ENGINEERING TRIPOS - PART IB - 1999

PAPER 7 : MATHEMATICAL METHODS

SOLUTIONS

(J.B.Young)

1\. a)

$$d\phi = \left(\frac{\partial\phi}{\partial x}\right)_y dx + \left(\frac{\partial\phi}{\partial y}\right)_x dy$$

$$= \left(\frac{\partial\phi}{\partial x}\right)_t dx + \left(\frac{\partial\phi}{\partial t}\right)_x dt$$

But $dt = \left(\frac{\partial t}{\partial x}\right)_y dx + \left(\frac{\partial t}{\partial y}\right)_x dy$ with $t = y/x$

$$\therefore \left(\frac{\partial\phi}{\partial x}\right)_y = \left(\frac{\partial\phi}{\partial x}\right)_t + \left(\frac{\partial t}{\partial x}\right)_y \left(\frac{\partial\phi}{\partial t}\right)_x = \left(\frac{\partial\phi}{\partial x}\right)_t - \frac{y}{x^2}\left(\frac{\partial\phi}{\partial t}\right)_x$$

$$\left(\frac{\partial\phi}{\partial y}\right)_x = \left(\frac{\partial t}{\partial y}\right)_x \left(\frac{\partial\phi}{\partial t}\right)_x = \frac{1}{x}\left(\frac{\partial\phi}{\partial t}\right)_x$$

$$\therefore x\left(\frac{\partial\phi}{\partial x}\right)_y + y\left(\frac{\partial\phi}{\partial y}\right)_x = x\left(\frac{\partial\phi}{\partial x}\right)_t - \frac{y}{x}\left(\frac{\partial\phi}{\partial t}\right)_x + \frac{y}{x}\left(\frac{\partial\phi}{\partial t}\right)_x$$

$$= x\left(\frac{\partial\phi}{\partial x}\right)_t$$

$$\therefore x\left(\frac{\partial\phi}{\partial x}\right)_t = n\phi$$

Thus $\dfrac{d\phi}{\phi} = n\dfrac{dx}{x}$ (along t = constant lines)

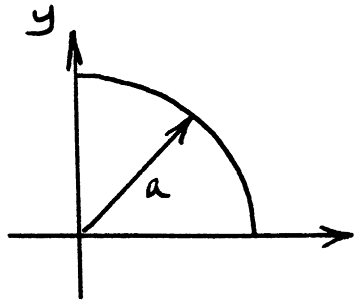
$$\therefore \ln\phi = n\ln x + \ln(f(t))$$

$$\therefore \phi = x^n f(t) = x^n f(y/x)$$

where $f(t)$ is an arbitrary function of $t = y/x$ 10

(b)
$$I_1 = \int_0^a \int_0^{+\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dy\, dx$$

Region of integration :



Change to polar co-ordinates :

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$$

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore \quad I_1 = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} e^{-r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{2} e^{-r^2} \Big|_0^a \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}(1-e^{-a^2})\, d\theta$$

$$= \frac{\pi}{4}(1-e^{-a^2})$$ 7

Let $a \to \infty$:

$$I_1 \to \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dy\, dx = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

$$\therefore \quad I_2 = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$ 3

2\. i)

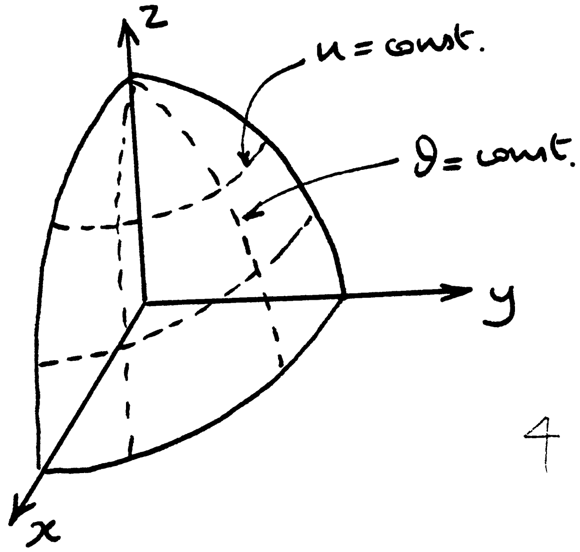
$$\left.\begin{array}{l} x = u\cos\theta \\ y = u\sin\theta \\ z = 4-u^2 \end{array}\right\} \quad \begin{array}{l} 0 \le u \le 2 \\ 0 \le \theta < 2\pi \end{array}$$

