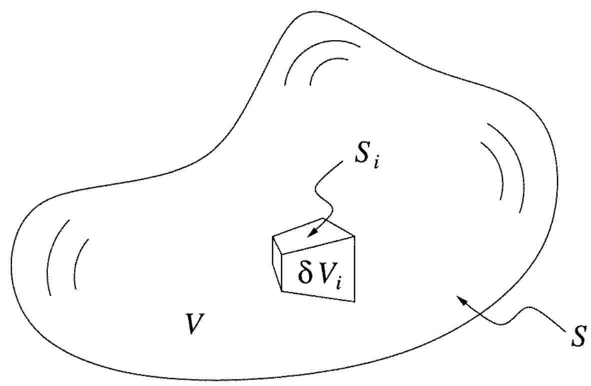
Paper 7: Mathematical Methods

Solutions to 2001 Tripos Paper

1. *Gauss's theorem and application to electric fields.*

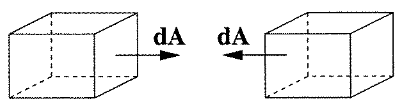
(a) Consider an arbitrary volume V divided into many smaller volumes δV_i.



From the definition of divergence, in the limit of infinitesimal δV_i, we have

$$(\nabla.\mathbf{u})\,\delta V_i = \oint_{S_i} \mathbf{u.dA} \Rightarrow \int_V \nabla.\mathbf{u}\,dV = \sum_{i=1}^{\infty} \oint_{S_i} \mathbf{u.dA}$$

Considering the sum of surface integrals on the right hand side, it is clear that contributions from adjacent "internal" surfaces cancel, as illustrated below.



Only contributions to $\sum_{i=1}^{\infty} \oint_{S_i} \mathbf{u.dA}$ from "external" surface elements remain uncancelled, so

$$\int_V \nabla.\mathbf{u}\,dV = \oint_S \mathbf{u.dA}$$ [7]

(b) Consider a small surface element with vector area $\mathbf{dA}$, where $\mathbf{dA}$ points away from the interior of V. From the definition of current density, the amount of current flowing out of V through this surface element is $\mathbf{J.dA}$. It follows that the total current flowing out of V is $\oint_S \mathbf{J.dA}$: the principle of charge conservation tells us that this is equal to the rate of *loss* of charge from V. Since the amount of charge in V is given by $\int_V \rho_e\,dV$, it follows that

$$\frac{\partial}{\partial t}\int_V \rho_e\,dV = -\oint_S \mathbf{J.dA}\,.$$

Rewriting the right hand side using Gauss's theorem, we have

$$\frac{\partial}{\partial t}\int_V \rho_e \, dV = -\int_V \nabla.\mathbf{J} \, dV \Leftrightarrow \int_V \left(\frac{\partial \rho_e}{\partial t} + \nabla.\mathbf{J}\right) dV = 0 \, ,$$

where we have used the fact that $\frac{\partial (dV)}{\partial t} = 0$, so $\frac{\partial}{\partial t}\int_V \rho_e \, dV = \int_V \frac{\partial \rho_e}{\partial t} \, dV$. Now, this result holds for *any* volume V, so the integrand must be zero. Thus

$$\frac{\partial \rho_e}{\partial t} = -\nabla.\mathbf{J} \, .$$ [8]

(c) Substituting Ohm's and Gauss's laws into the result from (b), we obtain

$$\frac{\partial \rho_e}{\partial t} = -\sigma \nabla.\mathbf{E} = -\frac{\sigma}{\epsilon}\rho_e \, .$$

This is a straightforward first order ordinary differential equation in ρ_e, which can be solved by separating the variables:

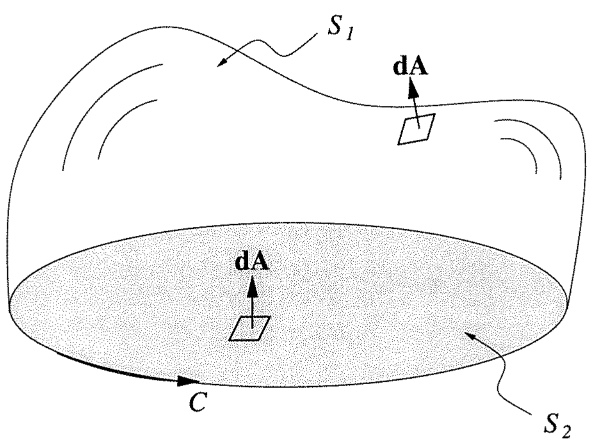
$$\int_{\rho_0}^{\rho_e} \frac{d\rho_e}{\rho_e} = -\int_0^t \frac{dt}{\tau} \, ,$$

where ρ_0 is the initial value of ρ_e and τ is the time constant ϵ/σ. Integrating, we find the solution $\rho_e = \rho_0 \exp(-t/\tau)$. So the charge declines exponentially in the interior of the conductor, which means it must be building up on the surface. [5]

