Paper 7: Mathematical Methods

Solutions to 2000 Tripos Paper

1. *Integral equations, solenoidal flows and field lines*

(a) The net heat flow by convection *into* the volume V is given by

$$-\iint_S (\rho c T \mathbf{u}).\mathbf{dS}$$

where the minus sign is necessary since $\mathbf{dS}$ points along the *outward* normal. The net heat flow by conduction *into* the volume V is given by

$$-\iint_S (-\lambda \nabla T).\mathbf{dS}$$

Note that the heat flow is in the direction $-\nabla T$, so $(-\lambda\nabla T).\mathbf{dS}$ is an *outward* heat flow, hence the need for the second minus sign. These two heat flows must sum to give the rate of change of energy contained in the volume V, which is given by

$$\frac{\partial}{\partial t}\iiint_V (\rho c T)\,\mathrm{dV}$$

Hence

$$\frac{\partial}{\partial t}\iiint_V (\rho c T)\,\mathrm{dV} = -\iint_S (\rho c T \mathbf{u}).\mathbf{dS} - \iint_S (-\lambda\nabla T).\mathbf{dS}$$

By Gauss' theorem,

$$\iint_S (\rho c T \mathbf{u}).\mathbf{dS} = \iiint_V \rho c \nabla.(T\mathbf{u})\mathrm{dV}$$

Gauss' theorem also tells us that

$$\iint_S (-\lambda\nabla T).\mathbf{dS} = \iiint_V -\lambda\nabla.(\nabla T)\mathrm{dV} = \iiint_V -\lambda\nabla^2 T\mathrm{dV}$$

Hence, the integral equation becomes

$$\frac{\partial}{\partial t}\iiint_V (\rho c T)\,\mathrm{dV} = -\iiint_V \rho c \nabla.(T\mathbf{u})\mathrm{dV} - \iiint_V -\lambda\nabla^2 T\mathrm{dV}$$ [6]

(b) Interchanging the order of differentiation and integration gives

$$\iiint_V \frac{\partial T}{\partial t}\,\mathrm{dV} = -\iiint_V \nabla.(T\mathbf{u})\mathrm{dV} - \iiint_V -\alpha\nabla^2 T\mathrm{dV}$$

where $\alpha = \lambda/\rho c$. Since this integral equation holds for every volume V, it follows that at every point

$$\frac{\partial T}{\partial t} = -\nabla.(T\mathbf{u}) + \alpha\nabla^2 T \Leftrightarrow \frac{\partial T}{\partial t} + \nabla.(T\mathbf{u}) = \alpha\nabla^2 T$$

However, we know that (data book)

$$\nabla.(T\mathbf{u}) = T\nabla.\mathbf{u} + \nabla T.\mathbf{u}$$

and also $\nabla.\mathbf{u} = 0$ for an incompressible fluid. Hence

$$\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T = \alpha\nabla^2 T$$ [7]

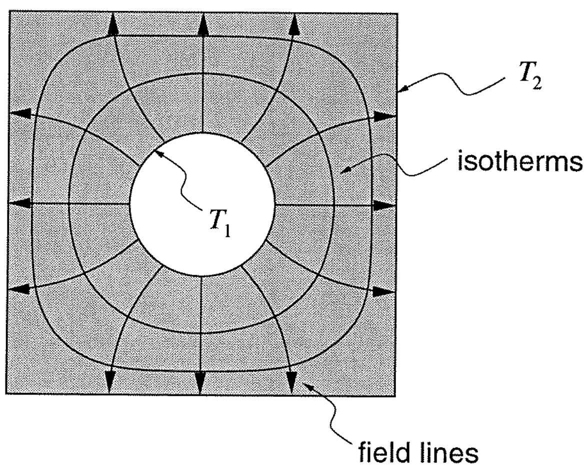
(c) The equation

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T$$

is an example of the **diffusion equation**. The divergence of $\mathbf{q}_1$ is given by