Eliminate $\theta \rightarrow \quad x^2+y^2 = u^2$

Eliminate $u \rightarrow \quad z = 4-(x^2+y^2)$

$\therefore \quad x^2+y^2+z = 4$

The surface is a paraboloid and the first octant is shown in the diagram:



4

ii) Vectorial element of area is given by,

$$d\underset{\sim}{S}_1 = \left(\frac{\partial \underset{\sim}{r}}{\partial u} \wedge \frac{\partial \underset{\sim}{r}}{\partial \theta}\right) du\, d\theta$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\theta & \sin\theta & -2u \\ -u\sin\theta & u\cos\theta & 0 \end{vmatrix} du\, d\theta$$

$$= \underline{i}\, 2u^2\cos\theta + \underline{j}\, 2u^2\sin\theta + \underline{k}\, u$$

(This is an outwardly directed vector)

$$\underset{\sim}{V}\cdot d\underset{\sim}{S}_1 = (2u^3\cos^2\theta + u^3)\, du\, d\theta$$

$$= u^3(2\cos^2\theta + 1)\, du\, d\theta$$

$$= u^3(2+\cos 2\theta)\, du\, d\theta$$

$$\therefore \quad I_1 = \int_{\theta=0}^{2\pi} \int_{u=0}^{2} u^3 (2+\cos 2\theta)\, du\, d\theta$$

$$= \int_{\theta=0}^{2\pi} 4(2+\cos 2\theta)\, d\theta$$

$$= 16\pi$$ 10

iii) $\nabla \cdot \underset{\sim}{V} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-2+4) = 0$

$\underset{\sim}{V}$ is therefore solenoidal.

Gauss's theorem gives:

$$\oiint \underset{\sim}{V} \cdot d\underset{\sim}{S} = \iiint \nabla \cdot \underset{\sim}{V}\, dv = 0$$

$\therefore$ Flux out through S_1 = flux in through S_2

$$\therefore \quad I_2 = \iint_{S_2} \underset{\sim}{V} \cdot d\underset{\sim}{S_2} = 16\pi$$ 6

3. i) The straight line joining (0,0,0) to (1,-2,-3) can be represented in parametric form by,

$$\left.\begin{aligned} x &= t \\ y &= -2t \\ z &= -3t \end{aligned}\right\}$$

Along this line,

$$\underset{\sim}{F} = 2xyz\,\underset{\sim}{i} + x^2z\,\underset{\sim}{j} + x^2y\,\underset{\sim}{k}$$

$$= 12t^3\,\underset{\sim}{i} - 3t^3\,\underset{\sim}{j} - 2t^3\,\underset{\sim}{k}$$

$$d\underset{\sim}{l} = (\underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k})\,dt$$

$$\therefore\ I = \int_{t=0}^{1} (12t^3 + 6t^3 + 6t^3)\,dt$$

$$= \int_0^1 24t^3\,dt$$

$$= 6t^4\Big|_0^1$$

$$= 6$$

7

ii) $$\nabla \wedge \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz & x^2z & x^2y \end{vmatrix}$$

$$= \underset{\sim}{i}(x^2 - x^2) + \underset{\sim}{j}(2xy - 2xy) + \underset{\sim}{k}(2xz - 2xz)$$

$$= 0$$

Hence $\underset{\sim}{F}$ is conservative with a scalar potential ϕ such that $\underset{\sim}{F} = \nabla\phi$.