Examiner's remarks: A fairly popular question, well-handled by many candidates. The most common error was sloppy integration of the simple differential equation in (c).

2. *Solenoidal fields and Stokes's theorem.*

(a) A solenoidal vector field $\mathbf{B}$ has zero divergence everywhere. Equivalently, $\oint_S \mathbf{B}.\mathbf{dA} = 0$ for all closed surfaces S.



Consider a vector field $\mathbf{D}$ passing through two open surfaces S_1 and S_2 that share a common rim. Stokes's theorem tells us that

$$\int_{S_1} \nabla \times \mathbf{D}.\mathbf{dA} = \int_{S_2} \nabla \times \mathbf{D}.\mathbf{dA} = \oint_C \mathbf{D}.\mathbf{dl}\,,$$

where the directions of the normals are fixed in accordance with the right hand screw rule. Now consider the closed surface $S = S_1 + S_2$ with outward pointing normal:

$$\oint_S \nabla \times \mathbf{D}.\mathbf{dA} = \int_{S_1} \nabla \times \mathbf{D}.\mathbf{dA} - \int_{S_2} \nabla \times \mathbf{D}.\mathbf{dA} = 0\,.$$

We have established that $\oint_S \nabla \times \mathbf{D}.\mathbf{dA} = 0$ for any closed surface S, so $\nabla \times \mathbf{D}$ must be solenoidal. [7]

(b) (i)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\mathbf{k}$$

$$\nabla.\mathbf{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$$ [3]

(ii) At any point on a field line, the direction vector $\mathbf{dr}$ is parallel to $\mathbf{F}$:

$$\mathbf{dr} = \begin{bmatrix} dx \\ dy \end{bmatrix} = \lambda \mathbf{F} = \begin{bmatrix} -y \\ x \end{bmatrix} \Leftrightarrow \frac{dx}{-y} = \frac{dy}{x} \Leftrightarrow x\,dx + y\,dy = 0\,.$$

Integrating both sides, we obtain $x^2 + y^2 =$ constant. The field lines are therefore concentric circles, centered on the origin. [3]

(iii) First using Stokes's theorem, we have

$$\int_S \nabla \times \mathbf{F}.\mathbf{dA} = \oint_C \mathbf{F}.\mathbf{dl} = \oint_C |\mathbf{F}|\,dl\,,$$

since $\mathbf{F}$ and $\mathbf{dl}$ are parallel. Noting that $|\mathbf{F}| = \sqrt{x^2 + y^2} = a$ on C, we have

$$\int_S \nabla \times \mathbf{F}.\mathbf{dA} = \oint_C a\,dl = 2\pi a^2\,.$$

Now using the result in (a), we know that $\nabla \times \mathbf{F}$ is solenoidal, so the flux of $\nabla \times \mathbf{F}$ through S must equal the flux of $\nabla \times \mathbf{F}$ through the disc enclosed by C:

$$\int_S \nabla \times \mathbf{F}.\mathbf{dA} = \int_{\text{disc}} \nabla \times \mathbf{F}.\mathbf{dA} = \int_{\text{disc}} 2\mathbf{k}.\mathbf{dA} = 2\pi a^2\,,$$

since $\mathbf{dA}$ and $\mathbf{k}$ are parallel. Now, we also know from (i) that $\mathbf{F}$ is solenoidal, so the flux of $\mathbf{F}$ through S must equal the flux of $\mathbf{F}$ through the disc enclosed by C, which is clearly zero since $\mathbf{F}$ lies in the plane of C. Hence

$$\int_S \mathbf{F}.\mathbf{dA} = 0\,.$$ [7]

Examiner's remarks: An extremely popular question with many excellent solutions. The most common errors were not using Stokes's theorem in (a) and careless presentation of the solution for the field lines in (b).

