

## ENGINEERING TRIPOS Part IB 2001

## Paper 1 – MECHANICS

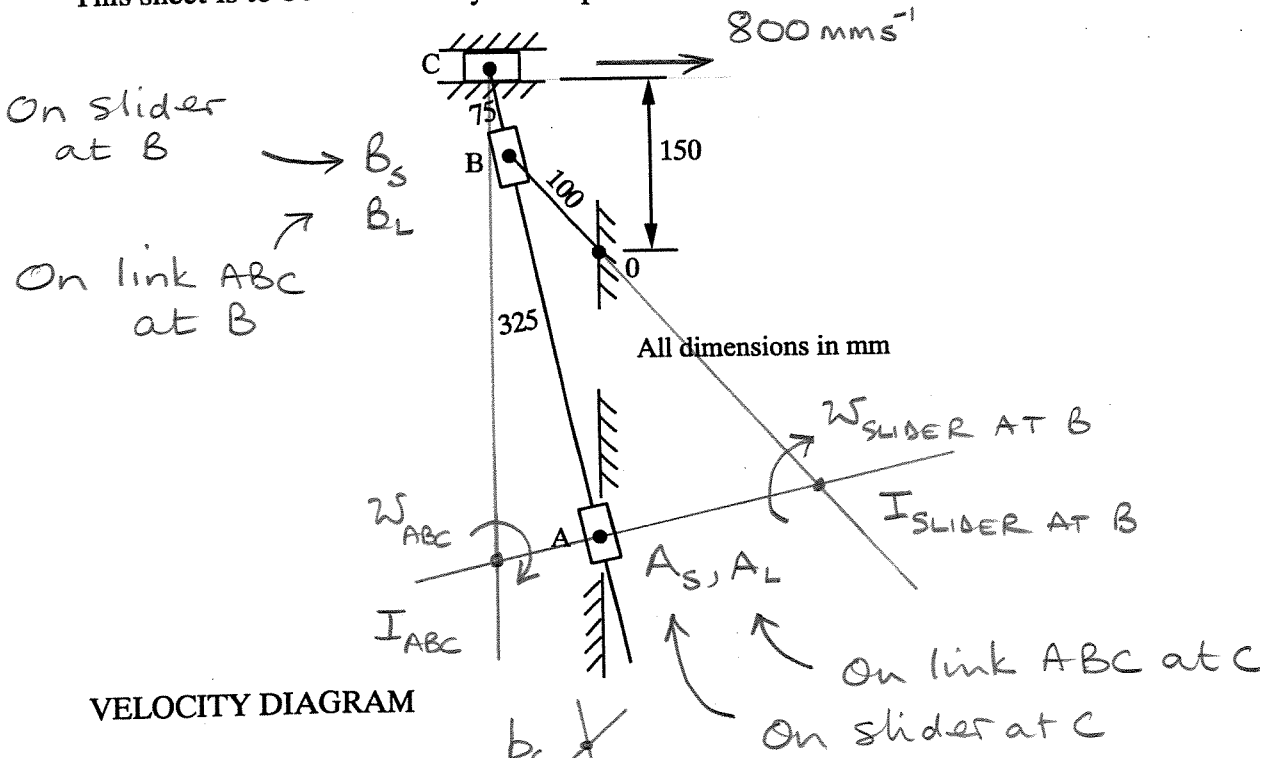
- Q1 (b) Relative sliding velocity at A =  $190 \text{ mms}^{-1}$   
 Relative sliding velocity at B =  $550 \text{ mms}^{-1}$   
 Angular velocity of link ABC =  $2.0 \text{ rads}^{-1}$ , clockwise  
 Angular velocity of link OB =  $7.4 \text{ rads}^{-1}$ , clockwise
- (c) Relative sliding acceleration at B =  $2700 \text{ mms}^{-2}$   
 Angular acceleration of link ABC =  $11.75 \text{ rads}^{-2}$ , clockwise  
 Angular acceleration of link OB =  $28.0 \text{ rads}^{-2}$ , clockwise

- Q2 (b)  $\mathbf{v}_B = \dot{\theta}(R + a \sin \phi) \mathbf{e}_\theta$   
 $\mathbf{a}_B = \ddot{\theta}(R + a \sin \phi) \mathbf{e}_\theta - \dot{\theta}^2(R + a \sin \phi) \mathbf{e}_r$
- (c)  $\omega_{DISC} = \dot{\phi} \mathbf{e}_\theta + \dot{\theta} \mathbf{k}$   
 $\dot{\omega}_{DISC} = -\dot{\phi}\dot{\theta} \mathbf{e}_r + \ddot{\phi} \mathbf{e}_\theta + \ddot{\theta} \mathbf{k}$
- (d)  $\mathbf{v}_B = \dot{\phi}a \cos \phi \mathbf{e}_r + \dot{\theta}(R + a \sin \phi) \mathbf{e}_\theta - \dot{\phi}a \sin \phi \mathbf{k}$   
 $\mathbf{a}_B = [-\dot{\theta}^2(R + a \sin \phi) + \ddot{\phi}a \cos \phi - \dot{\phi}^2 a \sin \phi] \mathbf{e}_r$   
 $+ [\ddot{\theta}(R + a \sin \phi) + 2\dot{\theta}\dot{\phi}a \cos \phi] \mathbf{e}_\theta$   
 $- [\ddot{\phi}a \sin \phi + \dot{\phi}^2 a \cos \phi] \mathbf{k}$

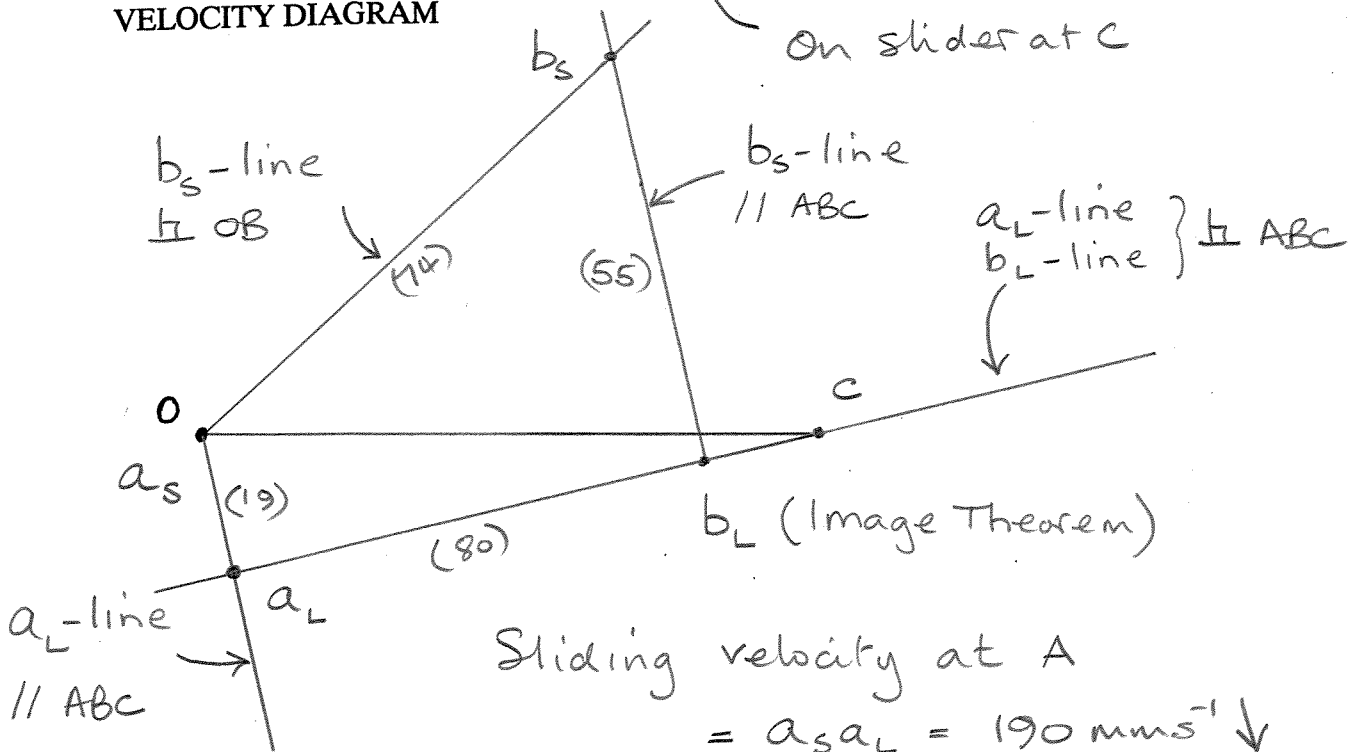
- Q3 (a)  $\omega_{BC} = \frac{\omega}{2}$ , anticlockwise  
 $\omega_{DC} = \omega$ , anticlockwise  
 $\mathbf{v}_E = \frac{l\omega}{2} \mathbf{i} + \frac{l\omega}{2} \mathbf{j}$   
 $\mathbf{a}_E = -\frac{l\omega^2}{4} \mathbf{i} + \frac{3l\omega^2}{4} \mathbf{j}$
- (b)  $T = \frac{mgl}{2} + \frac{ml^2\omega^2}{4}$
- (c)  $T_i = \frac{mgl}{2} + \frac{5ml^2\omega^2}{12}$
- (d)  $T_{ai} = 2Q$

