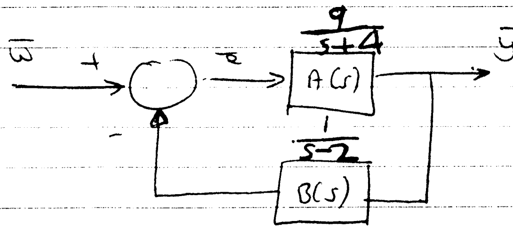
Engineering Tripos Part 1B Paper 6. Information Eng.

June 2002. IJW

1) a) In general,



$$\bar{e} = \bar{w} - B\bar{y}$$
and
$$\bar{y} = A\bar{e}$$

so the CLTF, ie

$$\frac{\bar{y}}{\bar{w}} = \frac{A}{(1+AB)}$$

where in this case $A = \frac{9}{s+4}$, $B = \frac{1}{s-2}$

so,

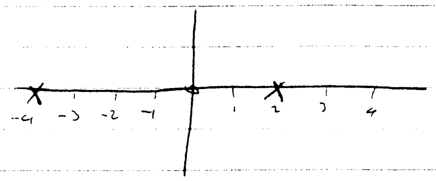
$$\frac{\bar{y}}{\bar{w}} = \frac{\frac{9}{s+4}}{1 + \frac{9}{(s+4)(s-2)}} = \frac{\frac{9}{s+4}}{\frac{(s+4)(s-2)+9}{(s+4)(s-2)}} = \frac{9(s+4)(s-2)}{(s+4)[(s+4)(s-2)+9]}$$

$$\frac{\bar{y}}{\bar{w}} = \frac{9(s-2)}{(s+4)(s-2)+9} = \frac{9(s-2)}{s^2-2s+4s-8+9} = \frac{9(s-2)}{s^2+2s+1}$$

$$\frac{\bar{y}}{\bar{w}} = \frac{9(s-2)}{(s+1)^2}$$

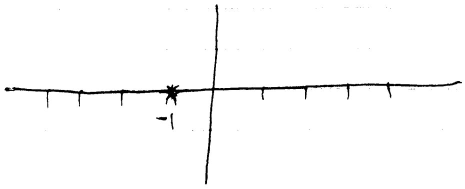
b) The OLTF $= AB = \frac{9}{s+4} \cdot \frac{1}{s-2} = \frac{9}{(s+4)(s-2)}$

The OLTF has poles at $s = -4$ and $s = 2$



$\therefore$ Unstable, ie 1 pole in the RHP.

The CLTF has poles at $s = -1$ (twice)



$\therefore$ Asymptotically stable.

2)

c) $\bar{y} = \dfrac{9(s-2)}{(s+1)^2} \cdot \bar{w}$

Now $\bar{w} = \dfrac{1}{s}$ ie, a unit step.

So,

$$\bar{y} = \frac{9(s-2)}{(s+1)^2} \cdot \frac{1}{s} = \frac{9(s-2)}{s(s+1)^2}$$

Partial fractions,

$$\frac{9(s-2)}{s(s+1)^2} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2}$$

$$9(s-2) = A(s+1)^2 + Bs(s+1) + Cs$$
$$9s - 18 = s^2(A+B) + s(2A+B+C) + A$$

const $\therefore A = -18$

s^2 $\therefore A + B = 0$

$\therefore B = 18$

s $\therefore 9 = 2A + B + C$

$\therefore C = 27$

So,

$$\bar{y}(s) = \frac{-18}{s} + \frac{18}{(s+1)} + \frac{27}{(s+1)^2}$$

$$\therefore y(t) = -18 + 18e^{-t} + 27te^{-t}$$

Final value - use final value theorem

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\bar{y}(s)$$

$$\lim_{s\to 0} s\frac{9(s-2)}{s(s+1)^2} = \lim_{s\to 0} \frac{9(s-2)}{(s+1)^2} = \underline{\underline{-18}}$$

(Note $y(t)$, $t\to\infty$ gives final value = -18)

Initial value

As above but $s\to\infty$

$$\lim_{s\to\infty} \frac{9(s-2)}{(s+1)^2} = \lim_{s\to\infty} \frac{9s-18}{s^2+2s+1} = \lim_{s\to\infty} \frac{9}{2s+2} = \underline{\underline{0}}$$

(Note $y(t)$, $t\to 0$ gives initial value 0)

