ENGINEERING TRIPOS PART IB

Friday 8 June 2001 9 to 11

Paper 7

MATHEMATICAL METHODS

*Answer not more than **four** questions.*

*Answer at least **one** question from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

(TURN OVER

SECTION A

*Answer at least **one** question from this section*

1 (a) The coordinate-free definition of divergence is

$$\nabla.\mathbf{u} = \lim_{\delta V \to 0} \frac{1}{\delta V} \oint_S \mathbf{u}.\mathbf{dA} \ ,$$

where S is the surface that encloses the small volume δV. Use this definition to prove Gauss's theorem. [7]

(b) Consider the electrical current density $\mathbf{J}$, which is defined such that the flow of current per unit area in the direction of a unit vector $\mathbf{n}$ is $\mathbf{J}.\mathbf{n}$. Use the principle of conservation of charge to show that, for an arbitrary volume V,

$$\frac{\partial}{\partial t} \int_V \rho_e \, dV = - \oint_S \mathbf{J}.\mathbf{dA} \ ,$$

where S is the surface that encloses V and ρ_e is the charge density. Now use Gauss's theorem to show that

$$\frac{\partial \rho_e}{\partial t} = -\nabla.\mathbf{J} \ .$$ [8]

(c) In a homogeneous conductor, ρ_e and $\mathbf{J}$ are related to the electric field $\mathbf{E}$ by Ohm's law and Gauss's law of electrostatics:

$$\mathbf{J} = \sigma \mathbf{E} \quad \text{and} \quad \nabla.\mathbf{E} = \rho_e / \epsilon \ ,$$

where σ is the conductor's conductivity and ϵ is its permittivity. Show that ρ_e decays exponentially in the interior of the conductor, implying that any charge present at time $t = 0$ moves to the boundary. [5]

2 (a) What is the definition of a solenoidal vector field? By applying Stokes's theorem to two surfaces that share a common rim, show that any vector field $\mathbf{B}$ of the form $\mathbf{B} = \nabla \times \mathbf{D}$ must be solenoidal. [7]

(b) Consider the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.

(i) Find $\nabla \times \mathbf{F}$ and $\nabla . \mathbf{F}$. [3]

(ii) Show that the field lines of $\mathbf{F}$ are concentric circles. [3]

(iii) It is required to evaluate the flux of $\nabla \times \mathbf{F}$ through an open surface S with rim curve C, where C is defined by $x^2 + y^2 = a^2$, $z = 0$, as shown in Fig. 1. Evaluate this flux in two ways, first using Stokes's theorem and second using the result in (a). What is the flux of $\mathbf{F}$ through S? [7]

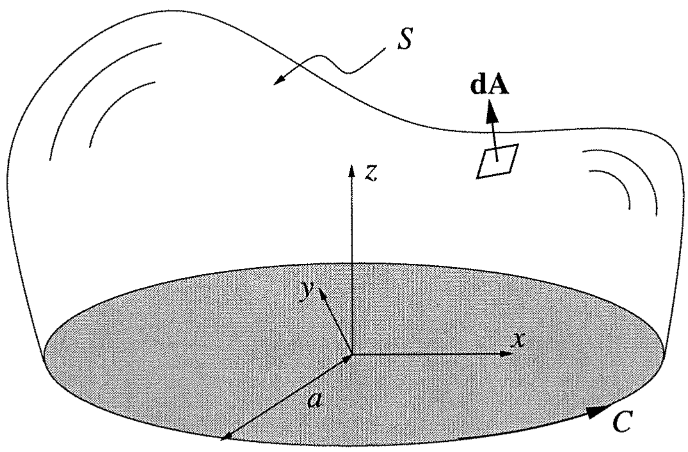

Fig. 1

3 The displacement $y(x,t)$ of a vibrating, stretched string satisfies the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \ .$$

The string is fixed so that $y = 0$ at $x = 0$ and $x = L$.

(a) Use the method of separation of variables to show that

$$y_n = (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin (n\pi x/L)$$

is a solution of the wave equation for any integer n. What is the relationship between the angular frequency ω_n and c? [12]

(b) At time $t = 0$, the string is displaced but held stationary, so that $\partial y/\partial t = 0$. Show that the general solution for y is then

$$y = \sum_{n=1}^{\infty} B_n \cos \omega_n t \, \sin (n\pi x/L) \ .$$

If the initial displacement is given by $y(x,0) = f(x)$, indicate how you would determine the coefficients B_n. [8]

SECTION B

*Answer at least **one** question from this section*

4 (a) Consider the differential equation $dy/dx = \lambda y$ with initial value $y(0) = A$. We wish to integrate this equation from $x = 0$ to $x = X$.

(i) Show that the Euler method results in the estimate

$$y(X) = A(1 + \lambda h)^{X/h} \ ,$$

where h is the step size. [4]

(ii) Using a Binomial series expansion, show that the Euler estimate converges to the true value of $y(X)$ in the limit $h \to 0$, and deduce how the error in the estimate varies with h. [6]

(b) The second-order Runge-Kutta method

$$y(x + h) = y(x) + h\, f\left(x + \frac{h}{2}\ ,\ y + \frac{h}{2} f(x, y)\right)$$

can be used to solve initial value problems of the form $dy/dx = f(x, y)$, $y(x_0) = y_0$.