- Q4 (b)  $\beta = \frac{3g}{5l}$ ; tension,  $T = \frac{\sqrt{2}}{5}mg$
- (c)  $S_B = \frac{-mg}{5}$ ;  $S_c = M_B = M_c = 0$
- (d)  $F = \frac{mg}{10l}(x - \frac{3x^2}{2l} + \frac{l}{2})$ ;  $M = \frac{mg}{20l}(-l^2 + lx + x^2 - \frac{x^3}{l})$
- Q5 (b)  $\omega_f = 171$  rpm; Energy lost = 0.9J.
- (c) No.
- (d)  $\omega_D = 343$  rpm
- Q6 (b) Relative to pulley A, B at  $90^\circ$  clockwise and C at  $233^\circ$  clockwise
- (c)  $F_Q = F_P = 76$ N
- (d) Radius = 0.23m; relative to pulley A, mass at C added at  $38^\circ$  clockwise and mass at A added at  $218^\circ$  clockwise

This sheet is to be attached to your script.



VELOCITY DIAGRAM



Sliding velocity at B =  $b_s b_s = \underline{\underline{550 \text{ mm/s}}}$  ↓

$\omega_{ABC} = \frac{a_c}{AC} = \frac{800}{400} = \underline{\underline{2.0 \text{ rads}^{-1}}}$

$\omega_{OB} = \frac{ob_s}{OB} = \frac{740}{100} = \underline{\underline{7.4 \text{ rads}^{-1}}}$

Coriolis at B:  
Slider relative to link

$$2\dot{r}\dot{\theta} = 2 \times 550 \times 2 = \underline{2200 \text{ mm s}^{-2}}$$

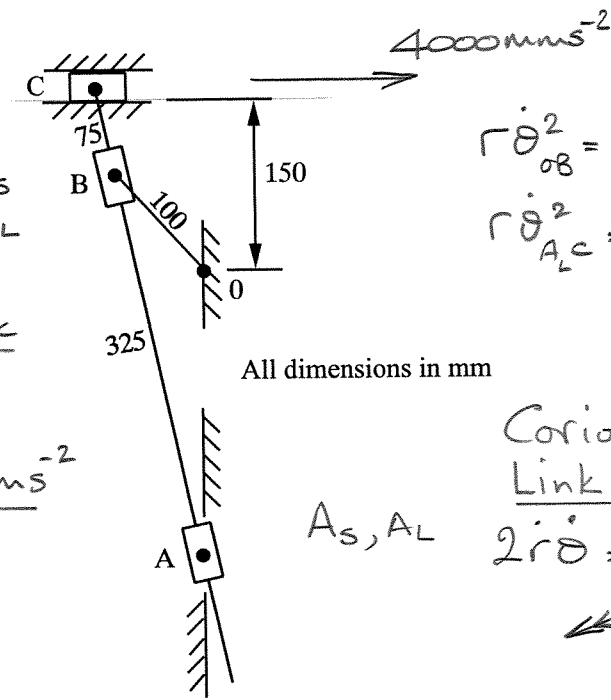


$$r\dot{\theta}^2 = 100 \times 7.4^2 = \underline{5500 \text{ mm s}^{-2}}$$

$$r\dot{\theta}^2 = 400 \times 2^2 = \underline{1600 \text{ mm s}^{-2}}$$

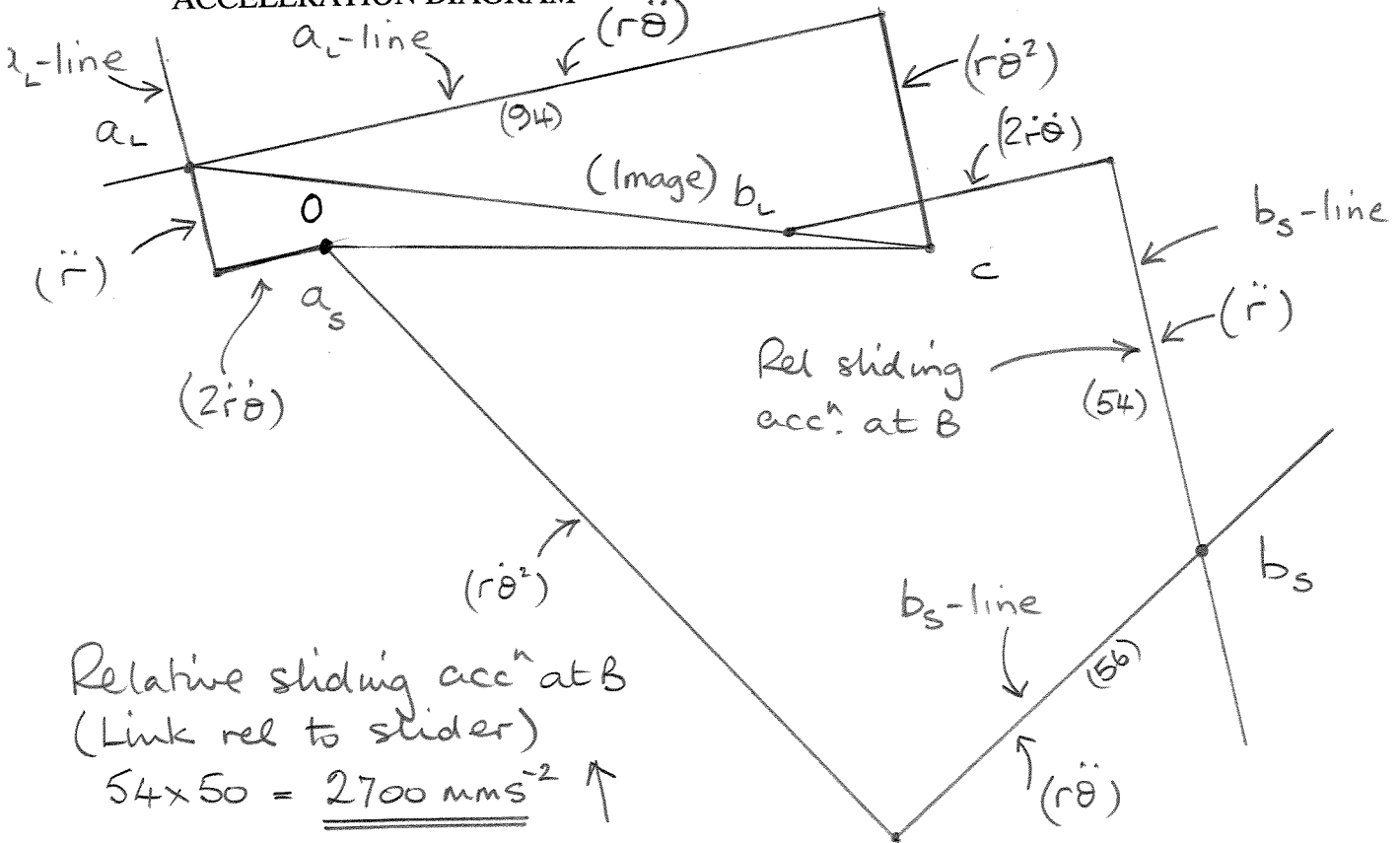
Coriolis at A:  
Link relative to slider

$$2\dot{r}\dot{\theta} = 2 \times 190 \times 2 = \underline{760 \text{ mm s}^{-2}}$$



