ENGINEERING TRIPOS PART IB

Friday 4 June 1999 9 - 11

Paper 7

MATHEMATICAL METHODS

Answer not more than **four** *questions.*

Answer at least **one** *question from each section.*

The **approximate** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

(TURN OVER

SECTION A

Answer at least **one** *question from this section.*

1 (a) By considering solutions of the form $\phi = \phi(x,t)$ where $t = y/x$, show that the partial differential equation,

$$x\left(\frac{\partial\phi}{\partial x}\right)_y + y\left(\frac{\partial\phi}{\partial y}\right)_x = n\phi$$

(where n is a constant), can be transformed into,

$$x\left(\frac{\partial\phi}{\partial x}\right)_t = n\phi \quad .$$

Hence find the general solution to the original partial differential equation. [10]

(b) Sketch the region of integration and apply a suitable change of variable to evaluate the double integral,

$$I_1 = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,dy\,dx$$ [7]

where a is a constant.

Hence evaluate,

$$I_2 = \int_0^{\infty} e^{-x^2}\,dx \quad .$$ [3]

2 (i) A surface is defined in terms of parameters u and θ by,

$$\begin{aligned} x &= u\cos\theta \\ y &= u\sin\theta \\ z &= 4 - u^2 \end{aligned}$$

for $0 \leq u \leq 2$ and $0 \leq \theta < 2\pi$. Find the equation of the surface in the form $f(x, y, z) = constant$, and sketch it, showing lines of constant-u and lines of constant-θ on your sketch. [4]

(ii) By direct integration, evaluate the surface integral,

$$I_1 = \iint_{S_1} \boldsymbol{V} \bullet \boldsymbol{dS}$$

where $\boldsymbol{V}$ is the vector field,

$$\boldsymbol{V} = x\boldsymbol{i} - (z-4)\boldsymbol{k}$$

and S_1 is the surface defined in (i), (with $\boldsymbol{dS}$ pointing in an outward direction). [10]

(iii) Using Gauss's theorem, evaluate the surface integral,

$$I_2 = \iint_{S_2} \boldsymbol{V} \bullet \boldsymbol{dS}$$

where S_2 is the plane surface defined by the intersection of S_1 and $z = 0$, and $\boldsymbol{dS}$ points in the positive z-direction. [6]

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3 (i) By direct integration, evaluate the line integral,

$$I = \int_L \boldsymbol{F} \bullet d\boldsymbol{L}$$

where $\boldsymbol{F}$ is the vector field,

$$\boldsymbol{F} = 2xyz\,\boldsymbol{i} + x^2 z\,\boldsymbol{j} + x^2 y\,\boldsymbol{k}$$

and L is the straight line joining the origin to the point $(1, -2, -3)$. [7]

(ii) Prove that $\boldsymbol{F}$ is a conservative field and find its scalar potential ϕ. Use this to verify the value of the line integral I found above. [7]

(iii) If $\psi = f(\phi)$ is an arbitrary differentiable scalar function of ϕ, show that the vector field $\psi\boldsymbol{F}$ is also conservative. [6]

SECTION B

Answer at least **one** *question from this section.*

4 (i) First order, initial value, ordinary differential equations of the form,

$$\frac{dy}{dt} = f(t, y)$$

can be solved numerically using the Euler or modified Euler (predictor-corrector) algorithms. For each of these methods, write down the formal mathematical procedure to advance the solution from $t = t_i$ to $t = t_{i+1}$. How do the truncation errors of the two methods vary with step size? [4]

(ii) Solve the following equation analytically and calculate the value of y at $t = 1{\cdot}0$:

$$\frac{d^2y}{dt^2} - y = 0 \qquad (y = 1{\cdot}0 \text{ and } dy/dt = -1{\cdot}0 \text{ at } t = 0)$$

Now integrate the equation numerically using firstly the Euler, and secondly the modified Euler method, each with a step size of 0·5, to obtain approximate values of y at $t = 1{\cdot}0$. [12]

(iii) Stating your reasoning clearly, estimate the step size required with each method to reduce the error at $t = 1{\cdot}0$ to $0{\cdot}0001$. [4]

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5 Five measurements (in scaled units) from a stress-strain experiment are listed below :

σ_i (stress)	2·25	3·58	4·25	4·40	4·20
ε_i (strain)	0·50	1·00	1·50	2·00	2·38

Theoretical considerations suggest that σ and ε are related by an equation of the form,

$$\sigma(\varepsilon) = k_1\varepsilon \exp(-k_2\varepsilon)$$

and it is desired to obtain the constants k_1 and k_2 using a least-squares procedure to minimise the function,

$$\sum_i r_i^2 = \sum_i [\sigma(\varepsilon_i) - \sigma_i]^2$$

where $r_i = \sigma(\varepsilon_i) - \sigma_i$ is the difference between the calculated and measured stress.