Initial value of slope of $y(t)$ is,

③

$$\lim_{t\to 0^+} \dot{y}(t) = \lim_{s\to\infty} s \times s\bar{y}(s) = \lim_{s\to\infty} s^2 \bar{y}(s)$$

$$= \lim_{s\to\infty} \frac{s^2\, 9(s-2)}{s(s+1)^2} = \lim_{s\to\infty} \frac{9s(s-2)}{(s+1)^2}$$

$$= \lim_{s\to\infty} \frac{9s^2 - 18s}{s^2 + 2s + 1}$$

Apply l'Hopitals rule

$$= \lim_{s\to\infty} \frac{18s - 18}{2s + 2}$$

and again,

$$= \lim_{s\to\infty} \frac{18}{2}$$

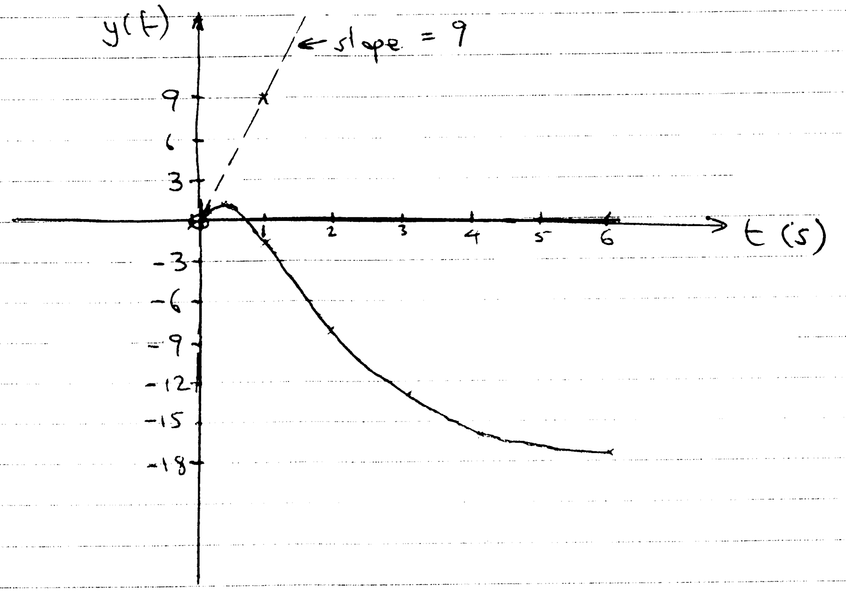
$$= \underline{\underline{9}}$$

OR

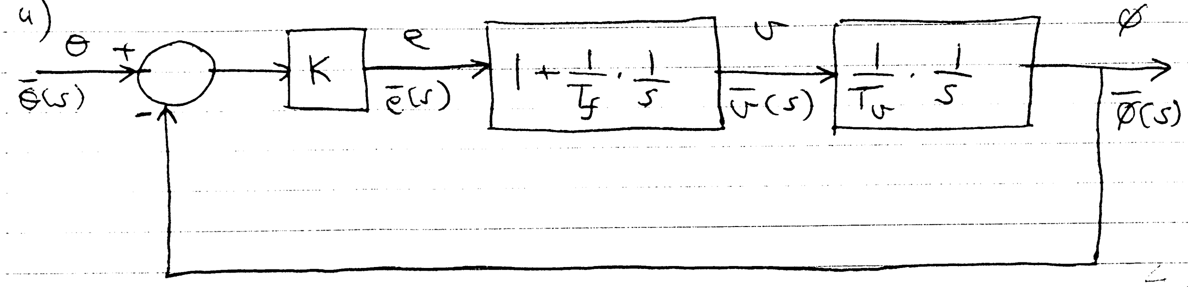
$$\lim_{s\to\infty} \frac{9(s^2 - 2s)}{s^2 - 2s + 4s + 1}$$

$$\lim_{s\to\infty} \frac{9(s^2-2s)}{(s^2-2s)\left(1 + \frac{4s+1}{s^2-2s}\right)}$$

$$\lim_{s\to\infty} \frac{9}{\left(1 + \frac{4s+1}{s^2-2s}\right)} = \underline{\underline{9}}$$



2) a)



Now, $\bar{e}(s) = K[\bar{\theta}(s) - \bar{\phi}(s)]$

$$\bar{v}(s) = \bar{e}(s)\left(1 + \frac{1}{T_f s}\right)$$

so $$\bar{v}(s) = K(\bar{\theta}(s) - \bar{\phi}(s))\left(1 + \frac{1}{T_f s}\right)$$

Also $$\bar{\phi}(s) = \bar{v}(s)\left(\frac{1}{T_v s}\right)$$

so, $$\bar{v}(s) = K\left(\bar{\theta}(s) - \frac{\bar{v}(s)}{T_v s}\right)\left(1 + \frac{1}{T_f s}\right)$$

$$\bar{v}(s) = K\bar{\theta}(s)\left(1 + \frac{1}{T_f s}\right) - \frac{K\bar{v}(s)}{T_v s}\left(1 + \frac{1}{T_f s}\right)$$

$$\bar{v}(s)\left(1 + \frac{K}{T_v s}\left(1 + \frac{1}{T_f s}\right)\right) = K\left(1 + \frac{1}{T_f s}\right)\bar{\theta}(s)$$

$$\bar{v}(s)\left(\frac{T_v T_f s^2 + K T_f s + K}{T_v T_f s^2}\right) = \frac{K(T_f s + 1)}{T_f s}\bar{\theta}(s)$$

$$\therefore \bar{v}(s) = \frac{K T_v s (T_f s + 1)}{T_v T_f s^2 + K T_f s + K}\bar{\theta}(s)$$