All dimensions in mm

ACCELERATION DIAGRAM



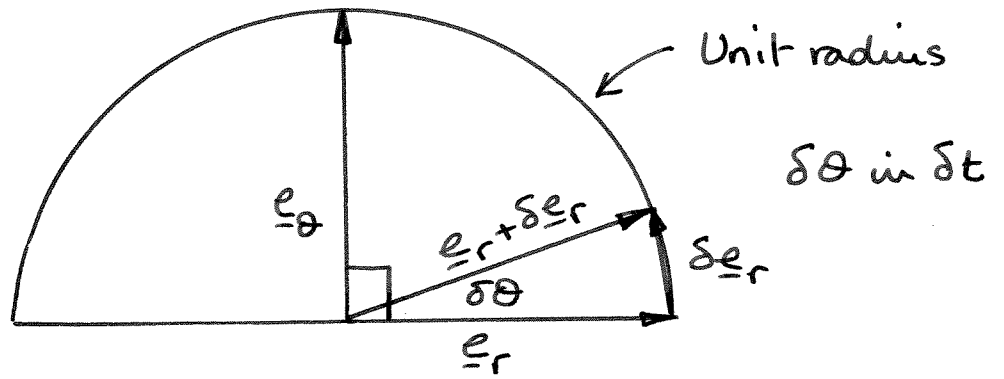
Rel sliding acc<sup>n</sup> at B

Relative sliding acc<sup>n</sup> at B  
(Link rel to slider)

$$54 \times 50 = \underline{2700 \text{ mm s}^{-2}}$$

$$\dot{\omega}_{ABC} = \frac{94 \times 50}{400} = \underline{11.75 \text{ rad s}^{-2}}$$

$$\dot{\omega}_{OB} = \frac{56 \times 50}{100} = \underline{28 \text{ rad s}^{-2}}$$

2  
(a)

$$\frac{d\underline{e}_r}{dt} = \lim_{\delta t \rightarrow 0} \frac{\delta \underline{e}_r}{\delta t}$$

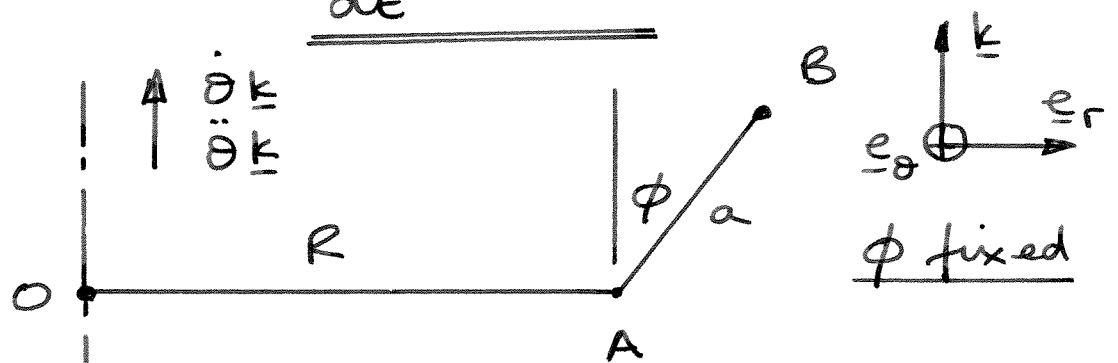
For small  $\delta\theta$ ,

$$\left. \begin{array}{l} \text{Magnitude of } \delta \underline{e}_r \rightarrow 1 \times \delta\theta \\ \text{Direction of } \delta \underline{e}_r \rightarrow \underline{e}_\theta \end{array} \right\} \delta \underline{e}_r = \delta\theta \underline{e}_\theta$$

$$\therefore \lim_{\delta t \rightarrow 0} \frac{\delta \underline{e}_r}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{\delta\theta}{\delta t} \underline{e}_\theta$$

$$\therefore \frac{d\underline{e}_r}{dt} = \dot{\theta} \underline{e}_\theta$$

(b)



By inspection:

$$\underline{v}_B = \dot{\theta} (R + a \sin \phi) \underline{e}_\theta$$

$$\underline{a}_B = \ddot{\theta} (R + a \sin \phi) \underline{e}_\theta - \dot{\theta}^2 (R + a \sin \phi) \underline{e}_r$$

$$(c) \quad \underline{\omega}_{Disc} = \underline{\dot{\theta} k + \dot{\phi} e_{\theta}}$$

$$\underline{\dot{\omega}}_{Disc} = (\ddot{\theta} \underline{k} + \dot{\theta} \underline{\dot{k}}) + (\ddot{\phi} \underline{e}_{\theta} + \dot{\phi} \underline{\dot{e}}_{\theta})$$

$\dot{\theta} \underline{\dot{k}} = 0$

$$\underline{\dot{\omega}}_{Disc} = \underline{-\dot{\phi} \dot{\theta} \underline{e}_r + \ddot{\phi} \underline{e}_{\theta} + \ddot{\theta} \underline{k}}$$

↑ Note this term

(d) One of three approaches can be adopted to solve this part of the question:

- (1) Start with the position vector  $\underline{OB}$  and differentiate twice
- (2) Use the Data Book expressions making the arm the rotating body
- (3) Use the Data Book expressions making the disc the rotating body.

Adopting (2), along with the 'coincident points' method, is the approach used below.

Data Book expressions:

$$\underline{v}_P = \underline{v}_Q + \underline{\omega} \times \underline{r} + \left[ \frac{d\underline{r}}{dt} \right]_R$$

$$\underline{a}_P = \underline{a}_Q + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + 2\underline{\omega} \times \left[ \frac{d\underline{r}}{dt} \right]_R + \left[ \frac{d^2 \underline{r}}{dt^2} \right]_R$$

Let the arm be the rotating body, so  $\underline{\omega} = \dot{\theta} \underline{k}$  and  $\frac{d\underline{\omega}}{dt} = \ddot{\theta} \underline{k}$

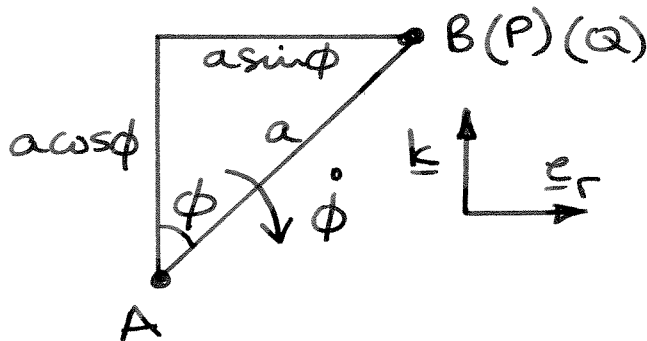
P will be point B on the disc

Place Q coincident with B (i.e. P), but in the rotating body (i.e. arm extended)