(i) Use this method to integrate the differential equation $dy/dx = 4y$, $y(0) = 1$, from $x = 0$ to $x = 1$: take the step size h to be 0.2. [4]

(ii) Use the fact that this method is second-order to predict the result of the same integration with a step size $h = 0.1$. Explain any difference between your prediction and the actual result of the numerical integration. [6]

5 (a) Using LU decomposition, **and no other method**, solve the following system of equations for x_1 and x_2:

$$\begin{bmatrix} 1.133 & 5.281 \\ 24.14 & -1.210 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.414 \\ 22.93 \end{bmatrix} .$$

To simulate the behaviour of a finite-precision computer, maintain only four significant figures throughout your calculations. [5]

(b) Repeat part (a), carrying out an *identical* sequence of operations, for the system

$$\begin{bmatrix} 24.14 & -1.210 \\ 1.133 & 5.281 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 22.93 \\ 6.414 \end{bmatrix} .$$

Remember to maintain only four significant figures throughout your calculations, and be sure to start from scratch: do *not* use your solution to part (a). [5]

(c) Comment on the relative accuracy of the calculations in (a) and (b). How would you ensure maximum accuracy when solving a larger system of equations by LU decomposition? [5]

(d) Discuss the relative advantages and disadvantages of LU decomposition and iterative techniques for solving systems of linear equations. [5]

SECTION C

*Answer at least **one** question from this section*

6 (a) If the Fourier transform of $x(t)$ is $X(\omega)$, prove that the Fourier transform of $x(t-T)$ is $e^{-i\omega T}X(\omega)$. [4]

(b)

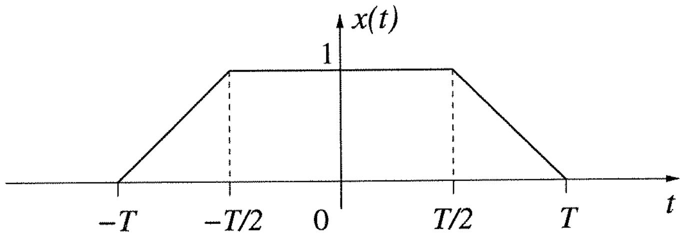

Fig. 2

Consider the signal $x(t)$ in Fig. 2, where $x(t) = 0$ for $|t| > T$. Use the result in (a) to show that the Fourier transform of $x(t)$ is given by

$$X(\omega) = \frac{T}{2}\left(1 + 2\cos\frac{\omega T}{2}\right)\operatorname{sinc}^2\left(\frac{\omega T}{4}\right) \ .$$ [5]

(c) Using Parseval's theorem, or otherwise, calculate the quantity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega \ ,$$

where $X(\omega)$ is given in (b). [5]

(d) Now consider the periodic signal $x_p(t) = \sum_{n=-\infty}^{\infty} x(t-2nT)$, where $x(t)$ is defined in Fig. 2. If the Fourier series representation of $x_p(t)$ is $\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$, where $\omega_0 = \pi/T$, use the data book formula for c_n to show that

$$c_n = \frac{1}{2T}\int_{-\infty}^{\infty} x(t)e^{-in\omega_0 t}dt$$

and hence express the coefficients c_n in terms of $X(\omega)$, the Fourier transform of $x(t)$. [6]

7 (a) The signal $x(t) = \cos(2\pi t)$ is sampled at times $t = 0$, $t = 0.25$, $t = 0.5$ and $t = 0.75$, producing the sequence $x_n = \{1, 0, -1, 0\}$.

(i) Calculate the discrete Fourier transform (DFT) of the sequence x_n. What frequencies do the individual terms in the DFT correspond to? [6]

(ii) Explain carefully why there are two non-zero terms in the DFT of x_n, even though $x(t)$ is a pure sinusoid. [4]

(b) A continuous probability density function $f(x)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \; , \quad -\infty < x < \infty \; .$$

(i) Show, by evaluating the appropriate integral, that the moment generating function of $f(x)$ is $e^{s^2/2}$. [5]

(ii) Use the moment generating function to find $\mathrm{E}[X^3]$. Verify your answer by inspection of the integral $\int_{-\infty}^{\infty} x^3 f(x)\, dx$. [5]

8 (a) Explain what is meant by *statistical significance.* [3]

(b) An examiner is marking a large pile of scripts. Last year, the average mark in the same exam was 61.5%. This year, the examiner awards marks above 61.5% to 8 out of the first 10 scripts, and therefore suspects that the mean has gone up.

(i) Stating any assumptions you make, calculate the probability that 8 or more out of any 10 scripts attract above-average marks. [4]

(ii) At the 5% level of significance, do the first 10 marks support the hypothesis that this year's average is greater than 61.5%? [2]

(c) After marking the first 35 scripts, the examiner stops to check the mean and standard deviation of the marks so far, finding the mean to be 65.4% and the standard deviation 12.0%

(i) Explain briefly why the average mark across a random sample of 35 scripts follows an approximately Normal distribution, with mean μ and standard deviation $\sigma/\sqrt{35}$, where μ and σ are the mean and standard deviation of the marks across *all* the scripts. [3]

(ii) Based on the sample of 35 scripts, the examiner concludes that this year's average will turn out to be above 61.5%. Perform a test of statistical significance at the 5% level to check the examiner's claim. State any assumptions you make. [5]

(d) Compare the outcomes of the significance tests in (b) and (c). From which test can you draw the strongest conclusions? [3]

END OF PAPER