3. *Partial differential equations, the wave equation.*

(a) To separate the variables, we assume the solution is of the form $y(x,t) = F(x)G(t)$. It follows that $\partial^2 y/\partial t^2 = F\ddot{G}$ and $\partial^2 y/\partial x^2 = F''G$. Substituting into the wave equation, we obtain $F\ddot{G} = c^2 F''G$, and rearranging this we obtain

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}\,.$$

Since the left hand side depends only on t and the right hand side depends only on x, it follows that both sides must equal a constant, k say. We thus obtain two ordinary differential equations for F and G:

$$F'' - kF = 0 \quad \text{and} \quad \ddot{G} - c^2 kG = 0\,.$$

The boundary conditions $y(0,t) = 0$ and $y(L,t) = 0$ tell us that $F(0)G(t) = 0$ and $F(L)G(t) = 0$ for all t. Since $G(t) = 0$ leads to the uninteresting solution $y(x,t) = 0$, it follows that

$$F(0) = F(L) = 0\,.$$

Considering the ordinary differential equation in F, we can now deduce that k must be strictly negative, say $k = -p^2$, since otherwise the boundary conditions for F imply $y(x,t) = 0$ as before. Thus $F'' + p^2 F = 0$, which has the general solution

$$F(x) = A\cos px + B\sin px\,.$$

Referring again to the boundary conditions for F, we obtain $F(0) = A = 0$ and then $F(L) = B\sin pL = 0$. We must take $B \neq 0$, since otherwise $y(x,t) = 0$, so $\sin pL = 0$ and hence $p = n\pi/L$ for any integer n. We thus obtain infinitely many solutions $F(x) = F_n(x)$, where

$$F_n(x) = B\sin\left(n\pi x/L\right)\,.$$

Turning now to the ordinary differential equation for G, and substituting $k = -p^2$, we have

$$\ddot{G} + \omega_n^2 G = 0 \quad \text{where} \quad \omega_n = \frac{cn\pi}{L}\,.$$

The general solution is $G_n(t) = A_n \sin\omega_n t + B_n \cos\omega_n t$. Combining the solutions for F and G, we obtain the following solution of the wave equation:

$$y_n(x,t) = \left(A_n \sin\omega_n t + B_n \cos\omega_n t\right)\sin\left(n\pi x/L\right)\,,$$

where $\omega_n = cn\pi/L$. [12]

(b) If y_n satisfies the wave equation, then so does an infinite sum of y_n over all integers n. The general solution of the wave equation is therefore:

$$y(x,t) = \sum_{n=-\infty}^{\infty} y_n(x,t) = \sum_{n=-\infty}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin(n\pi x/L) \; .$$

However, the $n = 0$ term is zero, and the $n < 0$ terms give essentially the same solutions as the $n > 0$ terms, up to a minus sign, so we can write

$$y(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin(n\pi x/L) \; .$$

We now turn to the boundary condition $\partial y/\partial t = 0$ at $t = 0$ for $0 \le x \le L$. Differentiating the general solution with respect to t, we obtain

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} (\omega_n A_n \cos \omega_n t - \omega_n B_n \sin \omega_n t) \sin(n\pi x/L) \; .$$

Substituting $t = 0$ and equating to zero, we have

$$\sum_{n=1}^{\infty} \omega_n A_n \sin(n\pi x/L) = 0 \;\; \text{for } 0 \le x \le L.$$

This is an odd Fourier series in x, and by inspection $A_n = 0$ for all n. Thus

$$y(x,t) = \sum_{n=1}^{\infty} B_n \cos \omega_n t \, \sin(n\pi x/L) \; ,$$

where $\omega_n = cn\pi/L$. At time $t = 0$, we have

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L) = f(x) \, .$$

We could proceed to find the coefficients B_n using standard Fourier series techniques. We would first extend $f(x)$ to obtain an odd function $f_o(x)$ of period $2L$, where $f_o(x) = f(x)$ in the range $0 \le x \le L$. We would then find the Fourier series for $f_o(x)$, which would require sine terms only, giving us the coefficients B_n. [8]

Examiner's remarks: The least popular of the vector calculus questions, though most candidates understood what they were doing. The principal error was in combining the single solution in (a) to the sum of a series in (b): commonly, no justification or argument was presented for this.