In this case  $\underline{r} = 0$ , and

$$\underline{v}_P = \underline{v}_Q + \left[ \frac{d\underline{r}}{dt} \right]_R$$

$$\underline{v}_Q = \dot{\theta} (R + a \sin \phi) \underline{e}_\theta \quad \leftarrow$$



$$\left[ \frac{d\underline{r}}{dt} \right]_R = \dot{\phi} a \cos \phi \underline{e}_r - \dot{\phi} a \sin \phi \underline{k} \quad \leftarrow$$

$$\underline{v}_P = \dot{\phi} a \cos \phi \underline{e}_r + \dot{\theta} (R + a \sin \phi) \underline{e}_\theta - \dot{\phi} a \sin \phi \underline{k}$$

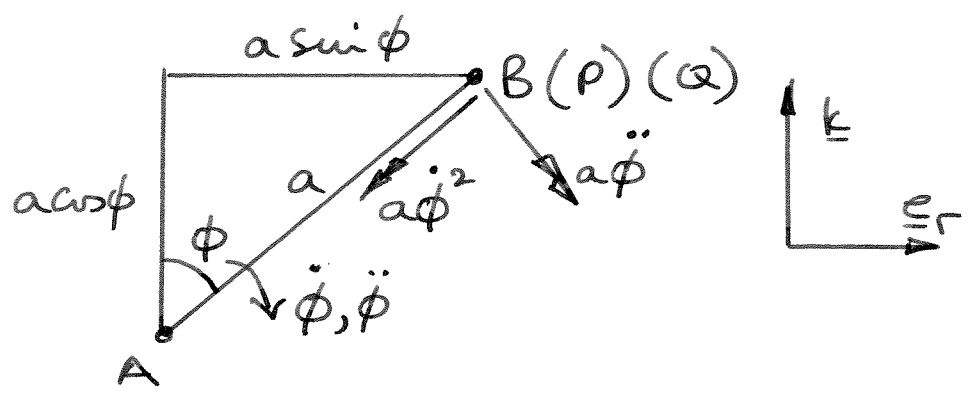
i.e. b

Since  $\underline{r} = 0$

$$\underline{a}_P = \underline{a}_Q + 2\underline{\omega} \times \left[ \frac{d\underline{r}}{dt} \right]_R + \left[ \frac{d^2\underline{r}}{dt^2} \right]_R$$

$$\underline{a}_Q = -\dot{\theta}^2 (R + a \sin \phi) \underline{e}_r + \ddot{\theta} (R + a \sin \phi) \underline{e}_\theta \quad \leftarrow$$

$$\begin{aligned} 2\underline{\omega} \times \left[ \frac{d\underline{r}}{dt} \right]_R &= 2\dot{\theta} \underline{k} \times (\dot{\phi} a \cos \phi \underline{e}_r - \dot{\phi} a \sin \phi \underline{k}) \\ &= 2\dot{\theta} \dot{\phi} a \cos \phi \underline{e}_\theta \quad \leftarrow \end{aligned}$$



$$\left[ \frac{d^2 \underline{r}}{dt^2} \right]_R = \left. \begin{aligned} & \ddot{\phi} a \omega \phi \underline{e}_r - \ddot{\phi} a \sin \phi \underline{k} \\ & - \dot{\phi}^2 a \sin \phi \underline{e}_r - \dot{\phi}^2 a \omega \phi \underline{k} \end{aligned} \right\} \leftarrow$$

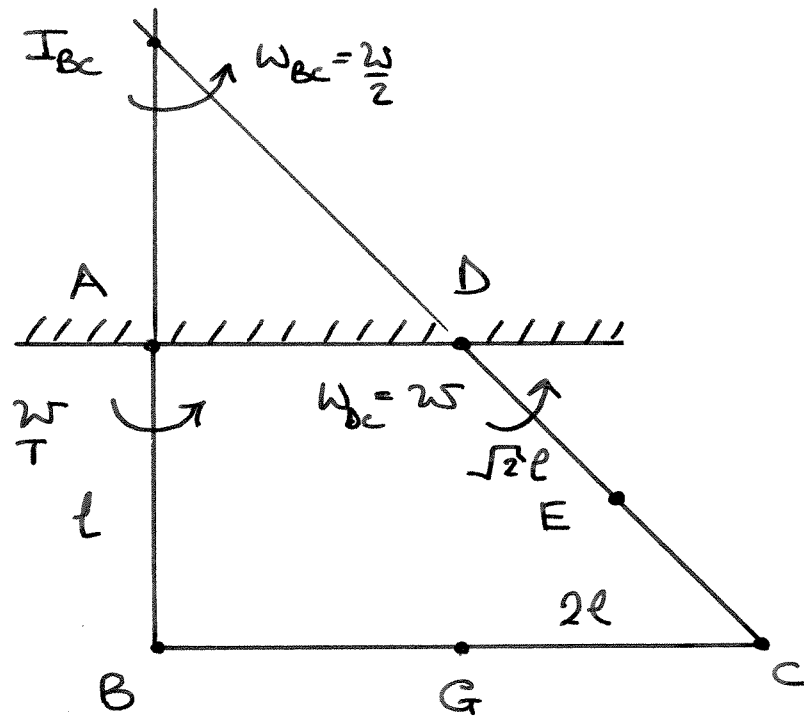
So, in components,  $\underline{a}_P =$   
↑ i.e. b

$\underline{e}_r$	$\underline{e}_\theta$	$\underline{k}$
$-\ddot{\theta}^2 (R + a \sin \phi)$	$\ddot{\theta} (R + a \sin \phi)$	
	$2 \dot{\theta} \dot{\phi} a \omega \phi$	
$(\ddot{\phi} a \omega \phi - \dot{\phi}^2 a \sin \phi)$		$-(\ddot{\phi} a \sin \phi + \dot{\phi}^2 a \omega \phi)$



3

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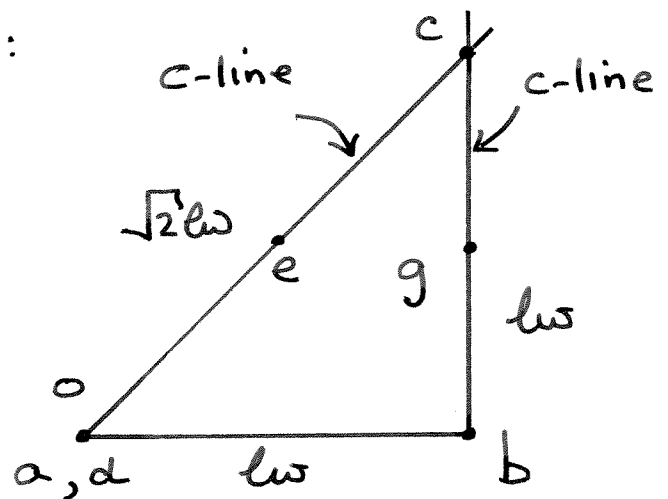
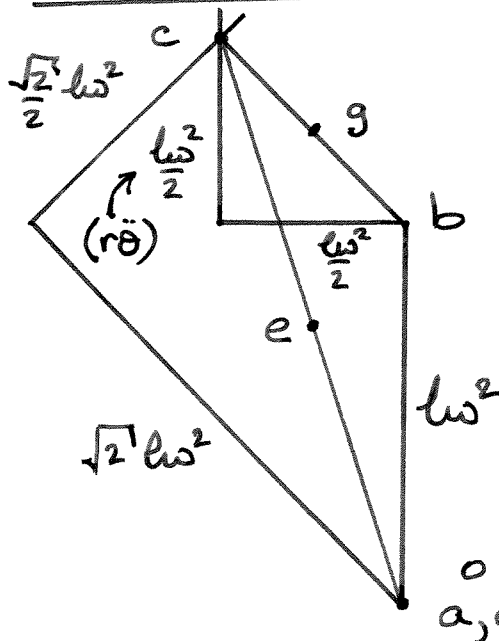
(a)

