ENGINEERING TRIPOS PART IB

Friday 9 June 2000 9 to 11

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer at least ***one*** *question from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

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SECTION A

*Answer at least **one** question from this section*

1 Consider a stationary surface S which encloses a volume V. The volume and the surrounding space are filled with an *incompressible*, moving fluid, the velocity field being $\mathbf{u}$. Energy is transported across the surface S by two mechanisms. First, there is a heat flux $\mathbf{q}_1 = -\lambda\nabla T$ by virtue of conduction, λ being the thermal conductivity and T the temperature field. Second, energy is transported across the surface by virtue of movement of the fluid. This results in an additional flux of $\mathbf{q}_2 = (\rho c T)\mathbf{u}$, where ρ is the density and c the specific heat capacity of the fluid. The fluid properties ρ, c and λ are constants.

(a) By explaining the meanings of the three terms, show that

$$\frac{\partial}{\partial t}\iiint_V (\rho c T)\,\mathrm{dV} = -\oiint_S (\rho c T\mathbf{u}).\mathbf{dS} - \oiint_S (-\lambda\nabla T).\mathbf{dS}\,.$$

Now use Gauss's theorem to rewrite the right hand side in terms of volume integrals. [6]

(b) Show that the integral equation above requires that, at every point in the fluid,

$$\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T = \alpha\nabla^2 T \quad , \quad \text{where } \alpha = \lambda/\rho c.$$ [7]

(c) In the limit of a vanishingly small $|\mathbf{u}|$, we obtain

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T\,.$$

Figure 1 shows a square plate containing a circular hole. The temperature of the inner, circular edge is maintained at T_1, while the outer edge of the plate is held at a lower temperature T_2. If the temperature field is steady, show that $\mathbf{q}_1$ is solenoidal. Sketch the isotherms and the field lines for $\mathbf{q}_1$. [7]

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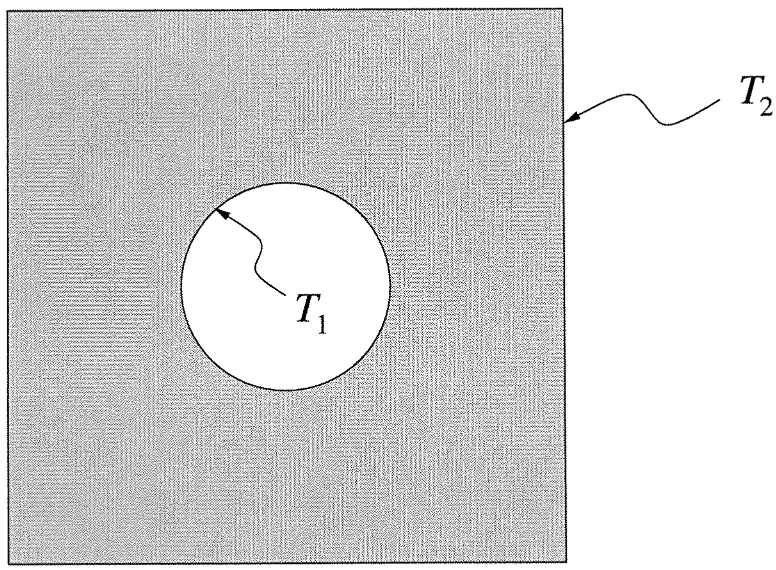

Fig. 1

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2 Consider the vector field

$$\mathbf{F}(x, y, z) = -3y\mathbf{i} + x\mathbf{j}\,.$$

(a) Confirm that $\mathbf{F}$ is solenoidal and that $\nabla \wedge \mathbf{F} = 4\mathbf{k}$. Evaluate the flux of $\nabla \wedge \mathbf{F}$ through the plane, circular surface S_1, where S_1 is shown in Fig. 2 and defined by $x^2 + y^2 \leq a^2, z = 0$. [6]