÷ through by K,

$$\bar{v}(s) = \frac{T_v s (1 + T_f s)}{\frac{T_v T_f}{K} s^2 + T_f s + 1}\bar{\theta}(s)$$

b). The CE is

$$\frac{T_f T_v}{K} s^2 + T_f s + 1 = 0$$

now $K = 1$ so, CE is

$$T_f T_v s^2 + T_f s + 1 = 0$$

The specified roots are $-0.1 \pm j0.1$

so,

$$(s + 0.1 - j0.1)(s + 0.1 + j0.1) = 0$$

ie $s^2 + 0.2s + 0.02 = 0$

comparing with the CE,

$$T_f T_v s^2 + T_f s + 1 = 0 \qquad \div \text{ through by } T_f T_v$$

$$s^2 + \frac{1}{T_v} s + \frac{1}{T_f T_v} = 0$$

ie, $\frac{1}{T_v} = 0.2 \quad \therefore \underline{\underline{T_v = 5}}$

and $\frac{1}{T_f T_v} = 0.02 \quad \therefore$

$$\frac{1}{T_f} = T_v \, 0.02 = 5 \times 0.02 = 0.1$$

$\therefore \underline{\underline{T_f = 10}}$

c). From (a),

$$\bar{v}(s) = \frac{T_v s \, (1 + T_f s)}{\frac{T_f T_v}{K} s^2 + T_f s + 1} \bar{Q}(s)$$

Now $\bar{\theta}(s) = \frac{-0.1}{s}$ so, ⑥

$$\bar{v}(s) = \frac{-0.1}{s} \cdot T_v s \frac{(1 + T_f s)}{\frac{T_f T_v}{K} s^2 + T_f s + 1}$$

$$\bar{v}(s) = \frac{-0.1 \, T_v (1 + T_f s)}{\frac{T_f T_v}{K} s^2 + T_f s + 1}$$

Sub for $K = 1$, $T_f = 10$ and $T_v = 5$,

$$\bar{v}(s) = \frac{-0.5 (1 + 10s)}{50s^2 + 10s + 1}$$

$$\bar{v}(s) = \frac{-0.5 (1 + 10s)}{50(s^2 + 0.2s + 0.02)}$$

$$\bar{v}(s) = \frac{-0.1 \, (s + 0.1)}{(s+0.1)^2 + 0.01}$$

From LT^{-1} tables, $a = 0.1$, $B = 0$, $A = 1$ and $w_0^2 = 0.01$ so,

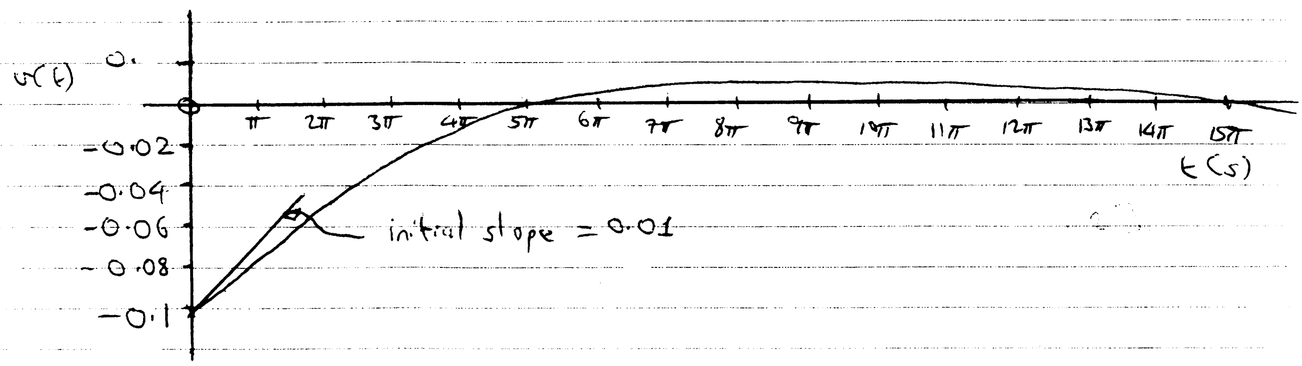
$$v(t) = -0.1 e^{-0.1t} \cos(0.1t)$$

$t = 0$, $v(0) = -0.1$
$t \to \infty$, $v = 0$

$v(t)$ has zero value when $0.1t = \frac{n\pi}{2}$ $n = \pm 1, \pm 3, \ldots$

$$t = 5n\pi$$

$n = 1$ $t = 5\pi$
$n = 3$ $t = 15\pi$



⑦

Now,

$$v(t) = -0.1 e^{-0.1t} \cos(0.1t)$$

$$\therefore \quad \frac{dv}{dt} = -0.1\left(e^{-0.1t}(-0.1 \sin 0.1t) + \cos(0.1t)(-0.1)e^{-0.1t}\right)$$

$$\frac{dv}{dt} = 0.01 e^{-0.1t}(\cos 0.1t + \sin 0.1t)$$

$$\frac{dv}{dt} = 0.01\sqrt{2}\, e^{-0.1t} \cos(0.1t - \pi/4)$$

At $t = 0$,

$$\frac{dv}{dt} = 0.01\sqrt{2} \cos -\pi/4$$

$$= 0.01\sqrt{2} \times \frac{1}{\sqrt{2}}$$

$$= \underline{\underline{0.01}}$$

⑧

3. a) The gain margin is a measure of how much the gain of the return ratio ($L(s) = K(s)G(s)$) can be increased before the closed-loop system becomes unstable

The phase margin is a measure of how much phase lag can be added to the return ratio before the closed-loop system becomes unstable.