4. *Numerical methods for initial value problems.*

(a) (i) The Euler method generates a series of y values spaced at intervals of h as follows:

$$y(x+h) = y(x) + hy'(x) = y(x) + h\lambda y(x) = y(x)(1+\lambda h)\,.$$

It follows that $y(kh) = y(0)(1+\lambda h)^k = A(1+\lambda h)^k$. Finally, writing $X = kh$, we obtain $y(X) = A(1+\lambda h)^{X/h}$ as required. [4]

(ii) The true value of $y(X)$ is of course $Ae^{\lambda X}$. A Binomial expansion of the Euler expression for $y(X)$ gives

$$\begin{aligned} y(X) &= A\left[1 + \frac{X}{h}\lambda h + \frac{\frac{X}{h}\left(\frac{X}{h}-1\right)}{2!}\lambda^2 h^2 + \frac{\frac{X}{h}\left(\frac{X}{h}-1\right)\left(\frac{X}{h}-2\right)}{3!}\lambda^3 h^3 + \ldots\right] \\ &= A\left[1 + \lambda X + \frac{\lambda^2 X^2}{2!} + \frac{\lambda^3 X^3}{3!} + \ldots + O(h)\right] = A\left[e^{\lambda X} + O(h)\right]\,. \end{aligned}$$

Note that we have used the standard power series expansion of e^x (data book) in the last step. Thus the Euler estimate converges to the true solution in the limit $h \to 0$, and the error is of order h, as we would expect for an Euler method. [6]

(b) (i) This is the same form of initial value problem as in (a), with $\lambda = 4$ and $A = 1$. Substituting $f(x,y) = \lambda y$, the second-order Runge-Kutta method becomes

$$\begin{aligned} y(x+h) &= y(x) + hf\left(x + \frac{h}{2}\,,\; y(x) + \frac{1}{2}h\lambda y(x)\right) \\ &= y(x) + h\left(\lambda(y(x) + \frac{1}{2}h\lambda y(x))\right) = y(x)(1+\lambda h + \frac{1}{2}h^2\lambda^2)\,. \end{aligned}$$

By the same reasoning as in (a), it follows that $y(X) = A(1+\lambda h + \frac{1}{2}h^2\lambda^2)^{X/h}$. Substituting $X = 1$, $\lambda = 4$, $A = 1$ and $h = 0.2$, we obtain the estimate $y(1) = 42.82$. [4]

(ii) Since this method is second-order, a Binomial expansion of the Runge-Kutta expression for $y(1)$ would give

$$y(1) = \left[e^4 + Ch^2 + O(h^3)\right]\,,$$

where C is a constant. With $h = 0.2$ we found in (i) that $y(1) = 42.82$. Ignoring terms in h^3 and above, we can estimate $C = (y(1) - e^4)/h^2 = (42.82 - e^4)/0.04 = -294.4$.

If we reduced the step size to 0.1, we would expect to find $y(1) = [e^4 + Ch^2] = [e^4 - 294.4 \times 0.01] = 51.7$, where we have again ignored terms in h^3 and above.

The actual result of the numerical integration is given by $y(X) = A(1+\lambda h + \frac{1}{2}h^2\lambda^2)^{X/h}$ with $X = 1$, $\lambda = 4$, $A = 1$ and $h = 0.1$, which comes out as 50.4. The difference between this value and the prediction of 51.7 is because we ignored terms in h^3 and above when coming up with the prediction. [6]

Examiner's remarks: This straightforward question proved too difficult for many candidates. Typical shortcomings included not being able to work out the exact solution to $dy/dx = \lambda y$, not knowing what the Euler method is and not appreciating the difference between a first- and second-order method. In contrast, those candidates who had even a basic grasp of these concepts scored close to full marks.