Velocity diagram:

$$\omega_{Bc} = \frac{bc}{Bc} = \frac{l\omega}{2l} = \frac{\omega}{2}$$

$$\omega_{Bc} = \frac{dc}{Bc} = \frac{\sqrt{2}l\omega}{\sqrt{2}l} = \omega$$

$$\underline{v_E = \frac{l\omega}{2} \rightarrow ; \frac{l\omega}{2} \uparrow}$$

Acceleration diagram:

$$(r\ddot{\theta})_{Bc} = 2l\left(\frac{\omega}{2}\right)^2 = \frac{l\omega^2}{2}$$

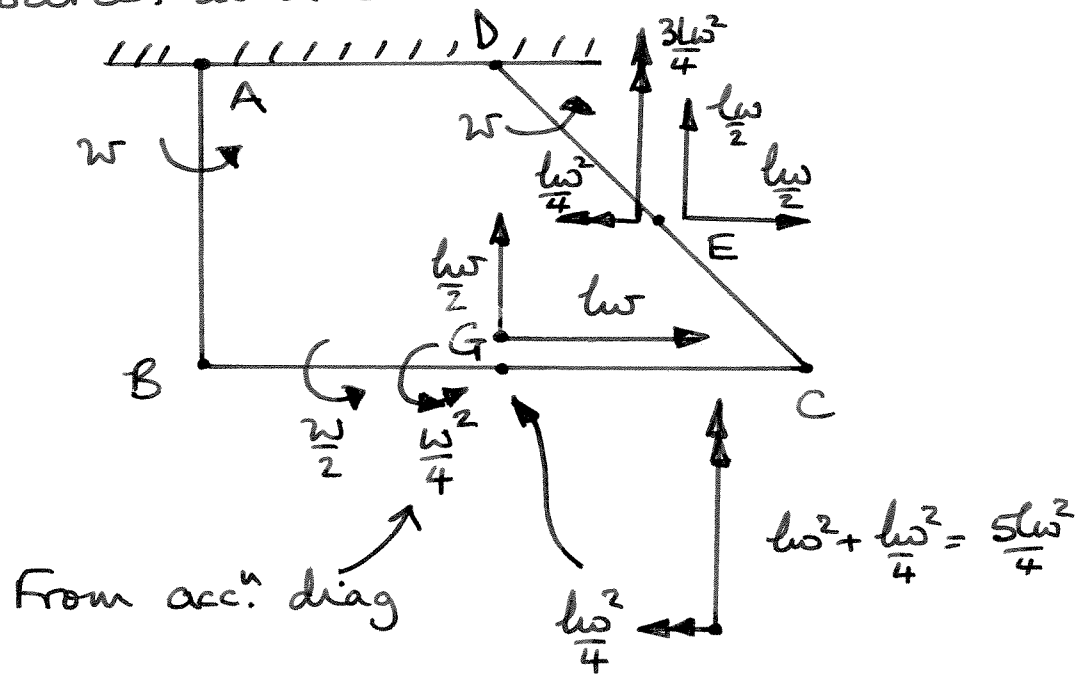
$$(r\ddot{\theta})_{c0} = \sqrt{2}l\omega^2$$

$$a_E = \frac{l\omega^2}{4} \leftarrow ; \frac{l\omega^2}{2} \left(1 + \frac{1}{2}\right) \uparrow$$

$$a_E = \frac{l\omega^2}{4} \leftarrow ; \frac{3}{4}l\omega^2 \uparrow$$

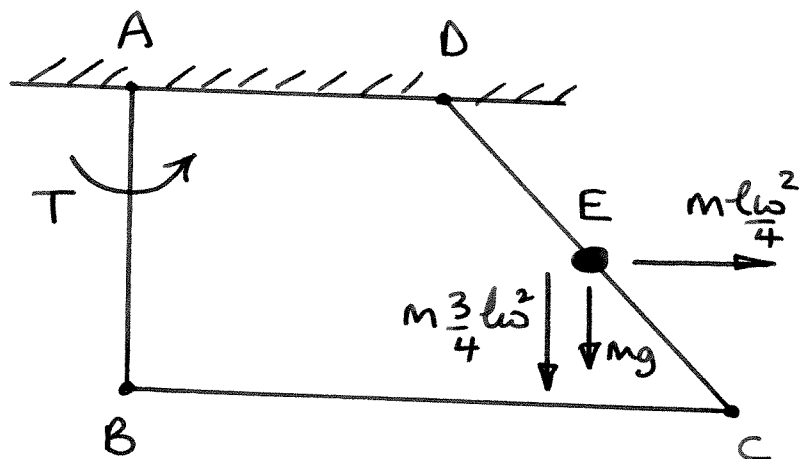
$$\underline{\underline{\dot{\omega}_{Bc} = \frac{l\omega^2}{2e} = \frac{\omega^2}{4} \curvearrowright}}$$

Velocities and accelerations :



(b)

Forces :



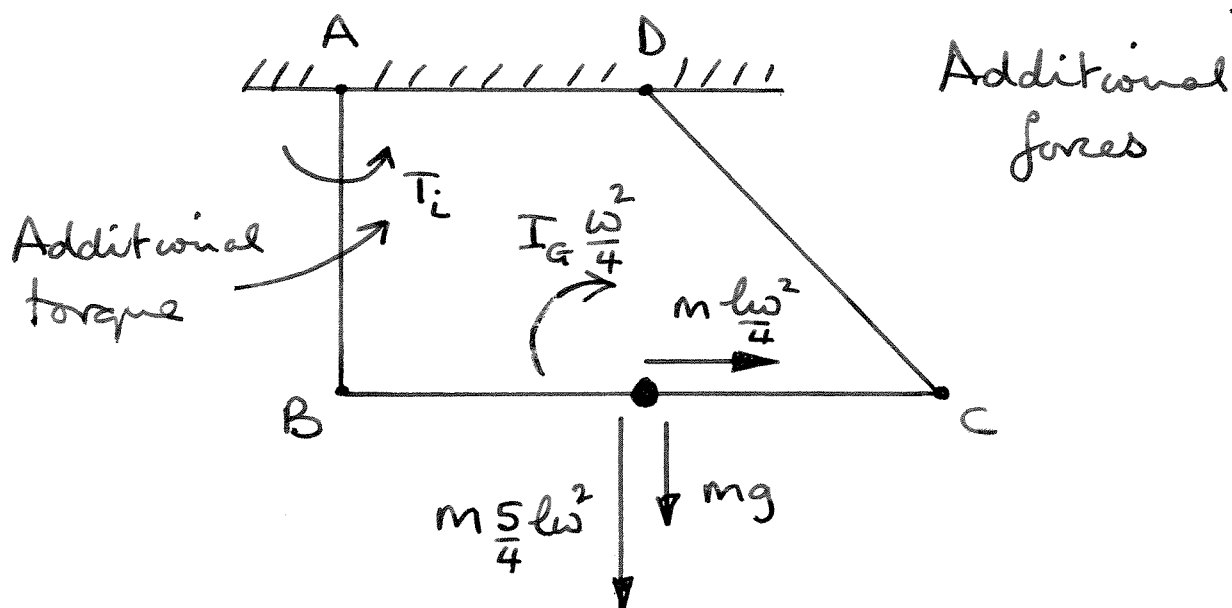
$$\sum \text{Instantaneous powers} = 0$$