$$\nabla.\mathbf{q}_1 = -\lambda\nabla.(\nabla T) = -\lambda\nabla^2 T$$

However, if the temperature field is steady, we know that $\partial T/\partial t = 0$ and the diffusion equation tells us that $\nabla^2 T = 0$ as well. It follows that $\nabla.\mathbf{q}_1 = 0$, and $\mathbf{q}_1$ is therefore solenoidal. We can sketch field lines for $\mathbf{q}_1$, which are perpendicular to the isotherms and flow from high to low temperature (since $\mathbf{q}_1 = -\lambda\nabla T$).



[7]

Examiner's remarks: This question was well answered by the majority of candidates who attempted it. There was a good level of understanding of Gauss's Law, field lines and isotherms. The average mark was brought down by a number of candidates who had very little comprehension of the concepts involved.

2. *Properties of vector fields*

(a) The divergence of $\mathbf{F}$ is given by

$$\nabla.\mathbf{F} = \frac{\partial}{\partial x}(-3y) + \frac{\partial}{\partial y}(x) = 0$$

Since $\nabla.\mathbf{F} = 0$ everywhere, it follows that $\mathbf{F}$ is solenoidal.

The curl of $\mathbf{F}$ is given by

$$\nabla \wedge \mathbf{F} = \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-3y)\right]\mathbf{k} = 4\mathbf{k}$$

The flux of $\nabla \wedge \mathbf{F}$ through S_1 is given by

$$\iint_{S_1} (\nabla \wedge \mathbf{F}).\mathbf{k}\ \mathrm{dA} = \iint_{S_1} 4\ \mathrm{dA} = 4\pi a^2$$ [6]

(b) Since $\nabla \wedge \mathbf{F}$ is solenoidal, it follows that the net flux into the closed hemisphere defined by S_1 and S_2 is zero. Hence the flux of $\nabla \wedge \mathbf{F}$ through S_2 must be $4\pi a^2$, the same as the flux of $\nabla \wedge \mathbf{F}$ through S_1.

A similar argument holds for the flux of $\mathbf{F}$. Since $\mathbf{F}$ is solenoidal, it follows that the flux of $\mathbf{F}$ through S_2 must be the same as the flux of $\mathbf{F}$ through S_1. The flux of $\mathbf{F}$ through S_1 is clearly zero, since $\mathbf{F}$ has no component normal to the plane of S_1. Hence the flux of $\mathbf{F}$ through S_2 is also zero.

Stokes' theorem tells us that

$$\iint_{S_1} (\nabla \wedge \mathbf{F}).\mathbf{k}\ \mathrm{dA} = \oint_c \mathbf{F}.\mathbf{dl}$$

where the line integral is taken in the sense shown in Fig. 2. We have already shown that the left hand side is equal to $4\pi a^2$, so it follows that $\oint_c \mathbf{F}.\mathbf{dl} = 4\pi a^2$. [8]

(c) If $f(z)$ is an arbitrary function of z, it follows that

$$\nabla.(f(z)\mathbf{F}) = \frac{\partial}{\partial x}(-3f(z)y) + \frac{\partial}{\partial y}(f(z)x) = 0$$

Hence $f(z)\mathbf{F}$ is solenoidal, and it follows that the flux of $f(z)\mathbf{F}$ through S_2 must be the same as the flux of $f(z)\mathbf{F}$ through S_1. The flux of $f(z)\mathbf{F}$ through S_1 is clearly zero, since $f(z)\mathbf{F} = -3f(z)y\mathbf{i} + f(z)x\mathbf{j}$ has no component normal to the plane of S_1. Hence the flux of $f(z)\mathbf{F}$ through the hemisphere S_2 is also zero. [6]

Examiner's remarks: This was perhaps the most straightforward question on the examination paper, attracting attempts from the vast majority of candidates. The level of performance was high, with candidates demonstrating a good grasp of both Gauss's and Stokes's theorems. Once again, the average mark was brought down by a number of candidates who had very little comprehension of the concepts involved.

