{d\theta}(\sin^2\theta) = 2\sin\theta\cos\theta$, then

$$\dot{\theta}^2 = \Omega^2\sin^2\theta + \frac{2g}{R}\cos\theta + C$$

$\dot{\theta} = 0$ when $\theta = \alpha$ so $C = -\Omega^2\sin^2\alpha - \frac{2g}{R}\cos\alpha$

$$\therefore\ \dot{\theta}^2 = \Omega^2(\sin^2\theta - \sin^2\alpha) + \frac{2g}{R}(\cos\theta - \cos\alpha)$$

To show this has a factor $(\cos\theta - \cos\alpha)$ it is NOT enough – as many candidates believed – to show there is a term containing the factor! But

$$\sin^2\theta - \sin^2\alpha = \cos^2\alpha - \cos^2\theta = (\cos\alpha - \cos\theta)(\cos\alpha + \cos\theta)$$

so $$\dot{\theta}^2 = (\cos\alpha - \cos\theta)\left\{\Omega^2(\cos\alpha + \cos\theta) - \frac{2g}{R}\right\}$$

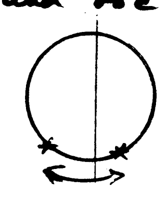
and bead can be at rest when $\cos\theta = \cos\alpha$ (i.e. $\theta = \pm\alpha$)

or when $\cos\theta = \frac{2g}{\Omega^2 R} - \cos\alpha$

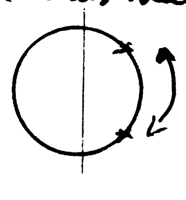
At low speeds, RHS $\gg 1$ and no solution exists, so only stationary points are $\theta = \pm\alpha$ – and since ring is frictionless, bead oscillates between them for ever.

At high speeds, $\cos\theta \sim -\cos\alpha$ is also a solution $\left\{\right.$ more accurately, $\cos\theta = -\cos\alpha + \frac{2g}{\Omega^2 R}$ gives $\theta \sim \pi - \alpha - \frac{2g}{\Omega^2 R}\operatorname{cosec}\alpha \ldots \left.\right\}$

and bead oscillates between α and this value.



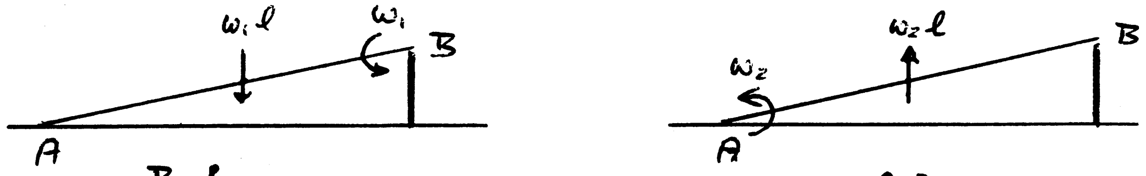
Low speed.



High speed.

p 7

Q4)



Before   After

{No marks were allocated for the above diagrams — but those who failed to draw them got the analysis wrong. }.

If the only impulse is at A, then moment of momentum about A is conserved.

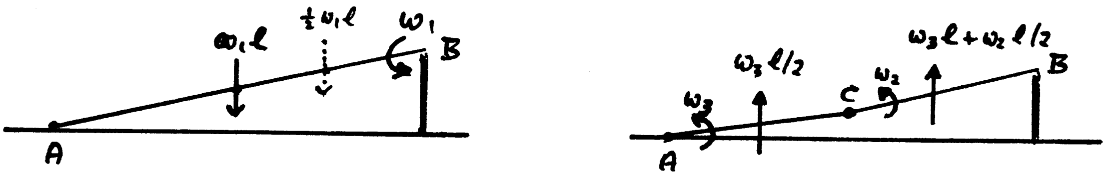
$$\circlearrowleft \quad \therefore \; I\omega_1 - m(\omega_1 l).\,l \;=\; I\omega_2 + m(\omega_2 l)\,l$$

where I is moment of inertia about centroid : $I = \frac{1}{3} m l^2$.

$$\therefore \; \tfrac{1}{3}\omega_1 - \omega_1 = \tfrac{1}{3}\omega_2 + \omega_2$$

ie $\underline{\omega_2 = -\frac{1}{2}\omega_1}$

ie the deck continues to rotate in such a way that end B still descends, crushing the pier — and since the top of the pier suddenly acquires a downward velocity, receiving an impulse which invalidates the analysis.