$$T \dot{\theta} - (mg + m \frac{3}{4} l \omega^2) \frac{l \dot{\theta}}{2} + (m \frac{l \omega^2}{4}) \frac{l \dot{\theta}}{2} = 0$$

$$T = \frac{1}{2} mgl + \frac{3}{8} ml^2 \omega^2 - \frac{1}{8} ml^2 \omega^2$$

$$\underline{\underline{T = \frac{1}{2} mgl + \frac{1}{4} ml^2 \omega^2}}$$

(c)



$$I_G = \frac{1}{12} m (2l)^2 = \frac{ml^2}{3}$$

$$\sum \text{instantaneous powers} = 0$$

$$T_i \omega - mg \left( \frac{l\omega}{2} \right) + m \frac{l\omega^2}{4} (l\omega) - m \frac{5}{4} l\omega^2 \left( \frac{l\omega}{2} \right) - \frac{I_G}{4} \frac{\omega^2}{2} \left( \frac{\omega}{2} \right) = 0$$

$$T_i = \frac{1}{2} mgl + ml^2 \omega^2 \left( \frac{1}{24} + \frac{15}{24} - \frac{6}{24} \right)$$

$$\underline{\underline{T_i = \frac{1}{2} mgl + \frac{5}{12} ml^2 \omega^2}}$$

(d)

The angular velocities of all the bars are in the same sense, i.e. anticlockwise

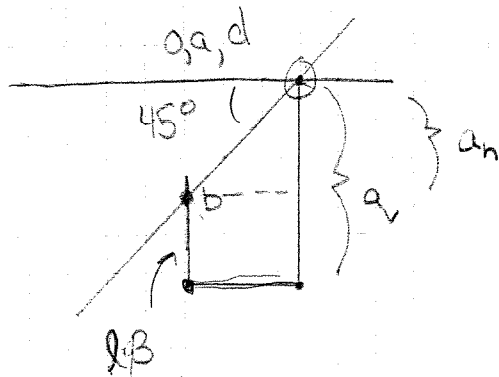
So "Subtract"

Additional input power = Friction power

$$T_{ai} \omega = \underbrace{Q \left( \omega - \frac{\omega}{2} \right)}_B + \underbrace{Q \left( \omega - \frac{\omega}{2} \right)}_C + \underbrace{Q \omega}_D$$

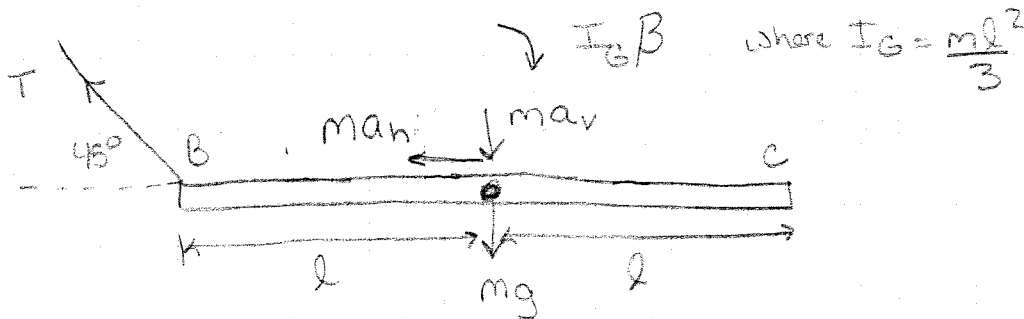
$$\underline{\underline{T_{ai} = 2Q}}$$

4. (a) Acceleration Diagram:



$$\therefore a = a_h + l\beta \quad (1)$$

(b)



$$+\uparrow T \sin 45 - ma_v - mg = 0 \quad (2)$$

$$+\leftarrow T \cos 45 + ma_h = 0 \quad (3)$$

$$\oplus_G T \cos 45 l + \left(\frac{ml^2}{3}\right) \beta = 0 \quad (4)$$

$$(3) \Rightarrow -T \cos 45 = ma_h$$

$$\text{from (2)} \quad -ma_h - ma_v = mg$$

$$a_h + a_v = -g \quad (5)$$

$$\text{from (4)} \quad -l a_h = -\frac{ml^2}{3} \beta$$

$$\therefore \beta = \frac{3a_h}{l} \quad (6)$$

$$\text{from (1)} \quad a_v = a_h + l\beta$$

$$(1), (5) \& (6) \Rightarrow a_v = (-g - a_v) + \frac{l \cdot 3}{l} (-g - a_v) = 4(-g - a_v)$$

$$\therefore a_v = -4g/5$$

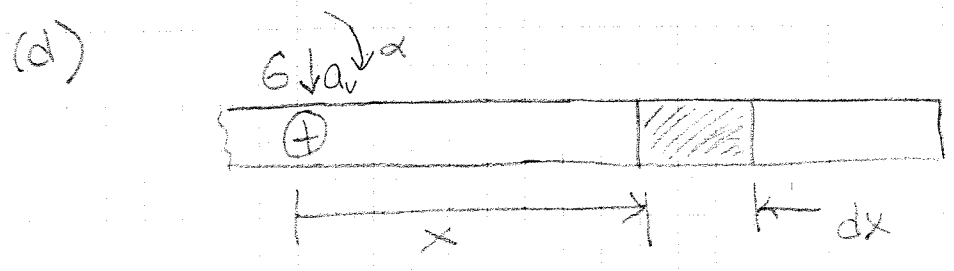
so,  $a_h = -g - a_v = -g + \frac{4g}{5} = -\frac{g}{5}$

from (3)  $T = \frac{-ma_h}{\cos 45} = \frac{mg}{5\cos 45}$  from (4)  $\beta = \frac{3h}{l}$

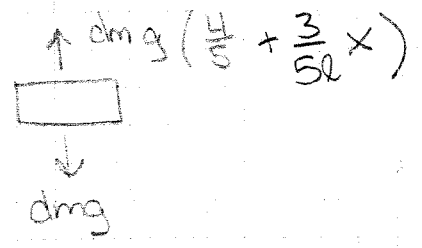
$\therefore T = \frac{\sqrt{2}mg}{5}$  &  $\beta = \frac{3g}{5l}$

(c)  $S_B = \frac{-T}{\sqrt{2}} = \frac{-mg}{5}$

$S_C = M_B = M_C = 0$



$a = a_v + x\beta = \frac{4}{5}g + \frac{3g}{5l}x$



where  $dm = \frac{m}{2l}dx$

$w = dm g (1 - \frac{4}{5} - \frac{3x}{5l}) \frac{1}{dx}$

$w = \frac{mg}{10l} (1 - \frac{3x}{2l})$

integrating

$\rightarrow F = \frac{mg}{10l} (x - \frac{3x^2}{2l} + A)$

$F=0$  at  $x=l \Rightarrow$

$0 = \frac{mg}{10l} (l - \frac{3l}{2} + A) \therefore A = \frac{l}{2}$

$\therefore F = \frac{mg}{10l} (x - \frac{3x^2}{2l} + \frac{l}{2})$  check  $F(x=l) = -\frac{mg}{5}$  ✓

integrate again

$$M = \frac{mg}{10l} \left( \frac{l^2 x}{2} + \frac{x^2}{2} - \frac{x^3}{2l} + B \right)$$

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but  $M=0$  when  $x=l$

$$0 = \frac{l^2}{2} + \frac{l^2}{2} - \frac{l^2}{2} + B \quad \therefore B = -\frac{l^2}{2}$$

$$\therefore M = \frac{mg}{20l} \left( -l^2 + lx + x^2 - \frac{x^3}{l} \right) \quad \text{check } M(x=l) = 0 \checkmark$$

-or-

$$M = -\frac{mgl}{20} + \frac{mgx}{20} + \frac{mgx^2}{20l} - \frac{mgx^3}{20l^2}$$

5. • linear impulse - integrating a force,  $F$ , over a period of time gives an impulse which using  $F = ma$  equates an impulse to a change in momentum.
- conservation of the moment of momentum - The moment of momentum about a point  $P$  is conserved if the moments of all external forces about  $P$  sum to zero.