3. *Partial differential equations*

(a) Assuming the solution is of the form

$$u = F(y)\cos\omega t + G(y)\sin\omega t$$

we obtain

$$\frac{\partial u}{\partial t} = -\omega F\sin\omega t + \omega G\cos\omega t \quad \text{and} \quad \frac{\partial^2 u}{\partial^2 y} = F''\cos\omega t + G''\sin\omega t$$

Substituting into the partial differential equation gives

$$-\omega F\sin\omega t + \omega G\cos\omega t = \nu\left(F''\cos\omega t + G''\sin\omega t\right)$$

Equating the coefficients of the sine and cosine terms gives

$$-\omega F = \nu G'' \quad \text{and} \quad \omega G = \nu F''$$

Hence

$$F = -(\nu/\omega)G'' \;\; \text{and} \;\; G = (\nu/\omega)F''$$

Combining the previous two equations, we obtain

$$F = -(\nu/\omega)^2 F^{(4)} \;\; \text{and} \;\; G = -(\nu/\omega)^2 G^{(4)}$$

Thus the fourth-order ordinary differential equations for F and G are identical. [8]

(b) Since the differential equation for G is the same as that for F, they must share the same general solution. Also, since

$$F(y) \to 0 \text{ as } y \to \infty \quad \text{and} \quad G(y) \to 0 \text{ as } y \to \infty$$

we can immediately rule out the positive exponential solutions, leaving

$$F(y) = \exp(-y/\delta)\left(A\cos(y/\delta) + B\sin(y/\delta)\right)$$

and

$$G(y) = \exp(-y/\delta)\left(C\cos(y/\delta) + D\sin(y/\delta)\right)$$

We now turn to other boundary conditions. The fluid in contact with the plate ($y = 0$) must share the same velocity $V \cos \omega t$ of the plate, hence

$$F(0) = V \quad \text{and} \quad G(0) = 0$$

Thus

$$F(y) = \exp(-y/\delta)\,(V\cos(y/\delta) + B\sin(y/\delta)) \tag{1}$$

and

$$G(y) = \exp(-y/\delta)\,(D\sin(y/\delta))$$

Differentiating twice, we obtain

$$G''(y) = -\frac{2D}{\delta^2}\exp(-y/\delta)\cos(y/\delta) = -\frac{\omega D}{\nu}\exp(-y/\delta)\cos(y/\delta)$$

But we know from earlier that

$$F = -(\nu/\omega)G'' = D\exp(-y/\delta)\cos(y/\delta)$$

Comparing with equation (1), we see that $D = V$ and $B = 0$. The velocity field is therefore

$$u = V\exp(-y/\delta)\,[\cos(y/\delta)\cos\omega t + \sin(y/\delta)\sin\omega t]$$ [12]

Examiner's remarks: A less popular question with mixed responses. Those who showed some competence tended to score very highly. However, there were a significant number who had almost no idea of how to proceed: formulating boundary conditions, and in particular realising that the motion of the fluid must decay with distance from the plate, was the most common problem.

4. *Newton-Raphson iteration and LU decomposition*

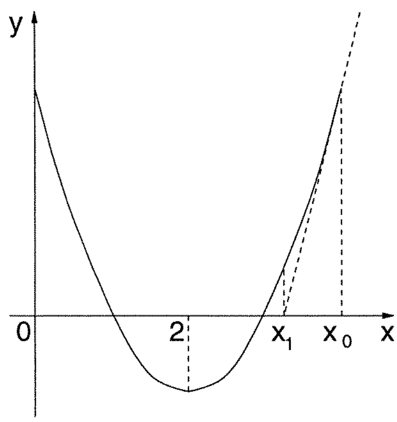
(a) (i) At iteration n, we have an approximate solution x_n where $f(x_n) \neq 0$. We require $f(x_{n+1}) = 0$. A Taylor series expansion around x_n gives

$$f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n)f'(x_n) + O(x_{n+1} - x_n)^2$$

