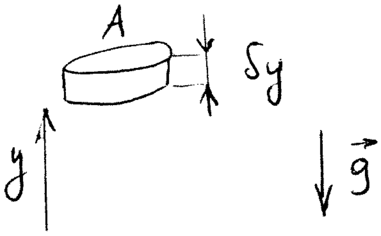
IB Fluids and Heat Transfer 2000
Solutions

Section A

Q1. (a) Equilibrium of a small element of thickness δy and cross section A at some height y.

The difference of pressure forces between top and bottom equilibrate the weight



$$A\left(p(y) - p(y+\delta y)\right) - \rho A \delta y\, g = 0$$

Dividing by $A\delta y$, in the limit of small δy, $\dfrac{dp}{dy} = -\rho g$

(b) z vertically downwards, so $\dfrac{dp}{dz} = -\dfrac{dp}{dy}$

$\dfrac{dp}{dz} = \rho g$ with $\rho = \rho_0(1+bz)$

Integrating once $p(z) = \int_0^z \rho(u)\, g\, du \ [+ P_{atm}]$ ← reference pressure plays no role

$$= \int_0^z \rho_0 g (1+bu)\, du$$

$$= \rho_0 g z \left(1 + \frac{bz}{2}\right)$$

(i) At bottom edge of the gate, $z = 5\,m$

$$p(B) = 1010 \times 9.81 \times 5 \times \left(1 + \frac{3.2\times10^{-3}\times 5}{2}\right)$$

$p(B) = 49940\ N.m^{-2}$ [above atmospheric]

(ii) The total force is the integral of pressure on gate surface

$F = \int p\, dA = w\int_A^B p(z)\, dz$ where $w = 1.5\ m$ is the width

$$F = w \rho_0 g \left[\frac{z^2}{2} + \frac{b z^3}{6} \right]_{z_A}^{z_B}$$

$$= 1.5 \times 1010 \times 9.81 \times \left[\frac{5^2 - 3^2}{2} + \frac{3.2 \times 10^{-3} \times (5^3 - 3^3)}{6} \right]$$

$$F \simeq 119700 \text{ N}$$

(iii) The moment of pressure forces on the gate about the hinge at A has to be balanced by the force applied at the bottom.

Moment of pressure forces:

$$M = \int (z - 3)\, p(z)\, dz$$

$$= w \int_A^B (z - 3)\, \rho_0 g z \left(1 + \frac{bz}{2}\right) dz$$

$$= w \rho_0 g \left[\frac{z^3}{3} + b \frac{z^4}{8} - 3 \frac{z^2}{2} - \frac{3bz^3}{6} \right]_{z_A}^{z_B}$$

$$\simeq 1.5 \times 1010 \times 9.81 \times \left[\frac{5^3 - 3^3}{3} + \frac{3.2 \times 10^{-3} (5^4 - 3^4)}{8} - \frac{3(5^2 - 3^2)}{2} - \frac{3.2 \times 10^{-3} (5^3 - 3^3)}{2} \right]$$

$$\simeq 129700 \text{ N.m}$$

$$F \times (z_B - z_A) = M$$

Therefore $F = \dfrac{M}{z_B - z_A} \simeq \dfrac{129700}{2} = 64850 \text{ N}$

Q2. (a) $Q_1 = 6.28 \ m^3 \cdot s^{-1}$

$Q_2 = 0.8 \times Q_1 \simeq 5.02 \ m^3 \cdot s^{-1}$

$Q_3 = 0.2 \times Q_1 \simeq 1.26 \ m^3 \cdot s^{-1}$

$$V_1 = \frac{Q_1}{A_1} = \frac{6.28}{\pi \frac{1^2}{4}} \simeq 8 \ m \cdot s^{-1}$$

$$V_2 = \frac{0.8 \times Q_1}{A_2} = \frac{0.8 \times 6.28}{\pi \frac{0.8^2}{4}} \simeq 10 \ m \cdot s^{-1}$$