{A major problem is reminding yourself that the half-decks do NOT have mass m and moment of inertia I — plan now how you will do it next time! }.

After impact, if rotation is again about A, C has upward velocity $\omega_3 l$, so centroid of CB has velocity $\omega_3 l + \frac{1}{2}\omega_2 l$.

Moment of momentum of whole system about A is conserved:

so $\circlearrowleft \quad I\omega_1 - m\omega_1 l^2$ (as before)

$$= I'\omega_3 + m'(\omega_3 l/2).(l/2) + I'\omega_2 + m'(\omega_3 l + \omega_2 \cdot \tfrac{l}{2})(\tfrac{3l}{2})$$

where $m' = m/2$: $I' = \frac{1}{3} m' (l/2)^2 = \frac{1}{24} m l^2$ (!!)

$$\therefore \; \tfrac{1}{3}\omega_1 - \omega_1 = \tfrac{1}{24}\omega_3 + \tfrac{1}{8}\omega_3 + \tfrac{1}{24}\omega_2 + \tfrac{3}{4}\omega_3 + \tfrac{3}{8}\omega_2$$

ie $\quad -16\omega_1 = \omega_3 + 3\omega_3 + \omega_2 + 18\omega_3 + 9\omega_2$

ie $\quad \underline{-8\omega_1 = 11\omega_3 + 5\omega_2}$ . . . . (1)

<u>Check!</u> $\omega_2 = \omega_3$ gives $-8\omega_1 = 16\omega_2$ as before.

p 8 4 cont. There will — or may be — impulses at C : but assume as before no impulse at B. Then moment of momentum of the half-deck CB about C is conserved:

$$Ⓒ \quad I'\omega_1 - m'(\tfrac{1}{2}\omega_1 l)(\tfrac{l}{2}) = I'\omega_2 + m'(\omega_3 l + \omega_2 l/2)(l/2)$$

$$\text{ie} \quad \tfrac{1}{24}\omega_1 - \tfrac{1}{8}\omega_1 = \tfrac{1}{24}\omega_2 + \tfrac{1}{4}\omega_3 + \tfrac{1}{8}\omega_2$$

$$\text{ie} \quad \underline{-2\omega_1 = 4\omega_2 + 6\omega_3} \quad \text{- - - -} ②$$

Solving ① and ② gives $\underline{\omega_3 = -{}^{11}/_{7}\,\omega_1}$: $\underline{\omega_2 = \tfrac{13}{7}\omega_1}$

ie left half continues towards the river bed, while the right half rears up in the air.

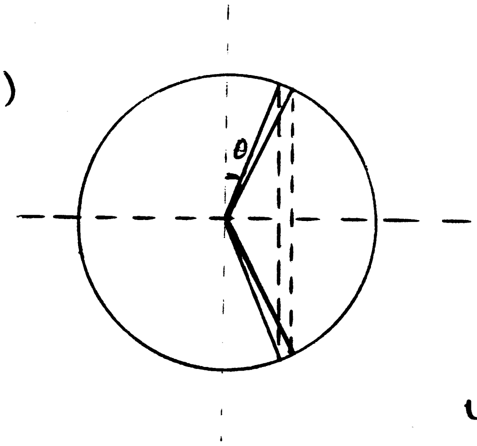
$$v_B = \omega_3 l + \omega_2 l = +\tfrac{2}{7}\omega_1 l$$

(which is not large compared with original impact velocity at A of $2\omega_1 l$).