(9)

b) From Bode plot,

- when Gain = 0dB, $\omega = 1.85\ rads^{-1}$
  gives $PM = 180° - 158° = \underline{\underline{22°}}$

<u>check</u> Using $G(j\omega)$, $\omega = 1.85$ gives Gain = 0.2 dB
Phase = $-157°$
$\therefore$ PM = $180 - 157 = 23°$

- when phase = 180°, $\omega = 4.3\ rads^{-1}$. At this freq, gain = -15 dB. So the GM is $\underline{\underline{15\ dB}}$

<u>check</u> Using $G(j\omega)$, $\omega = 4.3$ gives gain = -14dB
phase = $179°$
$\therefore$ GM = 14dB.

c) $$K(s) = \frac{6(5+s)}{30+s} = \frac{6\times5(1+\frac{1}{5}s)}{30(1+\frac{1}{30}s)} = \frac{(1+\frac{1}{5}s)}{(1+\frac{1}{30}s)}$$

Plot,

$(1+\frac{1}{5}s)$ and $\frac{1}{(1+\frac{1}{30}s)}$

corner freq = $5\ rads^{-1}$ corner freq = $30\ rads^{-1}$

At $\omega = 1.85\ rads^{-1}$ the compensator gain is $\approx$ 0dB. (actually = 0.5 dB). The phase of the compensator at this frequency is 16° when measured from the Bode plot. Consequently the phase margin rises from 22° to $22 + 16° = \underline{\underline{38°}}$

(10)

Over the range of interest (ie the combination of G(s) and K(s) gives a 180° the phase of K(s) is added to the phase of G(s) on the Bode (original) plot.

This time the 180° phase occurs at $\omega = 20 \text{ rads}^{-1}$.

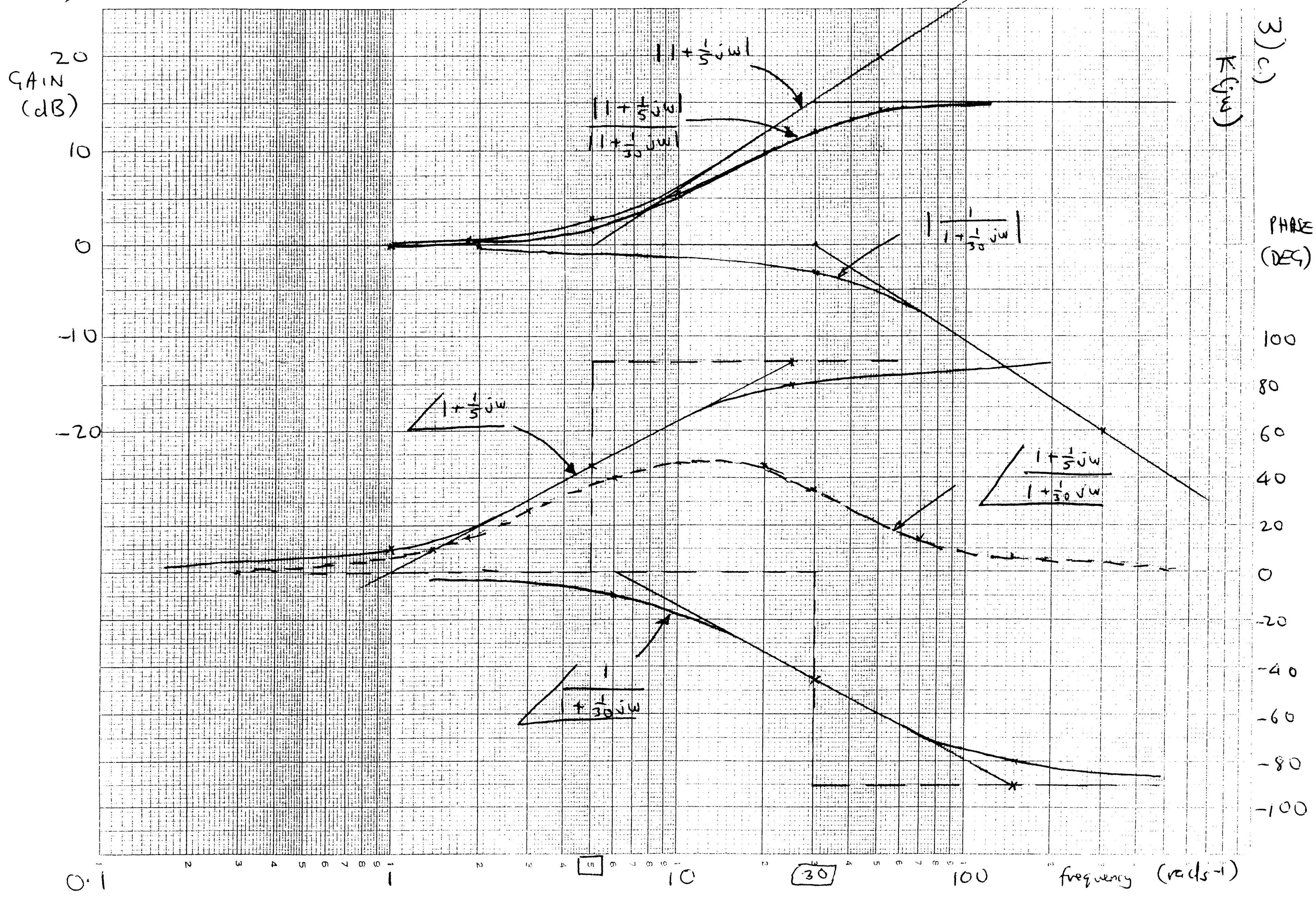
From the plots at this frequency the Gain of G(s) = -42.5dB
-43
and the gain of K(s) = 10 dB.
Thus the gain margin = $\underline{\underline{+32.5}}$ dB.