Substituting $f(x_{n+1}) = 0$ and truncating after the first derivative gives

$$0 = f(x_n) + (x_{n+1} - x_n)f'(x_n) \quad \Leftrightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ [5]

(ii)



The parabola $f(x) = x^2 - 4x + 3 = (x-3)(x-1)$ has roots at $x = 3$ and $x = 1$ and a turning point at $x = 2$. As the sketch shows, the Newton-Raphson algorithm proceeds by finding the tangent to $f(x)$ at x_n and setting x_{n+1} to the intersection of the tangent with the x axis. It is evident that this will converge to the root at $x = 3$ only if the starting point is to the right of the turning point, that is only if $x_0 > 2$. [4]

(iii)

n	x_n	$f(x_n)$	$f'(x_n)$
0	0.2	3.4×10^{-4}	8.1×10^{-3}
1	0.1580	1.144×10^{-4}	3.218×10^{-3}
2	0.1225	3.982×10^{-5}	

Despite running the Newton-Raphson iteration until the residual is very small (less than 10^{-4}), there remains a significant error in the estimated root $x_2 = 0.1225$: the true root is, of course, $x = 0$. This is because $f(x)$ is very flat around $x = 0$. The equation $f(x) = 0$ is in fact ill-conditioned. [5]

(b) Write the system of equations as $\mathrm{A}\mathbf{x} = \mathbf{b}$. The LU decomposition of A is

$$\begin{bmatrix} 10 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 & -2 \\ 0 & 4 & 0 \\ 0 & 0 & 18/5 \end{bmatrix}$$

The matrix equation can then be posed as $\mathrm{L}\mathbf{y} = \mathbf{b}$, where $\mathrm{U}\mathbf{x} = \mathbf{y}$. Solving first for $\mathbf{y}$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 72/5 \end{bmatrix}$$

Finally, solving for $\mathbf{x}$ gives

$$\begin{bmatrix} 10 & 0 & -2 \\ 0 & 4 & 0 \\ 0 & 0 & 18/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 72/5 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$ [6]

Examiner's remarks: This straightforward question attracted attempts by almost all candidates. Many answers to (i) were perfectly satisfactory, though some candidates were unable to manipulate a simple Taylor series expansion. Part (ii) was very easy for the vast majority of candidates who understood the principles of Newton-Raphson iteration. Part (iii) was less well answered, with many candidates wasting time by iterating until $|x_n| < 10^{-4}$ (instead of $|f(x_n)| < 10^{-4}$) and offering spurious explanations for the error in the solution. The LU decomposition part of the question was well answered by many candidates, though some apparently had no idea what LU decomposition was, while others confused the forms of the L and U matrices. Another common mistake was to perform the substitutions in the wrong order, having found L and U.

5. *Least squares*

(a) (i) We are looking for a minimum of E, which will occur when all three partial derivatives are zero:

$$\begin{aligned} \frac{\partial E}{\partial a} &= -2\sum_{i=1}^{n} x_i \left[z_i - (ax_i + by_i + c)\right] = 0 \\ \frac{\partial E}{\partial b} &= -2\sum_{i=1}^{n} y_i \left[z_i - (ax_i + by_i + c)\right] = 0 \\ \frac{\partial E}{\partial c} &= -2\sum_{i=1}^{n} \left[z_i - (ax_i + by_i + c)\right] = 0 \end{aligned}$$

Rearranging and writing in matrix form gives

$$\begin{bmatrix} \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i & \sum y_i & n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum x_i z_i \\ \sum y_i z_i \\ \sum z_i \end{bmatrix}$$

where all summations are over $i \in \{1 \ldots n\}$. [6]