5. *Numerical methods for solving systems of linear equations.*

(a) The system $\mathbf{AX} = \mathbf{B}$ can be written $\mathbf{LUX} = \mathbf{B}$, where L is a lower-triangular matrix with 1's along its leading diagonal, and U is an upper-triangular matrix:

$$\begin{bmatrix} 1.133 & 5.281 \\ 24.14 & -1.210 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

Working through the matrix equality row by row, we find:

$$\begin{aligned} u_{11} &= 1.133\ ,\ u_{12} = 5.281\ ,\ u_{11}l_{21} = 1.133 l_{21} = 24.14 \Leftrightarrow l_{21} = 21.31 \\ l_{21}u_{12} + u_{22} &= 21.31 \times 5.281 + u_{22} = -1.210 \Leftrightarrow u_{22} = -113.7 \end{aligned}$$

We now solve the system $\mathbf{LY} = \mathbf{B}$ using forward substitution:

$$\begin{bmatrix} 1.000 & 0.000 \\ 21.31 & 1.000 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 6.414 \\ 22.93 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 6.414 \\ -113.8 \end{bmatrix}$$

Finally, we solve $\mathbf{UX} = \mathbf{Y}$ using back substitution (remember to keep only four significant figures after *each* calculation):

$$\begin{bmatrix} 1.133 & 5.281 \\ 0.000 & -113.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.414 \\ -113.8 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.9956 \\ 1.001 \end{bmatrix}$$ [5]

(b) Repeating with the matrix rows reordered:

$$\begin{bmatrix} 24.14 & -1.210 \\ 1.133 & 5.281 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

Working through the matrix equality row by row, we find:

$$\begin{aligned} u_{11} &= 24.14\ ,\ u_{12} = -1.210\ ,\ u_{11}l_{21} = 24.14 l_{21} = 1.133 \Leftrightarrow l_{21} = 0.04693 \\ l_{21}u_{12} + u_{22} &= 0.04693 \times -1.210 + u_{22} = 5.281 \Leftrightarrow u_{22} = 5.338 \end{aligned}$$

We now solve the system $\mathbf{LY} = \mathbf{B}$ using forward substitution:

$$\begin{bmatrix} 1.000 & 0.000 \\ 0.04693 & 1.000 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 22.93 \\ 6.414 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 22.93 \\ 5.338 \end{bmatrix}$$

Finally, we solve $\mathbf{UX} = \mathbf{Y}$ using back substitution:

$$\begin{bmatrix} 24.14 & -1.210 \\ 0.000 & 5.338 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 22.93 \\ 5.338 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.000 \\ 1.000 \end{bmatrix}$$ [5]

(c) The correct solution to both sets of equations is (by inspection) $x_1 = 1$ and $x_2 = 1$. What went wrong in (a) was the creation of the large term l_{21}, with a truncation error of the order 0.005. The large magnitude of l_{21} in turn produced a large term u_{22} with a truncation error of the order 0.05. For larger systems, u_{22} would have been used to find subsequent l and u values, with significant knock-on errors.

For maximum precision, we should reorder the equations so that the largest magnitude terms appear along the leading diagonal of the coefficient matrix. The aim is to ensure that the multipliers in L remain small, and that the terms in U are of the same order as those in the original coefficient matrix. This strategy is equivalent to *partial pivoting* in Gaussian elimination. Even better is to reorder the equations so that the elements along the leading diagonal have large magnitudes relative to the other elements in their row. This is known as *scaled partial pivoting* or *equilibrating*. [5]

(d) LU decomposition can produce accurate solutions if care is taken to reorder the equations appropriately. In terms of speed, the triangular factorization is order N^3 for an $N \times N$ system, while the forward and back substitutions are relatively insignificant (order N^2). If the system of equations is to be solved repeatedly, with the same coefficients but different right hand sides, then the LU decomposition does not need to be recomputed and each extra solution requires only the $O(N^2)$ substitution stage.

Iterative techniques, like the Gauss-Seidel method, do not always converge (a necessary condition is that the coefficient matrix is strictly diagonally dominant), but can be efficient for certain large ($N \sim 10^5$) systems. If the coefficient matrix is sparse, and there is a pattern to the nonzero entries (eg. tridiagonal systems), then an iterative process could find a solution much more rapidly than LU decomposition. [5]

Examiner's remarks: Most of the candidates who attempted this question knew what LU decomposition was, though few were able to use it to solve a couple of 2×2 systems to the required precision: there were far too many algebraic slips. Intelligent comments on the accuracy of LU decomposition, and the effects of permuting the equations, were few and far between: the most common suggestion for improving accuracy was to use more significant figures, which completely misses the point. Discussion of the relative merits of LU decomposition and iterative techniques was also of a very poor standard: many candidates failed to even distinguish the two, while many wrote about how long it takes to do the calculations by hand, revealing a complete failure to appreciate that the course was about computer numerical analysis.