(Note calcs give $|G(j\omega)| = -43$dB and $|K(j\omega)| = 11$dB
∴ gain margin = 32dB).

So PM is raised from 22° → 33°
GM is raised from 14dB → 32.4dB
(5)

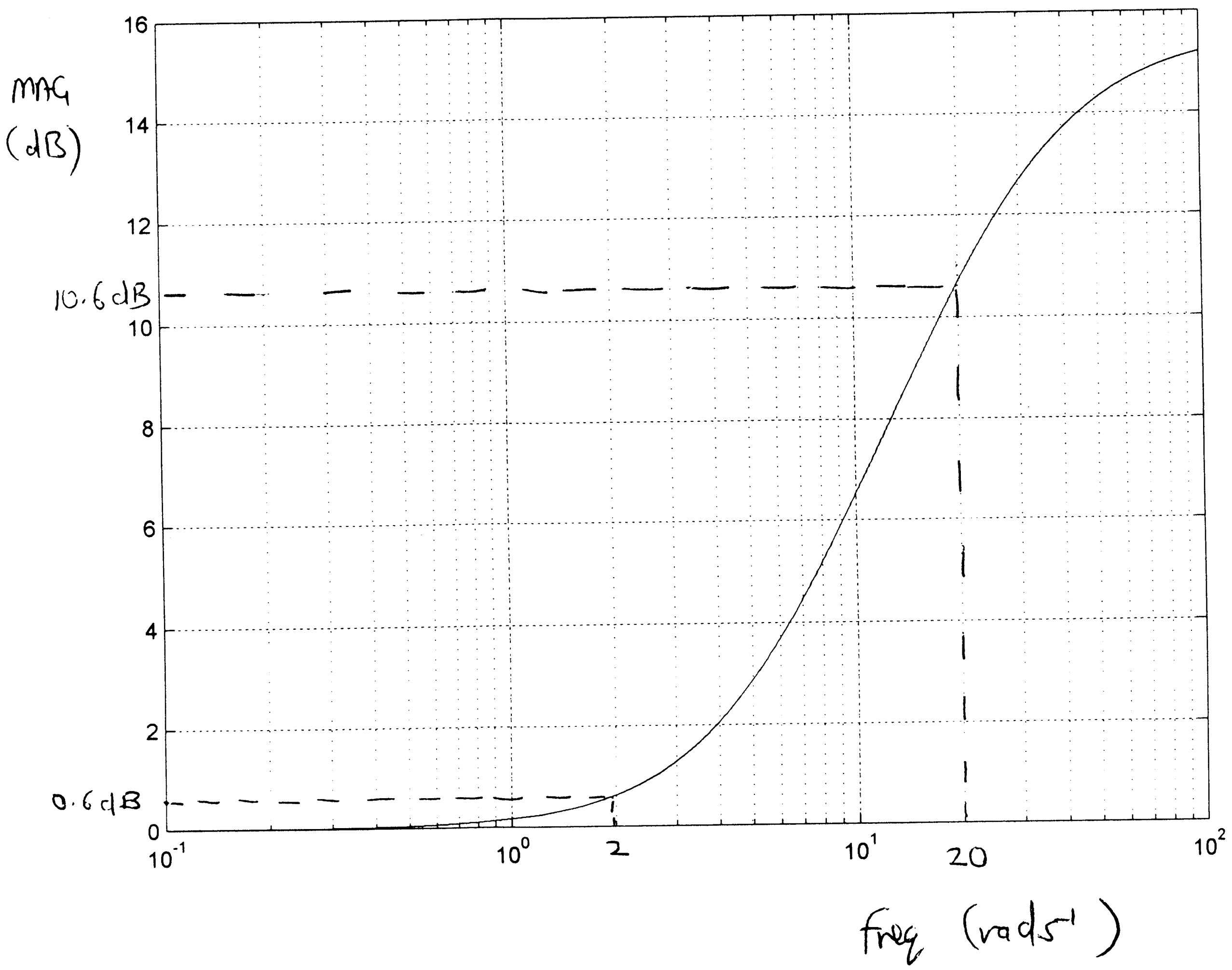
d) Consequently the damping is raised giving rise to less overshoot and a faster decaying output response to a step input.