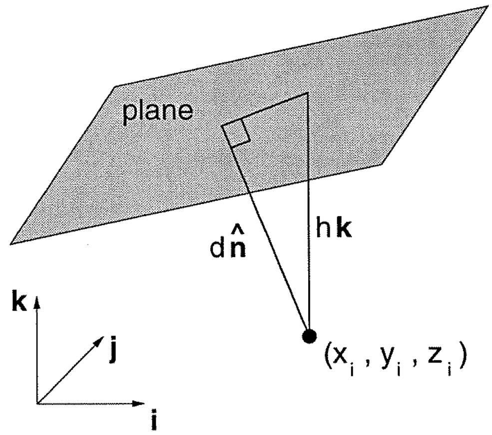
(ii) The matrix A will be singular if the points are collinear. This will occur every time if there are only two points, and also for collinear configurations of three or more points. There will then be an infinite number of planes which fit the points exactly. There are many other degenerate configurations of points that result in A being singular: for example, points evenly distributed on the surface of a sphere. [3]

(b) (i) With reference to the diagram, the orthogonal displacement from a point to the plane is $d\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit normal to the plane:

$$\hat{\mathbf{n}} = \frac{a\mathbf{i} + b\mathbf{j} - \mathbf{k}}{\sqrt{1 + a^2 + b^2}}$$

The vertical (z) displacement from the point to the plane is $h\mathbf{k}$, where

$$h = z_i - (ax_i + by_i + c)$$



It follows that

$$d = h\mathbf{k}.\hat{\mathbf{n}} = -\frac{z_i - (ax_i + by_i + c)}{\sqrt{1 + a^2 + b^2}}$$

and therefore

$$E' = \sum_{i=1}^{n} d^2 = \frac{1}{1 + a^2 + b^2} \sum_{i=1}^{n} [z_i - (ax_i + by_i + c)]^2$$ [6]

(ii) The orthogonal approach would be appropriate when errors are equally likely in the x, y and z directions. In contrast, the vertical approach would be appropriate when uncertain measurements z_i are taken at well-defined locations (x_i, y_i). The orthogonal approach leads to a system of equations for the plane's parameters which is quadratic in a and b (consider the form of $\partial E'/\partial a$ and $\partial E'/\partial b$). This contrasts with the straightforward, linear system for the vertical approach. [5]

Examiner's remarks: Almost all of the few candidates who attempted this questions were able to differentiate E to obtain the linear system of equations for the best fit plane. Very few were able to connect the singularity of the matrix A to the geometrical configuration of the points. Only one candidate managed to perform the simple dot product to derive E'. For the last part of the question, candidates who understood the significance of measurement uncertainty (about half of those who attempted the question) were able to discuss the two schemes intelligently.

6. *Fourier transforms and spectral estimation*

(a)

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt = \int_{-T/2}^{T/2} e^{-i\omega t}dt = \left[\frac{-e^{-i\omega t}}{i\omega}\right]_{-T/2}^{T/2} \\ &= \frac{2i\sin(\omega T/2)}{i\omega} = \frac{T\sin(\omega T/2)}{\omega T/2} = T\text{sinc}\left(\frac{\omega T}{2}\right) \end{aligned}$$ [3]

(b)

$$\begin{aligned} c(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]e^{i\omega t}d\omega \\ &= \frac{1}{2}\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right) \qquad \text{(sifting property of } \delta \text{ functions)} \\ &= \frac{1}{2}(2\cos\omega_0 t) = \cos\omega_0 t \end{aligned}$$ [3]

(c) (i) The Fourier transform of $x(t)$ is given by

$$X(\omega) = \pi\left[A\,\delta(\omega-\omega_1) + A\,\delta(\omega+\omega_1) + B\,\delta(\omega-\omega_2) + B\,\delta(\omega+\omega_2)\right]$$