$$\therefore \quad \frac{\partial \phi}{\partial x} = 2xyz \longrightarrow \phi = x^2yz + f(y,z)$$

$$\frac{\partial \phi}{\partial y} = x^2z \longrightarrow \phi = x^2yz + g(z,x)$$

$$\frac{\partial \phi}{\partial z} = x^2y \longrightarrow \phi = x^2yz + h(x,y)$$

$$\therefore \quad \phi = x^2yz + c$$

$$\phi(0,0,0) = c, \quad \phi(1,-2,-3) = 6 + c$$

$$I = \int_L d\phi = (6+c) - c = 6$$ 7

iii) If $\varphi \underline{E}$ is conservative, then $\nabla_\wedge(\varphi \underline{E}) = 0$

$$\nabla_\wedge(\varphi \underline{E}) = \varphi(\nabla_\wedge \underline{E}) + \nabla\varphi_\wedge \underline{E}$$

But $\nabla_\wedge \underline{E} = 0$ and $\nabla\varphi = \nabla f(\phi)$ as $\varphi = f(\phi)$

$$\nabla f(\phi) = \frac{df}{d\phi}\nabla\phi$$

$$\therefore \quad \nabla_\wedge(\varphi \underline{E}) = \frac{df}{d\phi}(\nabla\phi_\wedge \underline{E})$$

$$= \frac{df}{d\phi}(\nabla\phi_\wedge \nabla\phi) \qquad [\underline{E} = \nabla\phi]$$

$$= 0$$ 6

$\therefore$ $\varphi \underline{E}$ is conservative if $\varphi = f(\phi)$.

4.(i) Euler method :

$$y_{i+1} = y_i + \Delta t\, f(t_i, y_i) \qquad (\Delta t = t_{i+1} - t_i)$$

Modified Euler method :

$$y_{i+1}^{p} = y_i + \Delta t\, f(t_i, y_i) \qquad (\Delta t = t_{i+1} - t_i)$$

$$y_{i+1} = y_i + \frac{\Delta t}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_{i+1}^{p})\right]$$

The truncation errors of the Euler and modified Euler methods are proportional to Δt^2 and Δt^3 respectively. 4

(ii) $\frac{d^2y}{dt^2} - y = 0$ has general solution $y = Ae^{t} + Be^{-t}$

$$\therefore \left.\begin{array}{l} A+B = 1 \\ A-B = -1 \end{array}\right\} \qquad A = 0,\ B = -1.$$

$\therefore$ Solution is $y = e^{-t}$ $\qquad \therefore\ y = 0.368$ at $t = 1$ (exact value)

For the numerical solutions define $x = dy/dt$ and write the equation as :

$$\left.\begin{array}{l} \frac{dy}{dt} = x \\ \frac{dx}{dt} = y \end{array}\right\}$$

Euler :
$$\left.\begin{aligned} x_{i+1} = x_{i+1}^{p} &= x_i + y_i \Delta t \\ y_{i+1} = y_{i+1}^{p} &= y_i + x_i \Delta t \end{aligned}\right\} \Delta t = 0.5$$

t_i	x_i	y_i
0	−1	1
0.5	−0.50	0.50
1.0	−0.25	0.25

$y(1) = 0.250$

$|\text{error}| = 0.118$

Modified Euler :
$$x_{i+1} = x_i + \left(\frac{y_i + y_{i+1}^{p}}{2}\right)\Delta t$$
$$y_{i+1} = y_i + \left(\frac{x_i + x_{i+1}^{p}}{2}\right)\Delta t$$

t_i	x_i	y_i	x_{i+1}^{p}	y_{i+1}^{p}
0	−1	1	−0.5	0.5
0.5	−0.625	0.625	−0.3125	0.3125
1.0	−0.391	0.391		

$y(1) = 0.391$

$|\text{error}| = 0.023$ 12

(iii) Number of steps to get to $t = 1 = 1/\Delta t$

Euler error $\simeq k\Delta t^2 (1/\Delta t) = k\Delta t$. $k \sim \frac{0.118}{0.5} \sim 0.2$