3) c) K(jw)

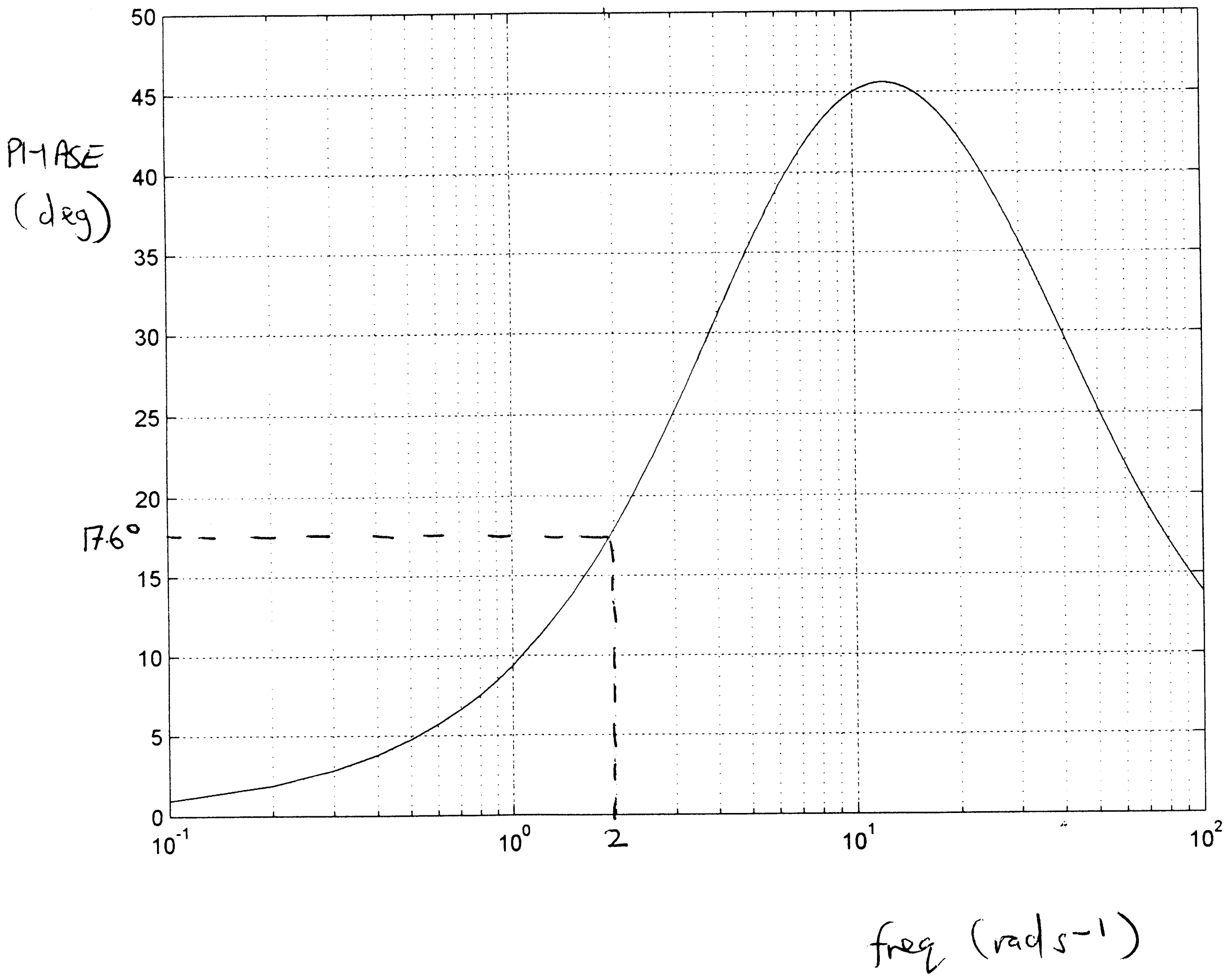

GAIN (dB)
20
10
0
-10
-20
PHASE (DEG)
100
80
60
40
20
0
-20
-40
-60
-80
-100
3) c.) K(jw)
$|1+\frac{1}{5}jw|$
$\frac{|1+\frac{1}{5}jw|}{|1+\frac{1}{30}jw|}$
$\left|\frac{1}{1+\frac{1}{30}jw}\right|$
$\angle 1+\frac{1}{5}jw$
$\angle \frac{1+\frac{1}{5}jw}{1+\frac{1}{30}jw}$
$\angle \frac{1}{1+\frac{1}{30}jw}$
0.1
1
5
10
30
100
frequency (rads^{-1})