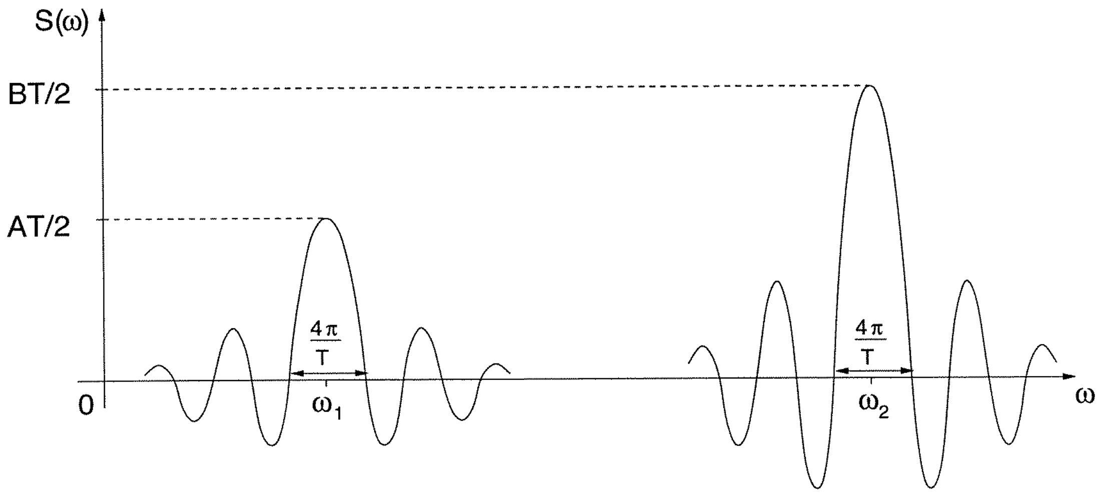
By the convolution theorem, the Fourier transform of $s(t) = x(t)h(t)$ is given by

$$\begin{aligned} S(\omega) &= \frac{1}{2\pi}X(\omega) * H(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega-\Omega)X(\Omega)d\Omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} T\text{sinc}\left(\frac{(\omega-\Omega)T}{2}\right)\times \\ &\quad \pi\left[A\delta(\Omega-\omega_1)+A\delta(\Omega+\omega_1)+B\delta(\Omega-\omega_2)+B\delta(\Omega+\omega_2)\right]d\Omega \end{aligned}$$

By the sifting property of δ functions, this simplifies to

$$\begin{aligned} S(\omega) &= \frac{T}{2}\left[A\,\text{sinc}\left(\frac{(\omega-\omega_1)T}{2}\right) + B\,\text{sinc}\left(\frac{(\omega-\omega_2)T}{2}\right)\right. \\ &\quad \left. + A\,\text{sinc}\left(\frac{(\omega+\omega_1)T}{2}\right) + B\,\text{sinc}\left(\frac{(\omega+\omega_2)T}{2}\right)\right] \end{aligned}$$

For the case $\omega_2 \gg \omega_1$, the positive half of the spectrum looks like this (see next page):



[6]

(ii) If $\omega_2 \approx \omega_1$, the two sinc functions overlap and it is difficult to resolve the individual components. [1]

(iii) Extending the observation period T reduces the width of the sinc pulses, making it easier to resolve the individual components. [2]

(iv) The spectrum of the triangular window function $w(t)$ is given by

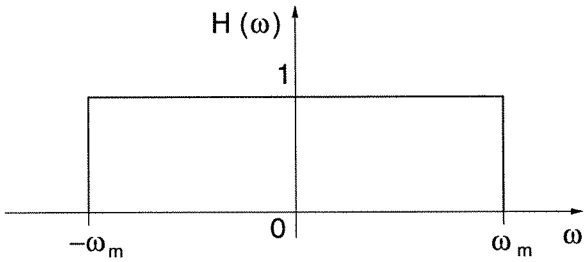
$$W(\omega) = \frac{T}{2}\,\text{sinc}^2\left(\frac{\omega T}{4}\right) \qquad \text{(data book)}$$