{ Surely the easiest question on the paper? The least popular and badly done. The collapse occurred during the erection of the Yarra (West Gate) bridge at Melbourne : it was reported a) that after impact the deck reared up in the air to an angle of over 45° b) that the pier at B fell before the deck returned to hit it (because its guy ropes had broken) and so reached the river bed substantially undamaged. The Royal Commission's report suggests that these two stories are not incompatible, but gives no details of their analysis }.

>9

Q5)



A ring at θ has a radius $r\cos\theta$ and a circumference $2\pi r\cos\theta$: if it subtends $d\theta$ at the centre its surface area is $2\pi r\cos\theta\,(r\,d\theta)$.

Area of a sphere is $4\pi r^2$, so the ring has mass $\frac{m'}{4\pi r^2}\cdot 2\pi r^2\cos\theta\,d\theta$.

∴ contributes $\frac{1}{2}m'\cos\theta\,d\theta\cdot(r\cos\theta)^2$ to the moment of inertia about a diameter.

$$\therefore I = \frac{1}{2}m'r^2\int_{-\frac{\pi}{2}}^{+\pi/2}\cos^3\theta\,d\theta = m'r^2\int_0^{\pi/2}\cos^3\theta\,d\theta$$

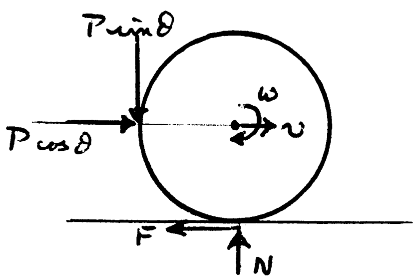
$$= m'r^2\cdot 2/3$$ (standard formula, or write $\cos^3\theta = \cos\theta(1-\sin^2\theta)$).

A solid sphere consists of shells of thickness dr, and so of mass $m' = \rho(4\pi r^2 dr)$ where $\rho = \frac{m}{\frac{4}{3}\pi R^3}$

$$\therefore m' = \frac{3m}{R^3}\cdot r^2 dr$$

so contribution to moment of inertia is $\frac{2}{3}m'r^2 = \frac{2m}{R^3}\cdot r^4 dr$.

$$\therefore I = \int_{r=0}^{R}\frac{2m}{R^3}\cdot r^4\,dr = \frac{2}{5}\cdot mR^2.$$



Linear momentum:

$$P\cos\theta - F = mv$$
$$P\sin\theta - N = 0$$

Angular momentum (about G)

$$F.a - P\sin\theta.a = I\omega$$

For rolling the point of the ball in contact with the ground (table?) must be at rest: $v = \omega a$ { many gave this with ω measured in the opposite sense! }. Since $I = \frac{2}{5}ma^2$, then

$$P\cos\theta - F = m\omega a$$
$$F - P\sin\theta = \frac{2}{5}m\omega a$$

so $P\cos\theta - F = \frac{5}{2}(F - P\sin\theta)$

and $F = \frac{2}{7}P\cos\theta + \frac{5}{7}P\sin\theta$.

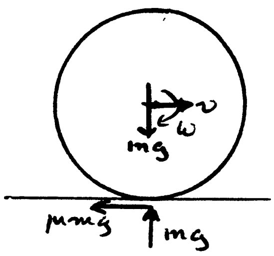
$$\therefore \frac{F}{N} = \frac{2}{7}\cot\theta + 5/7 \le \mu.$$

p10

Q5 contd. If μ is less than this, the ball initially skids, losing energy until rolling takes over. Immediately after the impact, the velocities are v_0, ω_0 where

$$m v_0 = P\cos\theta - \mu P\sin\theta$$
$$\tfrac{2}{5} m a \omega_0 = -P\sin\theta(1-\mu)$$

since we can assume $F = \mu P \sin\theta$. The initial K.E is therefore $\frac{1}{2} m v_0^2 + \frac{1}{2} I \omega_0^2 = \frac{m}{2}\left\{ v_0^2 + \frac{2}{5}(a\omega_0)^2 \right\}$