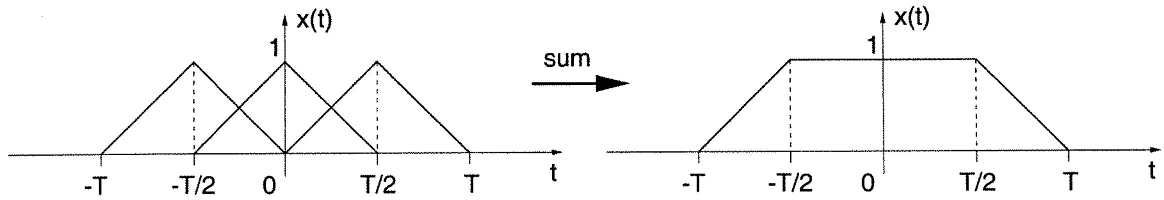
6. *Fourier transforms, Fourier series and Parseval's theorem.*

(a) The Fourier transform of $x(t-T)$ is calculated using a change of variable $u = t-T$:

$$\int_{-\infty}^{\infty} x(t-T)e^{-i\omega t}dt = \int_{-\infty}^{\infty} x(u)e^{-i\omega(u+T)}du = e^{-i\omega T}\int_{-\infty}^{\infty} x(u)e^{-i\omega u}du = e^{-i\omega T}X(\omega)\,,$$

where $X(\omega)$ is the Fourier transform of $x(t)$. [4]

(b) $x(t)$ is evidently the sum of three triangular pulses:



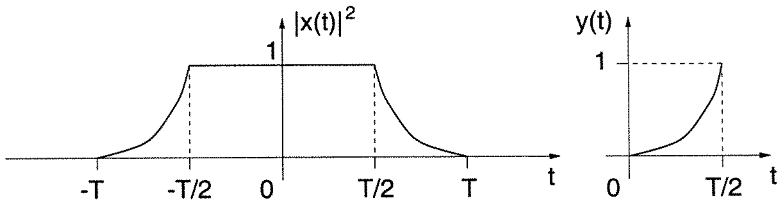
The Fourier transform of the central triangular pulse is $(T/2)\,\mathrm{sinc}^2(\omega T/4)$ (data book). The Fourier transforms of the outer two can be found using the time-shift theorem proved in (a): they are simply $e^{i\omega T/2}$ and $e^{-i\omega T/2}$ times the transform of the central pulse. Summing the individual transforms, we obtain

$$X(\omega) = \left(1 + e^{i\omega T/2} + e^{-i\omega T/2}\right) \frac{T}{2}\,\mathrm{sinc}^2\left(\frac{\omega T}{4}\right) = \frac{T}{2}\left(1 + 2\cos\frac{\omega T}{2}\right)\mathrm{sinc}^2\left(\frac{\omega T}{4}\right). \qquad [5]$$

(c) By Parseval's theorem,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = \int_{-\infty}^{\infty} |x(t)|^2\, dt\,.$$

The integral over t is much easier to evaluate than the integral over ω.



The area under $|x(t)|^2$ is the area under the rectangular section (which is obviously equal to T) plus twice the area under the parabola $y(t)$:

$$\int_0^{T/2} y(t)dt = \int_0^{T/2}\left(\frac{2t}{T}\right)^2 dt = \frac{4}{T^2}\left[\frac{t^3}{3}\right]_0^{T/2} = \frac{4}{T^2}\times\frac{T^3}{24} = \frac{T}{6}$$

Summing the three contributions, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = T + \frac{T}{6} + \frac{T}{6} = \frac{4T}{3}\,. \qquad [5]$$

(d) By the definition of the Fourier series (data book), the coefficients for a series of period $2T$ are given by

$$c_n = \frac{1}{2T}\int_{-T}^{T} x_p(t)e^{-in\omega_0 t}dt = \frac{1}{2T}\int_{-T}^{T} x(t)e^{-in\omega_0 t}dt = \frac{1}{2T}\int_{-\infty}^{\infty} x(t)e^{-in\omega_0 t}dt\,,$$

where $\omega_0 = \pi/T$ and we have used the fact that $x(t) = 0$ for $|t| > T$. Now compare this expression with the expression for the Fourier transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt\,.$$

It is immediately evident that

$$c_n = \frac{1}{2T}X(n\omega_0)\,.$$ [6]

Examiner's remarks: A popular question which produced a wide variety of solutions. Candidates with basic mathematical skills and some idea about Fourier transforms scored close to full marks. Too many other candidates performed poorly. In (a) some thought that $x(t-T) \equiv x(t) - x(T)$; in (b) some thought that functions added like jigsaws, so you could construct a trapezium by placing two triangles side by side and adding an inverted triangle to fit the space between the first two; in (c) many could not find the area under a parabola, a common mistake being to assume that $\int |x(t)|^2 dt \equiv (\int x(t)\,dt)^2$; and (d) totally confused most candidates.