(b) Polar moment of inertia of a disc =  $\frac{m r^2}{2}$

For discs A & B, the moment of momentum (angular momentum) is conserved about the axes of their shafts as the clutch C is engaged.

Before

$$J_A \omega_A = (J_A + J_B) \omega_F$$

$$J_A = \frac{0.5 \cdot 30^2 \cdot 10^{-6}}{2} \text{ kgm}^2 = 225 \cdot 10^{-6} \text{ kgm}^2$$

$$J_B = \frac{0.25 \cdot 20^2 \cdot 10^{-6}}{2} \text{ kgm}^2 = 50 \cdot 10^{-6} \text{ kgm}^2$$

$$\omega_F = \frac{225 \cdot 10^{-6} \text{ kgm}^2 \cdot 2000 \text{ rpm}}{(225 \cdot 10^{-6} + 50 \cdot 10^{-6}) \text{ kgm}^2}$$

$$\therefore \omega_F = \underline{1636 \text{ rpm}} \cdot \left(\frac{2\pi}{60}\right) = 171 \text{ rad} \cdot \text{s}^{-1}$$

$$\text{Kinetic energy} = \frac{1}{2} J \omega^2$$

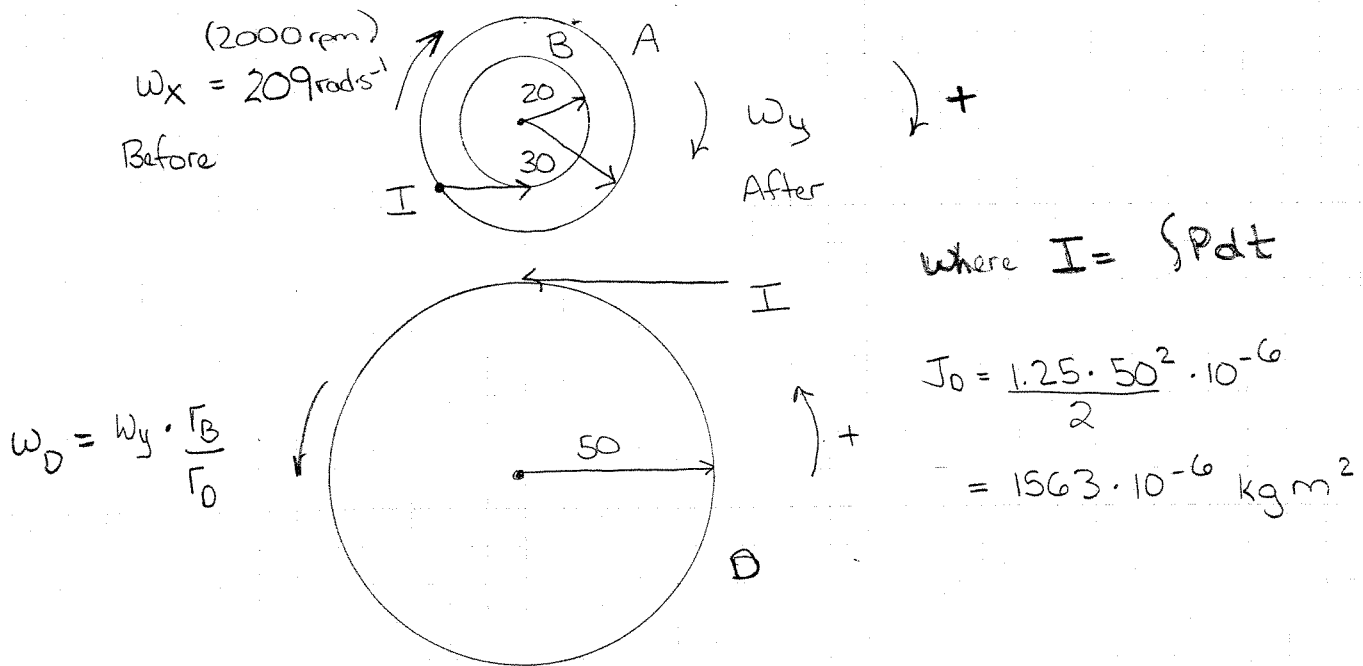
$$(KE)_{\text{BEFORE}} = \frac{1}{2} J_A \omega_A^2 = \frac{1}{2} \cdot 225 \cdot 10^{-6} \text{ kgm}^2 \cdot \overbrace{(2000 \text{ rpm} \cdot \frac{2\pi}{60})^2}^{209 \text{ rad} \cdot \text{s}^{-1}} = 4.9 \text{ J}$$

$$(KE)_{\text{AFTER}} = \frac{1}{2} (J_A + J_B) \omega_F^2 = \frac{1}{2} (225 \cdot 10^{-6} + 50 \cdot 10^{-6}) (171)^2 = 4.0 \text{ J}$$

$$\therefore \text{Energy lost} = 0.9 \text{ J}$$

(c) The speed of engagement makes no difference. The solution depends only on the angular momentum before being equal to the angular momentum after.

(d) There is no axis about which moment of momentum is conserved. At the point of contact between Band D the impulses are equal and opposite.



For A and B:  $[I_A r_A = J_A (\omega_x - \omega_y)]$  &  $[I_A r_A - I_B r_B = J_0 \omega_y]$

$-I_B r_B = (J_A + J_B) \omega_y - J_A \omega_x$

After Before

For D:

$I r_0 = J_0 \omega_y \cdot \frac{r_B}{r_0} - J_0 \cdot 0$

After Before

Inserting the numbers and eliminating I (to solve for  $\omega_y$ ):

$$\frac{J_A \omega_x}{r_B} - (J_A + J_B) \frac{\omega_y}{r_B} = J_0 \omega_y \cdot \frac{r_B}{r_0^2}$$

$$\frac{J_A \omega_x}{r_B} = \omega_y \left( \frac{J_A + J_B}{r_B} + \frac{J_0 r_B}{r_0^2} \right)$$



$$\frac{225 \cdot 2000}{20} = \omega_y \left( \frac{275}{20} + \frac{1563 \cdot 20}{50^2} \right)$$

$$22500 = \omega_y (13.75 + 12.5)$$

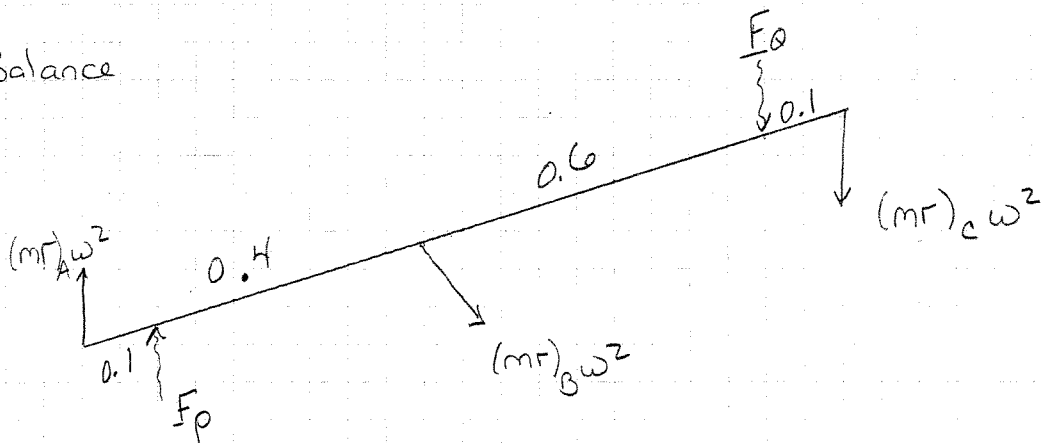
$$\therefore \omega_y = 857 \text{ rpm } (\sim 90 \text{ rad/s}^{-1})$$

$$\therefore \underline{\omega_0} = 857 \cdot \frac{20}{50} = \underline{343 \text{ rpm}} (\sim 36 \text{ rad/s}^{-1})$$

6. (a) statically balanced - the centre of mass of the rotating mass lies along the centreline so that the shaft will sit in equilibrium at any angular position

dynamically balanced - for constant rotation, the bearing forces are each equal to zero. This eliminates undesirable bearing reactions.