$\therefore$ for $|\text{error}| = 0.0001$, $\Delta t \sim 0.0005$

[Actual calculation gives $|\text{error}| = 9\times10^{-5}$ for $\Delta t = 0.0005$]

Modified Euler error $\simeq k\Delta t^3 (1/\Delta t) = k\Delta t^2$. $k \sim \frac{0.023}{0.25} \sim 0.$

$\therefore$ for $|\text{error}| = 0.0001$, $\Delta t \sim 0.03$

[Actual calculation gives $|\text{error}| = 10^{-4}$ for $\Delta t = 0.04$]

---

5. (i) Error is $r_i = k_1 \varepsilon_i e^{-k_2 \varepsilon_i} - \sigma_i$

$\therefore \quad \sigma_i \left(1 + \frac{r_i}{\sigma_i}\right) = k_1 \varepsilon_i e^{-k_2 \varepsilon_i}$

$\therefore \quad \ln \sigma_i + \ln\left(1 + \frac{r_i}{\sigma_i}\right) = \ln k_1 + \ln \varepsilon_i - k_2 \varepsilon_i$

If $|r_i| << \sigma_i$ then $\ln\left(1 + \frac{r_i}{\sigma_i}\right) \simeq \frac{r_i}{\sigma_i}$

$\therefore \quad \frac{r_i}{\sigma_i} \simeq \ln k_1 - k_2 \varepsilon_i - \ln\left(\frac{\sigma_i}{\varepsilon_i}\right)$

$\therefore \quad r_i \simeq \sigma_i \left[\ln k_1 - k_2 \varepsilon_i - \ln\left(\frac{\sigma_i}{\varepsilon_i}\right)\right]$ 6

(ii) Function to minimise is

$$\sum r_i^2 = \sum \sigma_i^2 \left[\ln k_1 - k_2 \varepsilon_i - \ln\left(\frac{\sigma_i}{\varepsilon_i}\right)\right]^2$$

$$\frac{\partial \sum r_i^2}{\partial (\ln k_1)} = \sum 2\sigma_i^2 \left[\ln k_1 - k_2 \varepsilon_i - \ln\left(\frac{\sigma_i}{\varepsilon_i}\right)\right] = 0$$

$$\frac{\partial \sum r_i^2}{\partial k_2} = \sum -\sigma_i^2 \varepsilon_i \left[\ln k_1 - k_2 \varepsilon_i - \ln\left(\frac{\sigma_i}{\varepsilon_i}\right)\right] = 0$$

Simultaneous equations for $\ln k_1$ and k_2 are :

$$\ln k_1 \left(\sum \sigma_i^2\right) - k_2 \left(\sum \sigma_i^2 \varepsilon_i\right) = \sum \sigma_i^2 \ln\left(\frac{\sigma_i}{\varepsilon_i}\right)$$

$$\ln k_1 \left(\sum \sigma_i^2 \varepsilon_i\right) - k_2 \left(\sum \sigma_i^2 \varepsilon_i^2\right) = \sum \sigma_i^2 \varepsilon_i \ln\left(\frac{\sigma_i}{\varepsilon_i}\right)$$ 7

iii)
$$\begin{pmatrix} 72.9 & -123.1 \\ 123.1 & -232.1 \end{pmatrix}\begin{pmatrix} \ln k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 68.1 \\ 102.7 \end{pmatrix}$$

LU decomposition of 2x2 matrix:

$$U = \begin{pmatrix} 72.9 & -123.1 \\ 0 & -24.23 \end{pmatrix} \qquad l_{21} = \frac{123.1}{72.9} = 1.689$$

$$L = \begin{pmatrix} 1 & 0 \\ 1.689 & 1 \end{pmatrix}$$

Now $LUx = b$ where $x = (\ln k_1, k_2)^t$

Writing $Ux = C \rightarrow LC = b$

$$\begin{pmatrix} 1 & 0 \\ 1.689 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 68.1 \\ 102.7 \end{pmatrix}$$

$\therefore \quad C_1 = 68.1, \quad C_2 = -12.32$

$$\therefore \quad \begin{pmatrix} 72.9 & -123.1 \\ 0 & -24.23 \end{pmatrix}\begin{pmatrix} \ln k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 68.1 \\ -12.32 \end{pmatrix}$$