7. *Discrete Fourier transforms, moment generating functions.*

(a) (i) The DFT of a 4-point sequence x_n is given by

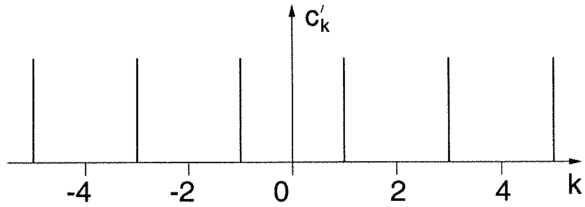
$$X_k = \sum_{n=0}^{3} x_n e^{-ikn\pi/2}\,.$$

Evaluating each term in turn, we obtain

$$\begin{aligned} X_0 &= \sum_{n=0}^{3} x_n e^0 = \sum_{n=0}^{3} x_n = 0\,, \\ X_1 &= \sum_{n=0}^{3} x_n e^{-in\pi/2} = e^0 - e^{-i2\pi/2} = 1-(-1) = 2\,, \\ X_2 &= \sum_{n=0}^{3} x_n e^{-in\pi} = e^0 - e^{-i2\pi} = 1-1 = 0\,, \\ X_3 &= \sum_{n=0}^{3} x_n e^{-i3n\pi/2} = e^0 - e^{-i6\pi/2} = 1-(-1) = 2\,. \end{aligned}$$

The terms $X_0 \ldots X_3$ correspond to the frequencies 0 Hz, 1 Hz, 2 Hz and 3 Hz respectively. [6]

(ii) The multiple non-zero terms in the DFT of x_n are a result of sampling. The complex Fourier series representation of $x(t) = \cos(2\pi t)$ is $\sum_{k=-\infty}^{\infty} c_k e^{ik2\pi t}$, where $c_1 = c_{-1} = 0.5$ and all other coefficients are zero. The DFT is a periodic version c_k' of the coefficient sequence, formed by repeating the coefficients at intervals of four (the sampling frequency), as shown below.



There is also a scale factor of four (the number of samples). [4]

(b) (i) The moment generating function of the Normal distribution is readily found by completing the square in the integral:

$$\begin{aligned} g(s) &= \int_{-\infty}^{\infty} e^{-sx} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-sx} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+2sx)/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-((x+s)^2-s^2)/2} dx = e^{s^2/2} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+s)^2/2} dx = e^{s^2/2}, \end{aligned}$$

since the final integral is a Normal distribution with mean $-s$ and unit variance, and the area under any probability density function is one. [5]

(ii) If we differentiate the moment generating function three times, we find

$$g'''(s) = -\int_{-\infty}^{\infty} x^3 e^{-sx} f(x) dx \text{ , and hence } \mathrm{E}[X^3] = \int_{-\infty}^{\infty} x^3 f(x) dx = -g'''(0) .$$

Differentiating the expression for $g(s)$ found in (i), we obtain

$$g'(s) = se^{s^2/2} \text{ , } \quad g''(s) = e^{s^2/2} + s^2 e^{s^2/2} \text{ , } \quad g'''(s) = se^{s^2/2} + 2se^{s^2/2} + s^3 e^{s^2/2} .$$

So $\mathrm{E}[X^3] = -g'''(0) = 0$. This is readily verified by direct evaluation of the integral $\int_{-\infty}^{\infty} x^3 f(x) dx$. Since x^3 is an odd function and $f(x)$ is an even function, the product $x^3 f(x)$ is odd and the integral, which is arranged symmetrically around the y axis, is therefore zero. [5]

Examiner's remarks: An unpopular question which produced some very good answers from the few candidates who understood the material. Unfortunately, the majority could not evaluate the most straightforward of DFTs in (a) and had little idea about what the DFT was measuring. Only one candidate thought to complete the square in (b), and surprisingly many had difficulty differentiating $\exp(s^2/2)$ three times to find $\mathrm{E}[X^3]$.