(i) Making the assumption that $r_i << \sigma_i$ show that,

$$r_i = \sigma_i\left[\ell n(k_1) - k_2\varepsilon_i - \ell n\left(\frac{\sigma_i}{\varepsilon_i}\right)\right]$$ [6]

(ii) Derive two simultaneous linear equations for $\ell n(k_1)$ and k_2 such that $\sum_i r_i^2$ is minimised. [7]

(iii) Solve the equations formally using LU decomposition to obtain values of k_1 and k_2. The following data (Σ representing summation over the five measurements) may be assumed without verification : [7]

$$\Sigma\sigma_i^2 = 72{\cdot}94 \qquad \Sigma\sigma_i^2\ell n(\sigma_i/\varepsilon_i) = 68{\cdot}05$$

$$\Sigma\sigma_i^2\varepsilon_i = 123{\cdot}14 \qquad \Sigma\sigma_i^2\varepsilon_i\,\ell n(\sigma_i/\varepsilon_i) = 102{\cdot}74$$

$$\Sigma\sigma_i^2\varepsilon_i^2 = 232{\cdot}08$$

SECTION C

Answer at least **one** *question from this section.*

6 (i) Show that the Fourier transform of the rectangular pulse $h(t)$ is $H(\omega)$ where,

$$h(t) = \begin{cases} b & \text{if } |t| \le T/2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad H(\omega) = bT\,sinc\left(\frac{\omega T}{2}\right)$$ [3]

(ii) Using the above result and the convolution theorem, show that the Fourier transform of the function,

$$f(t) = \begin{cases} \cos(\pi t) & \text{if } |t| \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$

is $F(\omega)$, where

$$F(\omega) = \frac{1}{2}\left\{ \text{sinc}\left(\frac{\omega + \pi}{2}\right) + \text{sinc}\left(\frac{\omega - \pi}{2}\right) \right\}$$

You may assume without proof the following Fourier transform pair:

$g(t) = \cos(\omega_0 t)$ has FT $G(\omega) = \pi\{\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\}$ [7]

(iii) Suppose now that we have a *time domain* signal

$$S(t) = \frac{1}{2}\left\{ \text{sinc}\left(\frac{t + \pi}{2}\right) + \text{sinc}\left(\frac{t - \pi}{2}\right) \right\}$$

Show that $S(t)$ is bandlimited. If $S(t)$ is sampled to give $S_S(t)$, sketch the Fourier transform of the sampled signal for the following values of the (angular) sampling frequency ω_S:

a) $\omega_S = 0{\cdot}75$, b) $\omega_S = 1$, c) $\omega_S = 2$. [10]

(TURN OVER

7 (i) The discrete Fourier Transform (DFT) of a sequence $\{f_0, f_1, ..., f_{N-1}\}$ is given by,

$$F_k = \sum_{n=0}^{N-1} f_n \, e^{-jkn2\pi/N} \qquad \text{for} \qquad 0 \le k \le N-1$$

Show that the inverse DFT is given by,

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k \, e^{jkn2\pi/N} \qquad \text{for} \qquad 0 \le n \le N-1$$

It may be assumed without proof that, $\sum_{k=0}^{M-1} r^k = \frac{(1-r^M)}{(1-r)}$ [10]

(ii) In the above expression for the inverse DFT, suppose that the F_k's are independent random variables with mean zero and standard deviation σ. Show that $E(f_n)$ (the expected value of f_n) is zero for all n and find the variance of f_n. How many samples must be taken in order to ensure that the standard deviation of f_1 is less than $\sigma/10$?

You may assume that the standard results for linear combinations of independent random variables also apply when the scalar constants are complex, i.e.,

$$E(\lambda_1 X_1 + \lambda_2 X_2) = \lambda_1 E(X_1) + \lambda_2 E(X_2)$$

$$\text{var}(\lambda_1 X_1 + \lambda_2 X_2) = |\lambda_1|^2 \, \text{var}(X_1) + |\lambda_2|^2 \, \text{var}(X_2)$$ [10]

8 (i) State the conditions for a discrete random variable to be represented by,

(a) The binomial distribution $B(n, p)$,
(b) The Poisson distribution $Po(\lambda)$.

State also the condition under which the Poisson distribution may be used to approximate the binomial distribution. Give the relationship between λ, n and p (no proof required). [6]

(ii) A manufacturer produces components at a cost of £0·90 each with a known failure rate of 50 per 100000 components. The retailer pays the manufacturer £100 for each box that contains at least 100 good components and nothing for any box that has less than 100 good components. Estimate the manufacturer's profit per box if,

(a) The box contains 100 components.
(b) The box contains 101 components. [10]

(iii) State, with any further justification required, what the manufacturer should do to maximise his profit. [4]

END OF PAPER