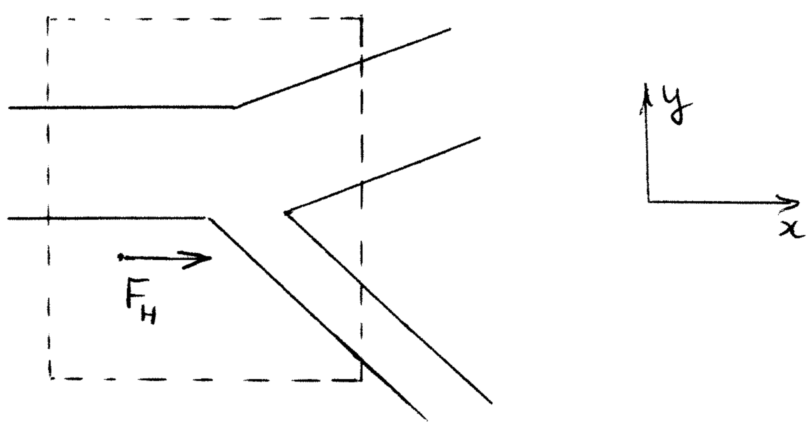
$$V_3 = \frac{0.2 \times Q_1}{A_3} = \frac{0.2 \times 6.28}{\pi \frac{0.3^2}{4}} \simeq 17.8 \ m \cdot s^{-1}$$

(b) Bernoulli applies between 1 and 2, or 1 and 3.

$$P_2 = P_1 + \frac{1}{2} \rho (v_1^2 - v_2^2)$$
$$= 150 \ kN \cdot m^{-2} + \frac{1}{2} \times 860 \times (8^2 - 10^2)$$
$$\simeq 134.5 \ kN \cdot m^{-2}$$

$$P_3 = P_1 + \frac{1}{2} \rho (v_1^2 - v_3^2)$$
$$\simeq 41.7 \ kN \cdot m^{-2}$$

(c) The following control volume is considered



$$\sum \text{forces} = \sum \dot{m}\,\vec{v}$$

The x-component can be written

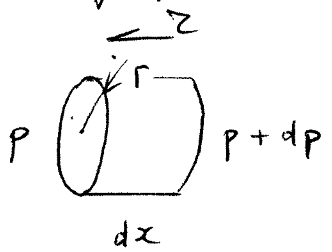
$$F_H + p_1 A_1 - p_2 A_2 \cos 25^\circ - p_3 A_3 \cos 70^\circ = \rho Q_2 v_2 \cos 25^\circ + \rho Q_3 v_3 \cos 70^\circ - \rho Q_1 v_1$$

$$\underset{(kN)}{F_H} \simeq -150 \times 3.14 \times 0.5^2 + \cos 25^\circ \times \left(134.5 \times 3.14 \times 0.4^2 + \frac{860 \times 0.8 \times 6.28 \times 10}{1000}\right)$$
$$+ \cos 70^\circ \times \left(41.3 \times 3.14 \times 0.15^2 + \frac{860 \times 0.2 \times 6.28 \times 17.8}{1000}\right)$$
$$- \frac{860 \times 6.28 \times 8}{1000}$$

$\simeq -53.0$ kN (to the left)

Q3. (a) The flow is parallel, therefore pressure is independent of r

(i) Equilibrium of forces on a horizontal cylinder of fluid



τ . shear stress

$$2\pi r \, dx \, \tau = -dp \, \pi r^2 \qquad \text{and} \quad \tau = \mu \frac{du}{dr}$$

So $\dfrac{du}{dr} = -\dfrac{r}{2\mu}\dfrac{dp}{dx}$

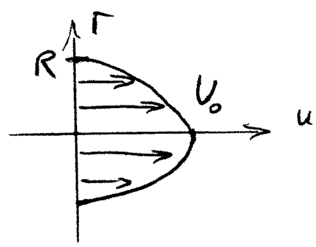
Integrating once $\quad u = -\dfrac{dp}{dx}\dfrac{r^2}{4\mu} + k$

The constant k is such that $u(R) = 0$ (no slip)

$$u = -\frac{dp}{dx}\frac{R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

And $U_0 = -\dfrac{dp}{dx}\dfrac{R^2}{4\mu}$ is the maximum velocity

(ii) Velocity profile



parabolic profile

Volume flux $= \bar{U} \times$ cross-section

$$\bar{U} = \frac{1}{\pi R^2}\int_0^R u(r)\, 2\pi r \, dr = \frac{2}{R^2} U_0 \int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] r \, dr$$

$$= 2U_0 \int_0^1 (1 - a^2)\, a \, da \qquad \left(a = \frac{r}{R}, \quad da = \frac{dr}{R}\right)$$