8. *Probability and hypothesis testing.*

(a) Statistical significance is a concept encountered in statistical tests of hypothesis. We assume that the statistical event in question is a random variable governed by a particular distribution: this is the *null hypothesis.* Using this hypothesis, we then calculate the probability of the observed event occurring. If this probability is less than ϵ, we say that the event is significant at the ϵ level and reject the null hypothesis in favour of an alternative. Typical confidence limits are $\epsilon = 0.05$ and $\epsilon = 0.01$. [3]

(b) (i) If we assume that the distribution of marks is symmetrically arranged around the mean (ie. the median is equal to the mean), then the probability of any one mark exceeding the mean is 0.5. This is not a bad assumption for typical distributions of marks. We must also assume that the first 10 scripts are independent, random samples and not specially selected in any way. Then the probability that m out of n marks exceed the mean follows a Binomial distribution:

$$\begin{aligned} p(8 \text{ or more} > \text{mean}) &= p(8 > \text{mean}) + p(9 > \text{mean}) + p(10 > \text{mean}) \\ &= {}^{10}C_8(0.5)^{10} +^{10} C_9(0.5)^{10} +^{10} C_10(0.5)^{10} = 0.0547 \end{aligned}$$ [4]

(ii) We can now test the significance of this event. The null hypothesis is that the mean mark is unchanged at 61.5%, the alternative hypothesis is that the mean has gone up. Under the null hypothesis, there is a 5.4% chance that 8 or more scripts out of 10 attract marks above 61.5%. Therefore, the event is not significant at the 5% level, and we have no grounds to reject the null hypothesis at this level of confidence. [2]

(c) (i) Let X be the mark awarded to any one script: we know that X is a random variable with mean μ and standard deviation σ. Now consider the random variable $Z = (X_1 + X_2 + \ldots + X_{35})/35$, the average mark across 35 randomly selected scripts. Z is the scaled sum of a large number of identically distributed random variables (35 is large enough) and the central limit theorem therefore applies: $Z \sim \mathrm{N}\left(\mu, \sigma/\sqrt{35}\right)$. [3]

(ii) The null hypothesis is that the mean of this year's marks is 61.5%: the alternative hypothesis is that the mean of this year's marks is greater than 61.5%. To get anywhere with the significance test, we need to assume that the sample standard deviation (12%) is a good estimate of the underlying population's standard deviation σ. So, under the null hypothesis, we assume $Z \sim \mathrm{N}\left(61.5, 12.0/\sqrt{35}\right) = \mathrm{N}\left(61.5, 2.028\right)$.

We now normalize the observed average mark of 65.4%, to see where it lies on the standard Normal curve:

$$\frac{65.4 - 61.5}{2.028} = 1.923 \quad \text{and} \quad \phi(1.923) = 0.973$$

Since $0.973 > 0.95$, the observation is significant at the 5% level and we can therefore reject the null hypothesis with 95% confidence. As well as the assumption about the population's standard deviation, we have also assumed that the 35 marked scripts are randomly drawn from the population and not specially selected in any way. [5]

(d) The test in (c) allows us to conclude that the mean has gone up with a known, 5% chance of being wrong. We cannot draw such powerful conclusions from the test in (b). Even though the test failed to reject the null hypothesis, we cannot subsequently *accept* the null hypothesis since we do not know the probability that such a decision is wrong (this probability depends on the true, unknown mean). The correct thing to do after the (b) test is to withhold judgement and collect more data.

Another argument against the test in (b) is that we had to assume that the median of the distribution of marks was the same as the mean. No such assumption was required for the test in (c). In addition, the outcome of the test in (c) was some way away from the confidence limit, whereas (b) was borderline. [3]

Examiner's remarks: A fairly popular question, well-handled by many candidates. Most understood the meaning of statistical significance and correctly associated it with hypothesis testing. Furthermore, most correctly applied the Binomial distribution in (b) and showed proficiency in the use of Normal tables in (c) (though often with the wrong variance, which is surprising given that the question indicated what variance to use). The most common mistake was drawing the wrong conclusion from a significance test: having calculated that the probability of an event was fairly large under the null hypothesis, candidates would reject the null hypothesis! Few candidates mentioned the central limit theorem in (c).

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