(b) Static Balance



$$(mr)_A = 0.018 \text{ kgm} \quad \times 500 \quad (9)$$

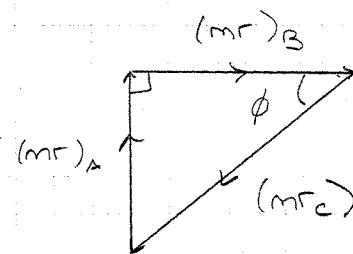
$$(mr)_B = 0.024 \text{ kgm} \quad (12)$$

$$(mr)_C = 0.030 \text{ kgm} \quad (15)$$

$$9 : 12 : 15$$

Drawn to scale 500:1

$$F_p + F_0 = 0$$



$$3:4:5 \Delta \Rightarrow \tan \phi = 3/4 \Rightarrow \phi = 36.9^\circ$$

$\therefore$  relative to pulley A, out of balance at B must be  $90^\circ$  clockwise  
out of balance at C  $(270 - \phi) = 233^\circ$  clockwise

(c) Dynamic Balance

Take moments about one bearing, say P. Since all moment arms lie along the shaft axis, moment vectors are all  $90^\circ$  from corresponding force vectors.  $\omega = 52 \text{ rad}\cdot\text{s}^{-1}$

Moments at P:

mass at A	$0.018 \cdot 0.1 \cdot \omega^2 = 0.0018 \omega^2 \text{ Nm}$	(4.867)	$\times 5000$
		(25.96)	(9)
mass at B	$0.024 \cdot 0.4 \cdot \omega^2 = 0.0096 \omega^2 \text{ Nm}$	(25.96)	(48)
		(89.23)	
mass at C	$0.030 \cdot 1.1 \cdot \omega^2 = 0.033 \omega^2 \text{ Nm}$	(89.23)	(165)

$$|PQ \times F_Q| \rightarrow |x F_Q$$

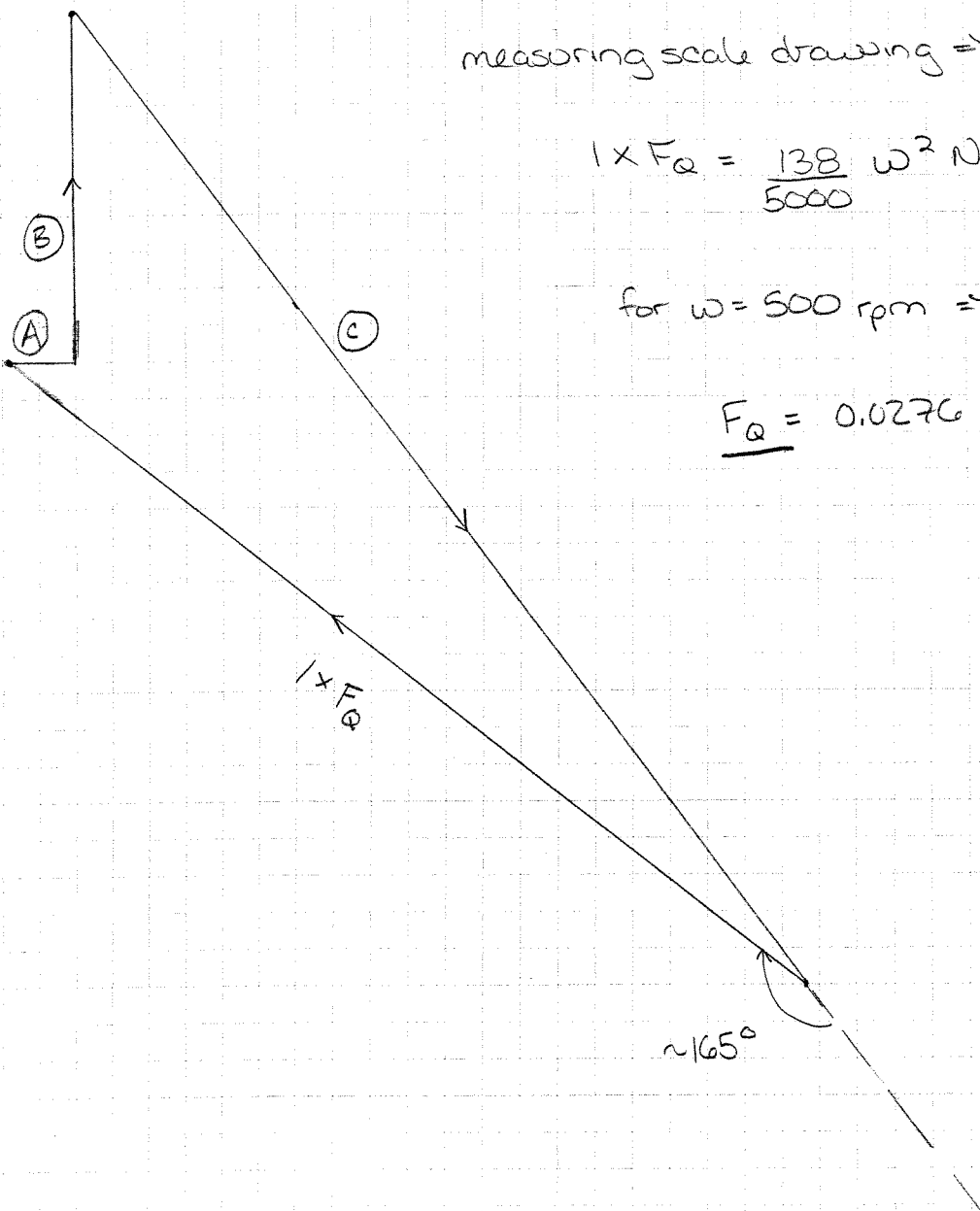
Drawn to scale 5000:1

measuring scale drawing  $\Rightarrow$

$$x F_Q = \frac{138}{5000} \omega^2 \text{ Nm} = 0.0276 \omega^2 \text{ Nm}$$

for  $\omega = 500 \text{ rpm} \Rightarrow$

$$F_Q = 0.0276 \left( \frac{500 \cdot 2\pi}{60} \right)^2 = \underline{76 \text{ N}}$$



To bring the shaft into dynamic balance while mounting static balance two masses must be added of equal magnitude, equal radius and  $180^\circ$  out of phase.

given  $m = 0.1 \text{ kg}$  (equal masses)

$$(0.1 \cdot r \cdot 1.1) \omega^2 + (0.1 \cdot r \cdot 0.1) \omega^2 = 0.0276 \omega^2$$

$$0.12 r = 0.0276$$

$$\therefore \underline{r = 0.23 \text{ m}}$$

Positions:

From sketch  $\angle$  between mass at C and  $F_{\text{eq}}$  is  $165^\circ$  clockwise.

So,  $233^\circ$  ( $\angle$  from A  $\rightarrow$  C) +  $165^\circ$  (C  $\rightarrow$  out of balance) =  $398^\circ = 38^\circ$  relative to position A.

$\therefore$  mass at C is added at position  $38^\circ$  rel. to A clockwise

mass at A is added at position  $38^\circ + 180^\circ = \underline{218^\circ}$

rel. to A clockwise