3.) c) K(jw)

MATLAB generated plot

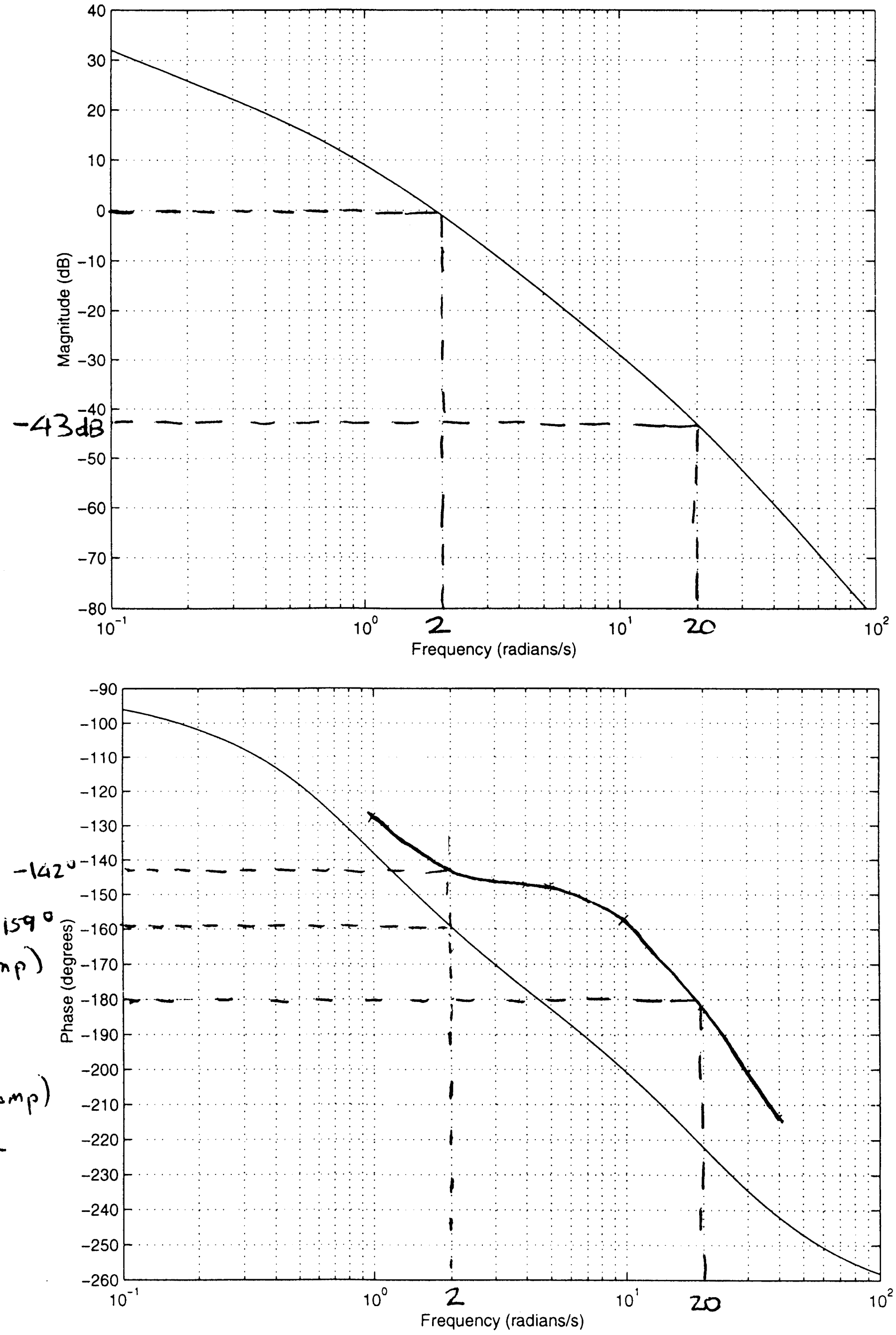


3) c) K(jw)
MATLAB GENERATED PLOT



ENGINEERING TRIPOS PART IB, Paper 6, Candidate number:

Extra copy of Fig.3 which may be annotated and handed in with your answer to Question 3.



GM = -43+10.6
= -32.6
i.e, 33dB

PM(no comp)
= 180-159
= 21°

PM(with comp)
= 180-142
= 38°

(15)

4. a) Since $G(s)$ is stable and $k=1$ then the feedback system (ie $\frac{G(s)}{1+G(s)}$) is asymptotically stable if and only if the -1 point is not encircled by the "full" Nyquist plot of $G(j\omega)$ (ie from $-\infty < \omega < \infty$).

b) $G(j\omega) = -0.28$ when $\omega = 1.484 \text{ rad s}^{-1}$

The limit of stability is at the -1 point.