$$= 2U_0 \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{U_0}{2}$$

(b) (i) Check Reynolds number is below transition value

$$Re = \rho \frac{\bar{U} D}{\mu} = \frac{2 \bar{U} R}{\nu}$$

$$\bar{U} = \frac{Q}{A} = \frac{10^{-5}}{\pi\, 0.005^2} = 0.127 \ m.s^{-1}$$

Hence $$Re = \frac{2 \times 0.127 \times 0.005}{10^{-6}} = 1270$$

$Re < 2000$, the flow is laminar, the results of (a) are relevant

(ii) As found in (a) (ii), for a laminar flow

$$U_{max} = U_0 = 2\bar{U}$$

So $U_0 = 0.254 \ m.s^{-1}$

Q4 (a) ρ density, σ surface tension) affect waves

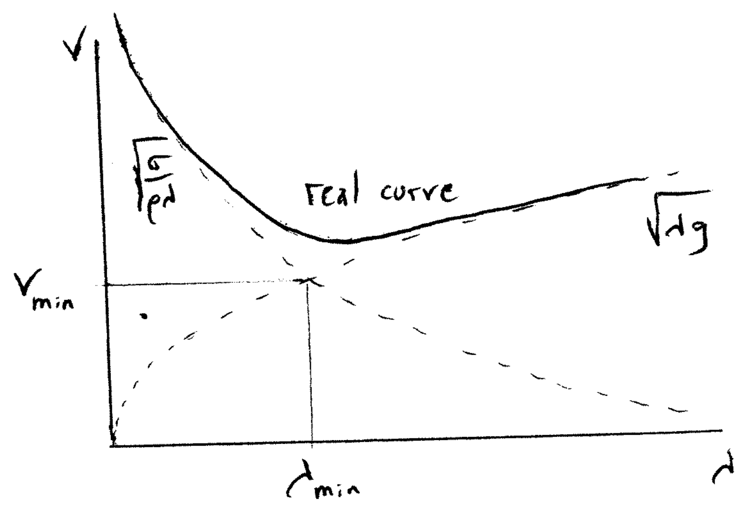
(b) d very large implies σ plays no role.
The parameters left are d, g, ρ. Three parameters involving 3 units m, kg, s. Therefore there are no dimensionless parameter. Calling v the wave velocity, ~~As~~ noting that $\sqrt{gd}$ has the dimension of a velocity, so $\frac{v}{\sqrt{gd}}$ is a pure constant (dependent on no parameter).

$$v \sim \sqrt{gd}$$

(c) d very small so g plays no role.
3 parameters d, ρ, σ : no dimensionless parameter.

So $v \sim \sqrt{\frac{\sigma}{\rho d}}$ (single group with dimension of a velocity)

(d)



d_{min} when $\sqrt{\frac{\sigma}{\rho d}} \simeq \sqrt{dg}$, so when $d \sim \sqrt{\frac{\sigma}{\rho g}}$

And, at that wavelength $v_{min} \sim \left[\frac{\sigma g}{\rho}\right]^{1/4}$

Q5 (a) For any control volume enclosed in a closed surface S

$$\oint_S \vec{q} \cdot d\vec{A} = 0 \qquad \text{where the heat flux density is } \vec{q} = -\lambda \vec{\nabla} T$$

By Gauss theorem $\displaystyle\int_{\mathcal{V}} \operatorname{div} \vec{q} \; d\mathcal{V} = 0$

This is true for any volume only if $\operatorname{div} \vec{q} = 0$ everywhere

$$\operatorname{div} \vec{q} = \cancel{\lambda \nabla^2 T} - \operatorname{div}(\lambda \vec{\nabla} T) = 0$$

In this axisymmetric problem $T = T(r)$ and $\vec{\nabla} T = \dfrac{dT}{dr} \vec{e}_r$

$$\operatorname{div}(\lambda \vec{\nabla} T) = \frac{1}{r} \frac{d}{dr} \left[r \lambda(r) \frac{dT}{dr} \right] = 0$$

(b) By the chain rule $\dfrac{dF}{dr} = \dfrac{dF}{dT} \dfrac{dT}{dr}$

F is the integral of λ so $\dfrac{dF}{dT} = \lambda$ and $\dfrac{dF}{dr} = \lambda \dfrac{dT}{dr}$

(c) The equation in (a) can be written

$$\frac{1}{r} \frac{d}{dr} \left[r \frac{dF}{dr} \right] = 0 \quad , \text{ so} \quad r \frac{dF}{dr} = k \quad \text{a constant}$$

and $\dfrac{dF}{dr} = \dfrac{k}{r}$

and $F = k \ln r + L$

(L another constant)

(d) Now $\lambda(T) = 0.01 + \dfrac{T - T_0}{T_1 - T_0} \times 0.99$ with $T_0 = 20°C$, $T_1 = 1000°C$

$$F(T) = \int_{T_0}^{T} \lambda(u)\, du = 0.01(T - T_0) + \frac{1}{2} \frac{0.99}{T_1 - T_0} (T - T_0)^2$$