Compared with the spectrum $H(\omega)$ of the square window function, $W(\omega)$ has faster decaying side lobes (ω^2 as opposed to ω) but a wider main peak ($8\pi/T$ as opposed to $4\pi/T$). The triangular window may help if one small component is swamped by the side lobes of a nearby, larger one. However, it will not help if the two components have very similar frequencies: if $|\omega_2 - \omega_1| < 8\pi/T$, the two main spectral peaks merge. [5]

Examiner's remarks: This question (the gist of which should have been familiar from examples papers and one of the Part IB experiments) was answered very well by many candidates and very poorly by many others. Common gaps in the knowledge of those candidates who struggled with this question include: the sifting properties of delta functions; the fact that windowing involves multiplication by the window function, not convolution; the difference between spectral leakage and aliasing; the general properties of sinc functions.

7. *Sampling and reconstruction, continuous random variables*

(a) (i) The signal can be recovered using an ideal low-pass filter $H(\omega)$ with cut-off frequency ω_m.



[2]

(ii) From the data book, the Fourier transform of a unit pulse of duration T centered at the origin is $T\text{sinc}\,(\omega T/2)$. Using the time shift theorem, the Fourier transform of $g(t)$ is

$$G(\omega) = Te^{-i\omega T/2}\,\text{sinc}\left(\frac{\omega T}{2}\right)$$

Let $W(\omega)$ be the frequency response of a filter which can be used to recover $x(t)$ from $x_g(t)$:

$$X(\omega) = X_g(\omega)W(\omega)$$

Recall from (i) that an ideal low-pass filter $H(\omega)$ can be used to recover $x(t)$ from $x_s(t)$:

$$X(\omega) = X_s(\omega)H(\omega)$$

Since $x_g(t)$ is formed from the convolution of $x_s(t)$ with $g(t)$, it follows that

$$X_g(\omega) = X_s(\omega)G(\omega)$$

Putting these equations together we get

$$\begin{aligned} X(\omega) &= X_s(\omega)H(\omega) = X_s(\omega)G(\omega)W(\omega) \\ \Rightarrow W(\omega) &= H(\omega)/G(\omega) = \begin{cases} \dfrac{e^{i\omega T/2}}{T\text{sinc}(\omega T/2)} & \text{for } |\omega| < \omega_m \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

In other words, the combination of $G(\omega)$ and $W(\omega)$ should be the same as $H(\omega)$. [8]

(b) (i) $\text{E}[X] = \int_{-\infty}^{\infty} xf(x)\,dx$ [2]

(ii) $\text{E}[\lambda X] = \int_{-\infty}^{\infty} \lambda xf(x)\,dx = \lambda \int_{-\infty}^{\infty} xf(x)\,dx = \lambda\text{E}[X]$

$$\text{E}[X-a] = \int_{-\infty}^{\infty} (x-a)f(x)\,dx = \int_{-\infty}^{\infty} xf(x)\,dx \; - \; a\int_{-\infty}^{\infty} f(x)\,dx = \text{E}[X] - a$$

since the area under a probability density function is 1. [3]

(iii)

$$
\begin{aligned}
E[(X - E[X])^2] &= \int_{-\infty}^{\infty} (x - E[X])^2 f(x)\, dx \\
&= \int_{-\infty}^{\infty} x^2 f(x)\, dx \;-\; 2E[X] \int_{-\infty}^{\infty} x f(x)\, dx \;+\; (E[X])^2 \int_{-\infty}^{\infty} f(x)\, dx \\
&= E[X^2] - 2E[X]E[X] + (E[X])^2 = E[X^2] - (E[X])^2
\end{aligned}
$$

[5]

Examiner's remarks: In (a), most candidates were able to sketch the ideal reconstruction filter for an impulse train and calculate the frequency response of the pulse-broadening filter. However, only a handful of candidates were able to deduce the ideal reconstruction filter for the broadened signal. Many candidates achieved perfect marks for (b), though others decided to use generating functions and got hopelessly confused. A few candidates seemed to confuse discrete random variables with their continuous counterparts, and offered a discrete analysis for (b).