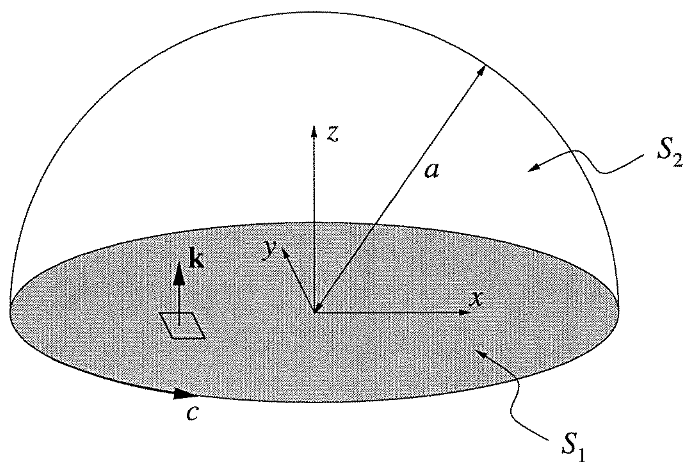

Fig. 2

(b) What is the flux of $\nabla \wedge \mathbf{F}$ through the hemisphere S_2, where S_2 is shown in Fig. 2 and defined by $x^2 + y^2 + z^2 = a^2, z \geq 0$? What is the flux of $\mathbf{F}$ through the same hemisphere? Evaluate the line integral $\oint_c \mathbf{F}.\mathbf{dl}$ along the curve $x^2 + y^2 = a^2$, $z = 0$, in the direction shown in Fig. 2. Note that no detailed calculation is required for (b), although you must justify your answers. [8]

(c) Show that $\nabla.(f(z)\mathbf{F}) = 0$, where $f(z)$ is an arbitrary function of z. What is the flux of $f(z)\mathbf{F}$ through the hemisphere S_2? [6]

3 An infinite flat plate is immersed in a viscous fluid and oscillates in its own plane with velocity $V \cos \omega t$, as shown in Fig. 3. The motion of the fluid is governed by the equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2},$$

where u is the horizontal component of velocity and ν is the kinematic viscosity.

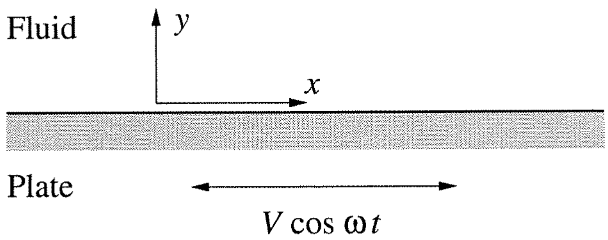

Fig. 3

(a) If the velocity field $u(y,t)$ takes the form

$$u = F(y) \cos \omega t + G(y) \sin \omega t\,,$$

show, by direct substitution, that

$$F = -(\nu/\omega) G'' \quad \text{and} \quad G = (\nu/\omega) F''\,.$$

Derive identical fourth-order ordinary differential equations for F and G. [8]

(b) The general solution for F is

$$F(y) = \exp(y/\delta)\,[A\cos(y/\delta) + B\sin(y/\delta)] + \exp(-y/\delta)\,[C\cos(y/\delta) + D\sin(y/\delta)]\,,$$

where $\delta = (2\nu/\omega)^{1/2}$ and A, B, C and D are constants. Use the no-slip condition at the plate (ie. $u(0,t) = V\cos \omega t$) and other boundary conditions to show that

$$G = k \exp(-y/\delta) \sin(y/\delta)\,,$$

where k is a constant, and hence find the velocity field $u(y,t)$. [12]

SECTION B

*Answer at least **one** question from this section*

4 (a) For an equation of the form $f(x) = 0$, the Newton-Raphson iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(i) Use a Taylor series expansion to derive the Newton-Raphson iteration formula. [5]

(ii) Consider the case $f(x) = x^2 - 4x + 3$. By sketching $f(x)$, find the range of initial vales x_0 for which the iteration converges to the root at $x = 3$. [4]

(iii) Now consider the case $f(x) = x^5 + 10^{-4}x$. Starting from an initial value $x_0 = 0.2$, apply the Newton-Raphson iteration formula n times until $|f(x_n)|$ is less than 10^{-4}. Explain why there remains a significant error in the estimated root x_n. [5]

(b) Using LU decomposition **and no other method**, solve the system of equations

$$\begin{bmatrix} 10 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix}$$

for the unknown quantities x_1, x_2 and x_3. [6]

5 (a) A plane of the form $z = ax + by + c$ is to be fitted to the set of points $(x_1, y_1, z_1) \ldots (x_n, y_n, z_n)$. The method of least squares is used to find a plane which minimises E, the sum of the squared distances in the z direction from each point to the plane:

$$E = \sum_{i=1}^{n} \left[z_i - (ax_i + by_i + c)\right]^2 .$$

(i) Show that the parameters of the plane can be found by solving the equation $\mathrm{A}\mathbf{r} = \mathbf{b}$ for $\mathbf{r}$, where

$$\mathrm{A} = \begin{bmatrix} \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i & \sum y_i & n \end{bmatrix} , \quad \mathbf{b} = \begin{bmatrix} \sum x_i z_i \\ \sum y_i z_i \\ \sum z_i \end{bmatrix} , \quad \mathbf{r} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and all summations are over the range $1 \leq i \leq n$. [6]

(ii) Under what circumstances might the matrix A be singular? [3]

(b) An alternative approach is to find the plane which minimises E', the sum of the squared orthogonal distances from each point to the plane.