At $r_2 = 0.2\,m$, continuity of heat fluxes

$$-\lambda \frac{dT}{dr} = h\,(T - T_0)$$

$$-\frac{dF}{dr} = 10\,(T - T_0) \quad , \text{ so } \quad -\frac{k}{r_2} = 10\,(T - T_0)$$

$$\text{and} \quad T - T_0 = -\frac{k}{2}$$

$$\begin{cases} \text{So } F = 0.01\left(-\frac{k}{2}\right) + \frac{1}{2}\,\frac{0.99}{T_1 - T_0}\left(-\frac{k}{2}\right)^2 \\ \text{and from (c) } F = k \ln r_2 + L \end{cases}$$

At $r_1 = 0.1\,m$ $\quad T = T_1 \Rightarrow F = 0.01\,(T_1 - T_0) + \frac{1}{2}0.99\,(T_1 - T_0)$

and from (c) $\quad F = k \ln r_1 + L$

so $\quad L = 0.01(1000-20) + \frac{1}{2} \times 0.99 \times (1000-20) - k \ln 0.1$

$$= 494.9 + 2.3026\,k$$

Substituting L one finds a quadratic equation for k

$$0.0001263\,k^2 - 0.6982\,k - 494.9 = 0$$

$$k = \frac{0.6982 - \sqrt{0.7375}}{2 \times 0.0001263} = -635.8 \quad \text{negative root}$$

and $\quad L = -969.0$

Heat flux per ~~second~~ meter length $\quad \phi = 2\pi r_2\, h\, (T - T_0)$

$$= -2\pi k \simeq 3995 \text{ W}$$

$T - T_0 = -\frac{k}{2} \simeq 317.9\ °C$, so $\quad T(r_2) = 337.9\ °C$

Q6. (a) At 100°C saturated steam pressure $P_s = 1.01325$ bar
(Haywood table 7)

Dalton's law $\quad P_{air} = P - P_s$

$= 0.9867$ bar

(b) $\omega = \dfrac{P_s}{P_{air}} \dfrac{M_{H_2O}}{M_{air}} = \dfrac{P_s}{P - P_s} \dfrac{18}{29} \simeq 0.637 \dfrac{kg_s}{kg_{air}}$

Mass fraction: $\quad m_s = \dfrac{1}{2} \times 0.637 = 0.3187 \dfrac{kg_{steam}}{kg_{mixture}}$

(c) $s = \dfrac{1}{2} s_{air} + 0.3187\, s_s + 0.1813\, s_\ell$
(steam) (liquid water)

Air at 300 K, 1 bar : from table 19 $\quad s_{air} = 3.892$ kJ.K⁻¹.kg⁻¹

$$s_{air}\Big|_{\substack{373.15\,K \\ 0.9867\,bar}} = s_{air}\Big|_{\substack{300\,K \\ 1\,bar}} + c_p \ln \frac{373.15}{300} - R \ln \frac{0.9867}{1}$$

$\simeq 4.1162$ $kJ.K^{-1}.kg^{-1}$

From table 7, $s_s = 7.355$ $kJ.K^{-1}.kg^{-1}$ and $s_\ell = 1.307$ $kJ.K^{-1}.kg^{-1}$
at 100°C and saturation.

So $\quad s = 0.5 \times 4.1162 + 0.3187 \times 7.355 + 0.1813 \times 1.307$

$= 4.639$ $kJ.K^{-1}.kg^{-1}$

(d) $\Delta s = 0$ in the expansion

Assuming steam is still saturated, and $P_s = 0.45$ bar $\left(\begin{array}{c}T = 78.7°C \\ \text{table 8}\end{array}\right)$

$\omega = \dfrac{P_s}{P - P_s} \dfrac{18}{29} = 0.6207$ with $p = 1$ bar

Similar calculation as in (c) gives $s = 4.349\ kJ.K^{-1}.kg^{-1}$

Starting now from $P_s = 0.5$ bar ($T = \cancel{80.9}\ 81.3\,°C$ table 8)

one finds $s = 4.6932\ kJ.K^{-1}.kg^{-1}$

Take a linear interpolation between $P_s = 0.45$ bar and 0.5 bar so that $\Delta s = 0$

$$\frac{P_s - 0.45}{P_s - 0.5} = \frac{4.639 - 4.349}{4.639 - 4.6932} \longrightarrow P_s = 0.492 \text{ bar}$$

The interpolation on temperatures yields $T = 80.9\,°C$

$$\omega = \frac{P_s}{P - P_s}\,\frac{18}{29} = 0.6011$$

so $m_s = 0.3006 \dfrac{\text{kg steam}}{\text{kg mixture}} < 0.5$ still saturated.