7

$\therefore \quad k_2 = 0.508, \quad \ln k_1 = 1.792 \rightarrow k_1 = 6.00$

Least squares fit is:

$$\sigma = 6.00\varepsilon e^{-0.508\varepsilon}$$

ε_i	σ_i	σ_{LS}
0.50	2.25	2.32
1.00	3.58	3.61
1.50	4.25	4.20
2.00	4.40	4.34
2.38	4.20	4.26

6. (i)

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{+T/2} b\, e^{-j\omega t}\, dt$$

$$= -\frac{b}{j\omega} e^{-j\omega t} \Big|_{-T/2}^{+T/2}$$

$$= \frac{b}{j\omega}\left[e^{j\frac{\omega T}{2}} - e^{-j\frac{\omega T}{2}}\right]$$

$$= \frac{2b}{\omega} \sin\left(\frac{\omega T}{2}\right)$$

$$= \frac{bT}{(\omega T/2)} \sin\left(\frac{\omega T}{2}\right)$$

$$= bT \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$ 3

(ii) Write $f(t) = g(t)\, h(t)$ where,

$$g(t) = \cos(\pi t); \qquad h(t) = \begin{cases} 1 & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Inverse convolution theorem:

$$2\pi F(\omega) = G(\omega) * H(\omega)$$

$$\therefore \quad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega')\, H(\omega - \omega')\, d\omega'$$

From given FT pairs:

$$H(\omega) = \operatorname{sinc}\left(\frac{\omega}{2}\right) \qquad (b=1,\ T=1)$$

$$G(\omega) = \pi\{\delta(\omega+\pi) + \delta(\omega-\pi)\} \qquad (\omega_0 = \pi)$$

$$\therefore \quad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi\left[\delta(\omega'+\pi) + \delta(\omega'-\pi)\right] \operatorname{sinc}\left(\frac{\omega-\omega'}{2}\right) d\omega'$$

$$\therefore \quad F(\omega) = \frac{1}{2}\left[\operatorname{sinc}\left(\frac{\omega+\pi}{2}\right) + \operatorname{sinc}\left(\frac{\omega-\pi}{2}\right)\right]$$

(Using the "sifting" property of the δ-function.) 7

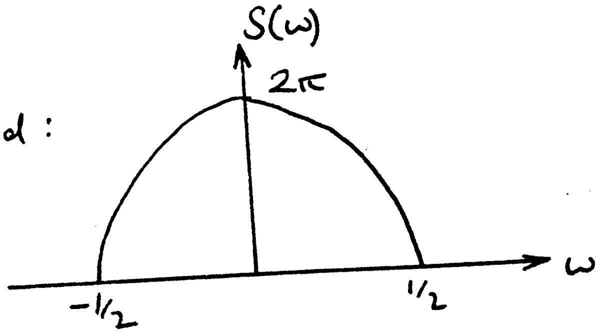
(iii) If $f(t) \leftrightarrow F(\omega)$, then,

$$2\pi f(-\omega) \leftrightarrow F(t)$$

$\therefore$ If $s(t) = \frac{1}{2}\left[\mathrm{sinc}\left(\frac{t+\pi}{2}\right) + \mathrm{sinc}\left(\frac{t-\pi}{2}\right)\right]$, then,

$$S(\omega) = \begin{cases} 2\pi\cos(-\pi\omega) = 2\pi\cos(\pi\omega) & |\omega| \leq 1/2 \\ 0 \quad \text{otherwise.} \end{cases}$$