(i) Show that E' is given by the following expression:

$$E' = \frac{1}{1 + a^2 + b^2} \sum_{i=1}^{n} \left[z_i - (ax_i + by_i + c)\right]^2 .$$ [6]

(ii) Discuss the relative advantages and disadvantages of minimising E' instead of E. [5]

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SECTION C

*Answer at least **one** question from this section*

6 (a) Show that the Fourier transform of the rectangular pulse $h(t)$ is $H(\omega)$, where

$$h(t) = \begin{cases} 1 & \text{if } |t| \leq T/2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H(\omega) = T\text{sinc}\left(\frac{\omega T}{2}\right) .$$ [3]

(b) Use the formula for the inverse Fourier transform to show that the infinite cosine wave $c(t)$ has Fourier transform $C(\omega)$, where

$$c(t) = \cos \omega_0 t \quad \text{and} \quad C(\omega) = \pi \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] .$$ [3]

(c) A real signal of the form $x(t) = A \cos \omega_1 t + B \cos \omega_2 t$ is observed over the period $-T/2 \leq t \leq T/2$. Spectral analysis is to be used to estimate A, B, ω_1 and ω_2.

(i) Use the convolution theorem to obtain the Fourier transform of the truncated signal $x(t)h(t)$, where $h(t)$ is defined in (a). Sketch the positive half ($\omega \geq 0$) of the transform for the case $\omega_2 \gg \omega_1$. [6]

(ii) What difficulties can you envisage in estimating A, B, ω_1 and ω_2 if $\omega_1 \approx \omega_2$? [1]

(iii) What advantage might be gained by extending the observation period T? [2]

(iv) What advantage might be gained by multiplying $x(t)$ by the triangular pulse $w(t)$ before estimating the spectrum? [5]

$w(t)$ is defined as follows:

$$w(t) = \begin{cases} 1 - 2|t|/T & \text{if } 0 \leq |t| \leq T/2, \\ 0 & \text{otherwise.} \end{cases}$$

7 (a) A bandlimited signal $x(t)$ has spectrum $X(\omega)$, where $X(\omega) = 0$ for $|\omega| \geq \omega_m$. $x(t)$ is sampled by the impulse train $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$. The sampling period T is set to π/ω_m, so that aliasing is just avoided.

(i) Sketch the frequency response of an ideal filter which can be used to recover $x(t)$ from the weighted impulse train $x_s(t) = s(t)x(t)$. [2]

(ii) $x_s(t)$ is passed through a pulse-broadening filter with impulse response

$$g(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq T, \\ 0 & \text{otherwise,} \end{cases}$$

producing the stepped signal $x_g(t) = g(t) * x_s(t)$. By first obtaining the Fourier transform of $g(t)$, derive an expression for the frequency response of an ideal filter which can be used to recover $x(t)$ from $x_g(t)$. [8]

(b) Consider the continuous random variable X with probability density function $f(x)$.

(i) Define the expectation $\mathrm{E}[X]$ in terms of $f(x)$. [2]

(ii) Show that $\mathrm{E}[\lambda X] = \lambda \mathrm{E}[X]$ and $\mathrm{E}[X - a] = \mathrm{E}[X] - a$, where λ and a are constants. [3]

(iii) Show that $\mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$. [5]

8 (a) Explain what is meant by the terms *discrete random variable* and *continuous random variable.* Give one everyday example of each type of random variable. [4]

(b) A weighing machine produces a reading r which does not accurately reflect the true mass m. The error, $m - r$, is Normally distributed with mean 1 g and standard deviation 10 g. The machine is used by a food company to fill nominal 1 kg bags of rice: when the weighing machine reads f, filling stops and the bag is sealed. If the company wishes only 1% of bags to be underweight, what should f be? [5]

(c) I have a choice of the bus or tube to travel to work. A daily return ticket costs £2 by bus and £4 by tube. Since one-way tickets are relatively expensive, I always use the same mode of transport to go to work and come home. Every day, I wait precisely 5 minutes at the bus stop: if a bus has not arrived after 5 minutes, I resort to the tube. On average, there are 12 buses every hour.

(i) Assuming that the number of buses arriving in any 5 minute interval is well modelled by the Poisson distribution, show that the probability I take the bus on any particular day is approximately 0.632. [2]

(ii) Comment on the validity of the Poisson assumption in (i). [4]

(iii) Find the probability that I use the bus for precisely 15 return journeys in a 20 working-day month. State any assumptions you make. [3]

(iv) Calculate my expected monthly travel costs. [2]

END OF PAPER