During skidding there is a friction force $\mu m g$: but we can avoid introducing this by noting that the moment of momentum about a fixed point on the table is conserved, so that

$$m v_0 . a + I . \omega_0 = m v_f . a + I \omega_f$$

Since finally $v_f = a\omega_f$ we have

$$v_0 + \tfrac{2}{5} a\omega_0 = v_f + \tfrac{2}{5} a\omega_f = \tfrac{7}{5} v_f$$

$$\therefore \text{ loss of KE } = \frac{m}{2}\left\{ v_0^2 + \tfrac{2}{5}(a\omega_0)^2 - v_f^2 - \tfrac{2}{5}(a\omega_f)^2 \right\}$$
$$= \frac{m}{2}\left[v_0^2 + \tfrac{2}{5}(a\omega_0)^2 - \tfrac{7}{5}\cdot\left(\tfrac{5}{7}\right)^2\left(v_0 + \tfrac{2}{5} a\omega_0\right)^2 \right]$$
$$= \frac{m}{70}\left[35 v_0^2 + 14(a\omega_0)^2 - 25 v_0^2 - 20 v_0 . a\omega_0 - 4 a^2\omega_0^2 \right]$$
$$= \frac{m}{70}\left[10 v_0^2 - 20 v_0 . a\omega_0 + 10(a\omega_0)^2 \right] = \frac{m}{7}\left[(v_0 - a\omega_0)^2 \right]$$

This can be written in terms of $v_0 = \frac{P}{m}(\cos\theta - \mu\sin\theta)$ and $a\omega_0 = -\frac{5}{2}\cdot\frac{P}{m}\cdot(1-\mu)\sin\theta$ to give

$$v_0 - a\omega_0 = \frac{P}{m}\left[\cos\theta - \mu\sin\theta + \tfrac{5}{2}(1-\mu)\sin\theta \right]$$

and energy loss is $\frac{m}{7}\cdot\left(\frac{P}{m}\right)^2\left[\cos\theta + \left(\frac{5}{2} - \frac{7\mu}{2}\right)\sin\theta \right]^2$

But is this algebra really necessary?

During skidding, friction force $\mu m g$ means that

$$m \frac{dv}{dt} = -\mu m g \ : \ I \frac{d\omega}{dt} = +\mu m g a$$

ie $\frac{dv}{dt} = -\mu g$: $\frac{d(a\omega)}{dt} = +\frac{5}{2}\mu g$

The sliding velocity is $v_s = v - a\omega$, so

$$\frac{dv_s}{dt} = -\mu g - \tfrac{5}{2}\mu g = -\tfrac{7}{2}\mu g$$

and sliding takes place for a time $T = (v_0 - a\omega_0)/(\frac{7}{2}\mu g)$

(so that as before, $v_f = a\omega_f = v_0 - \mu g T = \frac{1}{7}(5 v_0 + 2 a\omega_0)$).

p11 Q5 (cont). Since the sliding velocity decreases at a constant rate $\frac{7}{2}\mu g$, distance slid can be found from "$v^2 = u^2 + 2fs$" and so is $(v_0 - a\omega_0)^2/7\mu g$, and the work done is $\mu mg \times$ distance slid

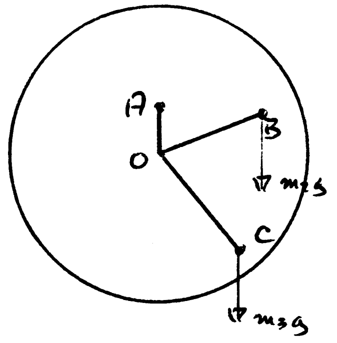
ie $\frac{m}{7}(v_0 - a\omega_0)^2$ as before.

Thus, work done is NOT force × distance moved : it is force × distance slid.