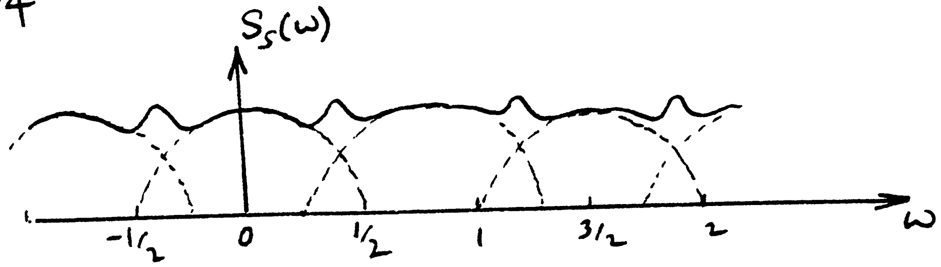
$S(\omega)$ is obviously bandlimited:



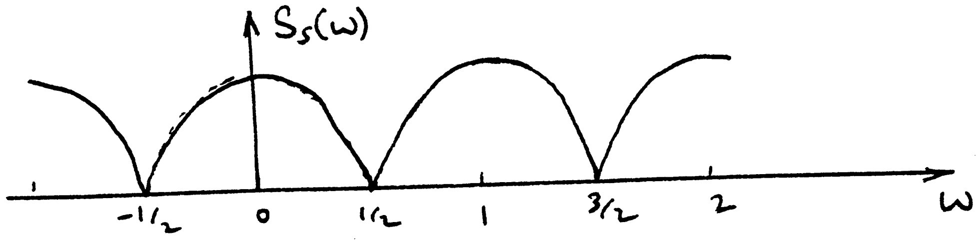
4

The FT of a sampled signal is the FT of the original signal repeated every interval of the sampling frequency. The Nyquist frequency (twice the highest frequency present) is $\omega_{NY} = 1$.

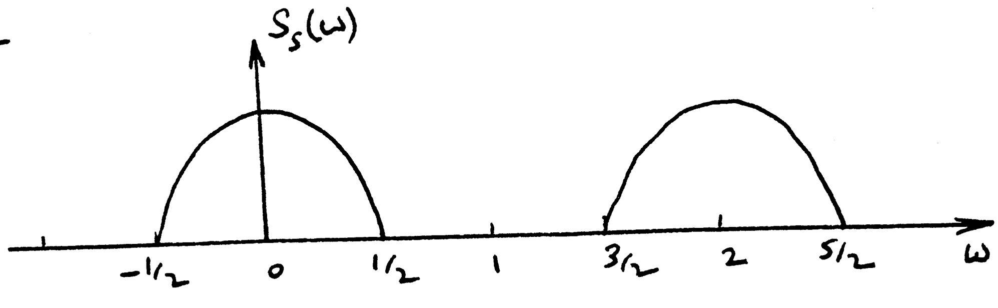
a) $\omega_s = 3/4$



b) $\omega_s = 1$ (Nyquist frequency)



c) $\omega_s = 2$



6

7. (i) $$F_k = \sum_{n=0}^{N-1} f_n e^{-jkn\frac{2\pi}{N}}$$

Multiply both sides by $e^{jkm\frac{2\pi}{N}}$:

$$F_k e^{jkm\frac{2\pi}{N}} = \sum_{n=0}^{N-1} f_n e^{j\frac{2\pi}{N}k(m-n)}$$

Sum over $k=0$ to $N-1$:

$$\sum_{k=0}^{N-1} F_k e^{jkm\frac{2\pi}{N}} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} f_n e^{j\frac{2\pi}{N}k(m-n)}$$

$$= \sum_{n=0}^{N-1} f_n \left\{ \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-n)} \right\}$$

For $m \neq n$, the bracketed series is a geometric progression of form $\sum_{k=0}^{N-1} ar^k$ with $a = 1$, $r = e^{j\frac{2\pi}{N}(m-n)}$

$$\therefore \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-n)} = \left[\frac{1-e^{2\pi j(m-n)}}{1-e^{\frac{2\pi}{N}j(m-n)}} \right] = 0 \text{ if } m \neq n.$$

because $e^{2\pi j(m-n)} = 1$.

Hence, the only contribution is when $n = m$:

$$\sum_{k=0}^{N-1} F_k e^{jkm\frac{2\pi}{N}} = f_m \sum_{k=0}^{N-1} 1 = N f_m$$

$$\therefore \quad \underline{f_m = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{j\frac{2\pi mk}{N}}}$$

10

(Alternatively, substitute the given expression for F_k into the expression for f_n which is to be verified.)