8. *Probability*

(a) A variable is a characteristic that varies over time and/or for different objects under consideration. A variable is random if the value that it assumes, corresponding to the outcome of an experiment, is a random event. Discrete random variables can assume one of a finite number of values. Continuous random variables can assume any value on a continuous scale. An everyday example of a discrete random variable is the number of days it rains in a given month. An everyday example of a continuous random variable is the total amount of rainfall in a given month. [4]

(b) Suppose the weighing machine is reading f. We know the error $m - f$ is Normally distributed with mean 1 g and standard deviation 10 g. Hence

$$m - f \sim N(1, 10) \;\; \Leftrightarrow \;\; m \sim N(1 + f, 10)$$

We require $p(m < 1000) = 0.01$. The cumulative Normal distribution table (data book) gives $\phi(2.326) = 0.99$, so $\phi(-2.326) = 0.01$. Scaling and shifting for $m \sim N(1 + f, 10)$ gives

$$-2.326 = \frac{1000 - (1 + f)}{10} \;\; \Leftrightarrow f = 1022.26$$

So the bags should be filled until the weighing machine reads 1022.26 g. [5]

(c) (i) On average, there is one bus every 5 minutes. Assuming x, the number of buses arriving in any 5 minute interval, is distributed according to the Poisson distribution with mean 1, then

$$p(\text{take the tube}) = p(x = 0) = e^{-1}\frac{1^0}{0!} = 0.368 \;\; \text{(to 3 significant figures)}$$

So the probability I take the bus is approximately $1 - 0.368 = 0.632$. [2]

(ii) The Poisson distribution is applicable when the counts (in this case, the number of buses in each 5 minute interval) occur randomly and independently of each other.

This would certainly not be the case for buses running on-time to a timetable: assuming regular buses, the probability of one bus in a 5 minute interval is one, and the probability of any other number of buses in a 5 minute interval is zero, which is not a Poisson distribution!

The Poisson distribution might be a better model if the buses do not leave the terminus on time, the route is subject to random traffic delays and my bus stop is some distance from the terminus. In this case, buses would come at an average rate of one every 5 minutes, but the particular number of buses arriving in any 5 minute interval would be random. We still need to assume independence between bus arrival times. This would require the buses to leave the terminus independently (possible, especially if they are turning around after random traffic delays on earlier journeys). Alternatively, the buses could be held up in independent traffic jams on the way to my bus stop (unlikely). [4]

(iii) Assuming that the probability I take the bus on any one day is equal to 0.632 and independent of the probability I take the bus on any other day, the number of days I take the bus in a 20 working-day month is Binomially distributed with $n = 20$ and $p = 0.632$. The probability I take the bus for precisely 15 return journeys is therefore

$$^{20}C_{15}\, 0.632^{15}\, 0.368^{5} = \frac{20!}{15! \times 5!}\, 0.632^{15}\, 0.368^{5} = 0.107 \;\; \text{(to 3 significant figures)}$$ [3]

(iv) The mean of the Binomial distribution is $np = 12.64$. So I expect to take the bus 12.64 days in a 20 working-day month. My expected monthly travel costs are therefore

$$12.64 \times 2 + (20 - 12.64) \times 4 = \pounds 54.72 \;\; \text{(to the nearest penny)}$$ [2]

Examiner's remarks: Most candidates were able to distinguish between continuous and discrete random variables, and one or two even attempted to define a random variable. Most candidates understood how to use the cumulative Normal tables, though relatively few were careful enough when it came to using the tabulated value: the most common mistake was to incorporate the 1g bias with the wrong sign. Again, most candidates were able to successfully manipulate the Poisson and Binomial distribution formulae, and a good number also offered intelligent comments regarding the validity of the Poisson assumption.

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