[? Obvious if the process is observed from a moving frame of reference in which the point of contact on the ground is stationary : the distance slid then becomes the only distance over which the friction does work.]

p12

Q6a



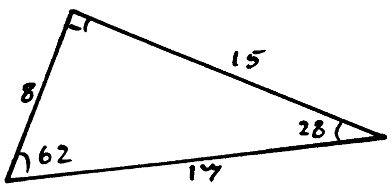
Static balancing requires that

$$\sum m_i \underline{r}_i \times \underline{g} = 0$$

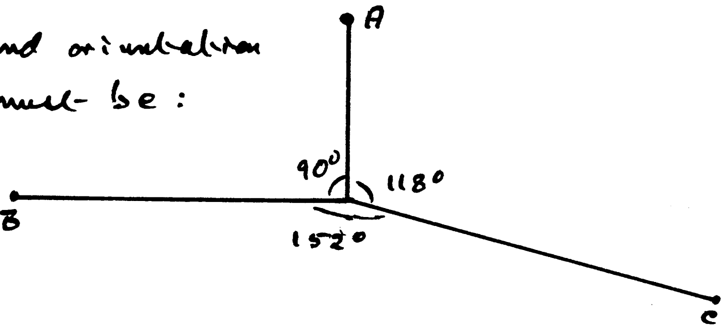
whatever the orientation of the shaft: so condition is $\sum m_i \underline{r}_i = 0$.

[and essential for a rotating shaft since the force on the bearings will be $\sum m_i \omega^2 \underline{r}_i$, so sum of forces on bearings is zero if $\sum m_i \underline{r}_i = 0$].

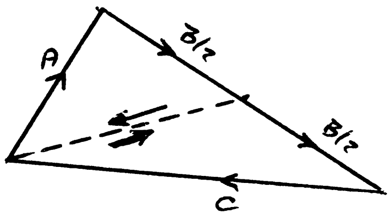
For three unbalances, need to orient the rotors so that these form a closed triangle. Here, conveniently, $8^2 + 15^2 = 17^2$ ($\tan^{-1} 8/15 = 28{\cdot}1°$)



and orientation must be:



To find the out-of-balance couple, can use the standard trick of splitting the central out-of-balance in two and moving half to A, half to C. Then the two out-of-balances are each the median of the triangle:



Length is $\sqrt{(8^2 + (15/2)^2)} = 10{\cdot}97 \times 10^{-3}$ Kg m
so out-of-balance forces are
$\pm\ 10{\cdot}97 \times 10^{-3}\ \omega^2$ (N)

Distance AC is 200 mm, so for
$\omega = 400$ rad/sec,

out-of-balance couple is 350.9 Nm.

{ Alternatively, taking moments about C — to take advantage of the fact that the centrifugal forces at A, B are at right angles — couple has components

$0{\cdot}1 \times 15 \times 10^{-3} \omega^2$, $0{\cdot}2 \times 8 \times 10^{-3} \omega^2$ ie (240, 256) Nm
and the same magnitude. }.

Can reduce the out-of-balance couple to some extent by putting the largest out-of balance in the middle:

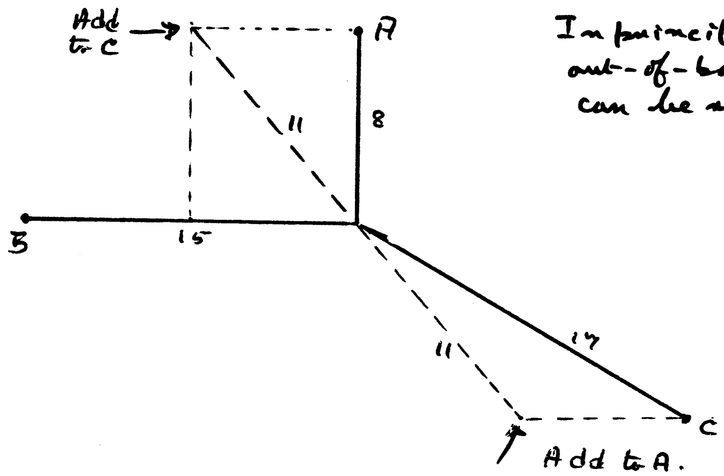
p13 (particularly obvious by the 'median' method: this will reduce the components (still taking moments about C) to

$$(0.1\times 15\times 10^{-3}\omega^2,\ -0.1\times 8\times 10^{-3}\omega^2) \quad \text{ie } (240, 128)$$

and the magnitude to $\underline{272\ \mathrm{Nm}}$.