(ii)
$$E(f_n) = E\left\{\frac{1}{N}\sum_{k=0}^{N-1} F_k e^{jkn\frac{2\pi}{N}}\right\}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} E(F_k) e^{jkn\frac{2\pi}{N}} \quad (\text{from } E(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 E(x_1) + \lambda_2 E(x_2))$$

$$= 0 \quad \text{as } E(F_k) = 0 \text{ for all } k.$$

$\therefore \; E(f_n) = 0$ for all n.

$$\mathrm{var}(f_n) = \mathrm{var}\sum_{k=0}^{N-1}\left(\frac{1}{N} e^{jkn\frac{2\pi}{N}} F_k\right)$$

$$= \sum_{k=0}^{N-1}\left|\frac{e^{jkn2\pi/N}}{N}\right|^2 \mathrm{var}(F_k) \quad (\text{from } \mathrm{var}(\lambda_1 x_1 + \lambda_2 x_2) = |\lambda_1|^2 \mathrm{var}(x_1) + |\lambda_2|^2 \mathrm{var}(x_2))$$

$$= \sum_{k=0}^{N-1}\frac{1}{N^2}\sigma^2$$

$$= \frac{1}{N^2}(N\sigma^2)$$

$$= \frac{\sigma^2}{N}$$

$\therefore \; \mathrm{var}(f_n) = \frac{\sigma^2}{N}$ for all n. 7

Now $\mathrm{var}(f_1) = \sigma^2/N$, hence we require

$$SD(f_1) = \frac{\sigma}{\sqrt{N}} < \frac{\sigma}{10}$$

i.e., $\sqrt{N} > 10 \rightarrow N > 100$

More than 100 samples are required to ensure SD of f_1 is less than $\sigma/10$ 3

8(i) Conditions for binomial distribution:

1) Each "trial" has probability of success p and failure $q = 1 - p$.

2) p and q are the same for each "trial".

3) The trials are independent.

Conditions for Poisson distribution:

1) Each event is independent.

2) No two events happen at the same time.

3) Average number of events per unit time is λ. 4

The Poisson distribution may be used to approximate the binomial distribution provided,

$n \to \infty$, $p \to 0$ such that $\lambda = np$ is fixed.

(Generally $n > 50$, $p < 0.1$) 2

(ii) $$\left.\begin{array}{l} p = 50/100000 = 0.0005 \\ n = 100 \text{ or } 101 \end{array}\right\} \text{Hence use the Poisson approximation}$$

(a) Boxes of 100 components. $\lambda = 100 \times 0.0005 = 0.05$

Prob. (no failures) $= \dfrac{\lambda^0}{0!} e^{-\lambda} = e^{-0.05} = 0.95123$

Profit per box $= 0.95123 \times £100 - 100 \times £0.90 = £5.12$

5

(ii) Boxes of 101 components. $\lambda = 101 \times 0.0005 = 0.0505$

$$\text{Prob. (no failures)} = \frac{\lambda^0}{0!}e^{-\lambda} = e^{-0.0505} = 0.95075$$

$$\text{Prob. (1 failure)} = \frac{\lambda^1}{1!}e^{-\lambda} = 0.0505e^{-0.0505} = 0.04801$$

$$\text{Prob. (100 or more good components)} = \underline{\overline{0.99876}}$$

Profit per box = 0.99876 × £100 − 101 × £0.90 = £8.98 5

(iii) We need to do further calculations to find if this is the maximum profit per box.

Boxes of 102 components $\lambda = 102 \times 0.0005 = 0.051$

Probability (100 or more good components) =

$$\left(1 + 0.051 + \frac{0.051^2}{2}\right)e^{-0.051} = 0.99998$$

Profit per box = 0.99998 × £100 − 102 × £0.90 = £8.20 4

Hence the manufacturer should pack 101 components per box to maximise his profit.