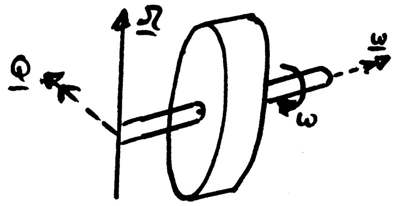
Alternatively, add additional out-of-balances to maintain static balance while opposing the dynamic out-of-balance. For example, if placed on A and C (equal and 180° out of phase for static balance) they must be (from the median diagram) $\pm 10.97\times 10^{-3}$ kgm, oriented at $\tan^{-1}(B/2A) = \tan^{-1}(15/16) = 43.2^\circ$ to OA

ie



In principle, the out-of-balance couple can be reduced to zero.

Q6b A rotor spinning with angular velocity $\underline{\omega}$ has an angular momentum $J\underline{\omega}$. If the spinning rotor rotates about a second axis with angular velocity $\underline{\Omega}$, the rate of change of angular momentum is



$$\frac{dh}{dt} = \frac{d}{dt}(J\underline{\omega}) = J\dot{\underline{\omega}} + \underline{\Omega}\times(J\underline{\omega})$$

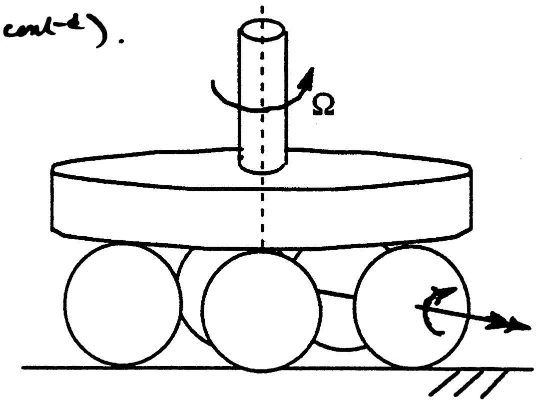
so that even when the magnitude of $\underline{\omega}$ does not change, the angular momentum does. Since

$\underline{Q} = \frac{dh}{dt}$, a couple is required to produce this change: the gyroscopic "moment" $\underline{\Omega}\times J\underline{\omega}$.

Direction given by usual right hand rule for vector products, so as shown in figure.

>14

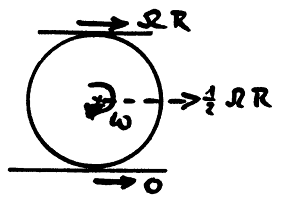
Q6b (cont'd).



Since the balls roll, the top moves round the circle with speed ΩR, while the bottom is stationary.

Hence the centre of the ball moves at $\frac{1}{2}\Omega R$, and the precession angular velocity is $\frac{1}{2}\Omega$

(i.e it takes 2 revolutions of the shaft to make a ball describe one circle around it.)



If ball rotates ('spins') with velocity ω, top moves at $\frac{1}{2}\Omega R + \omega r$; bottom at $\frac{1}{2}\Omega R - \omega r$ so that $\omega = \Omega R/2r$

Angular momentum is $J\omega = \frac{2}{5}mr^2 \cdot \frac{\Omega R}{2r}$
$= \frac{1}{5} m r R \Omega$

and so gyroscopic moment is $\frac{1}{10} m r R \Omega^2$.

{ Most candidates casually gave $\frac{2}{5} m r R \Omega^2$: some saw one of the factors 2: no-one saw both. }

A moment requires a direction to specify it. The figure given is unhelpful since the double arrow on the ball conflicts with the arrow indicating the spin: in fact $\underline{\omega}$ points radially inwards. Accordingly $\underline{\Omega} \times \underline{\omega}$ points back along the track (and is provided by an inward friction force on the top of the ball and an out-ward one on the bottom.).