$\therefore K_{upper} = \frac{-1}{-0.28} = 3.57$

$G(j\omega) = 0.58$ when $\omega = 0$.

$K_{lower} = \frac{-1}{0.58} = -1.72$

$\therefore -1.72 \leq K \leq 3.57$

When $K = 3.57$ then the system is marginally stable

So $s = 0 + j\,1.484$ since $1 + L(1.484j) = 0$
$= 0 - j\,1.484$ or $1 + L(s) = 0$ at $s = 1.484j$.

The closed-loop frequency of oscillation is 1.484 rad s^{-1}.

c) GM when $K = 1$ is $\frac{1}{0.28} = 3.57$

So GM $= \beta$ when $K = k_1$

$\therefore$ GM $= \beta \frac{k_1}{k_2}$ when $K = k_2$

$\beta = 3.57$ and $k_1 = 1$

want new GM $= 1.5$, ie

$1.5 = 3.57 \times \frac{1}{k_2}$

$\therefore k_2 = \frac{3.57}{1.5} = 2.38$

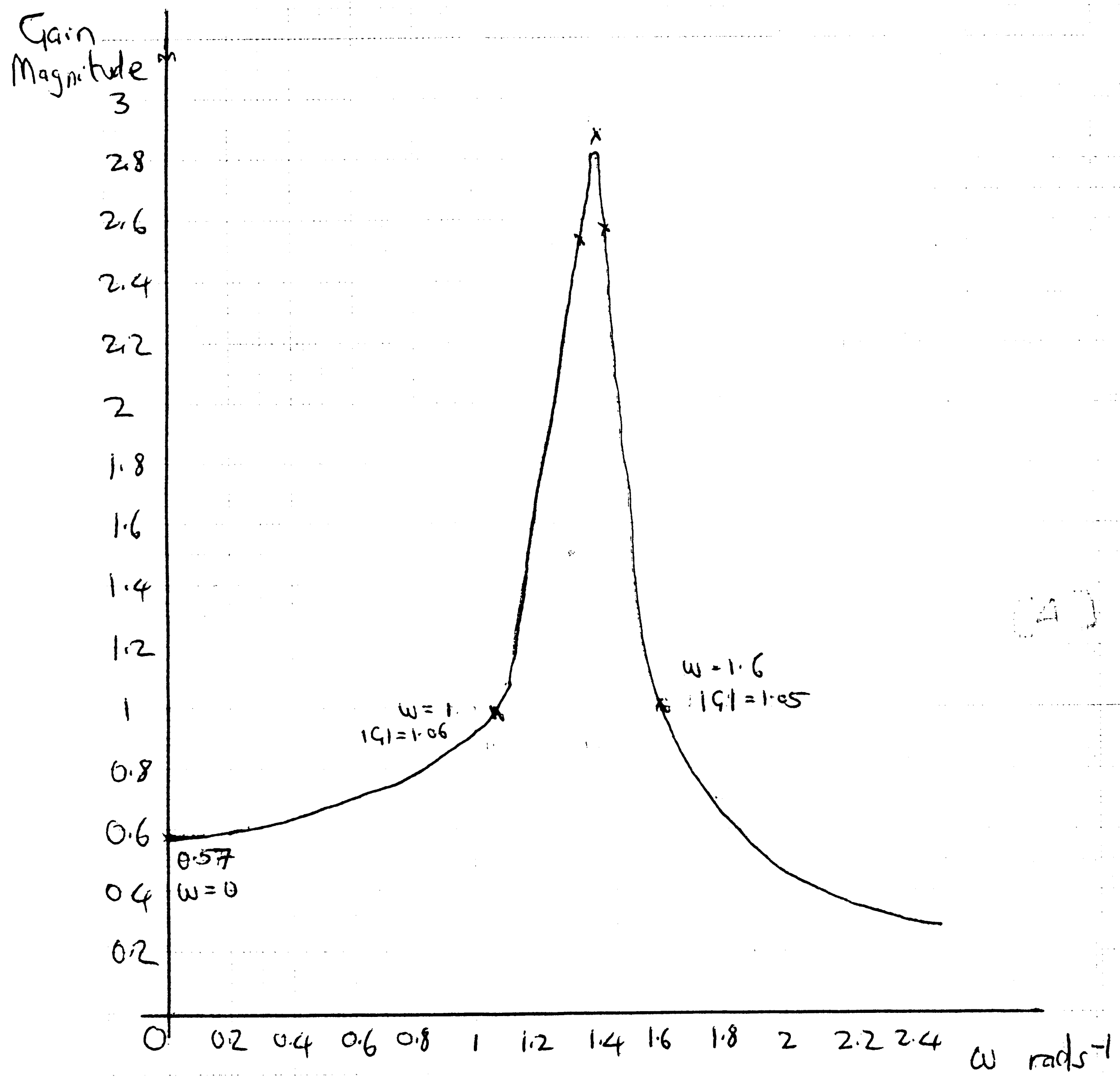
We need to move the -1 point.

Move to $\frac{-1}{2.38} = -0.42$

(in exam of 2)

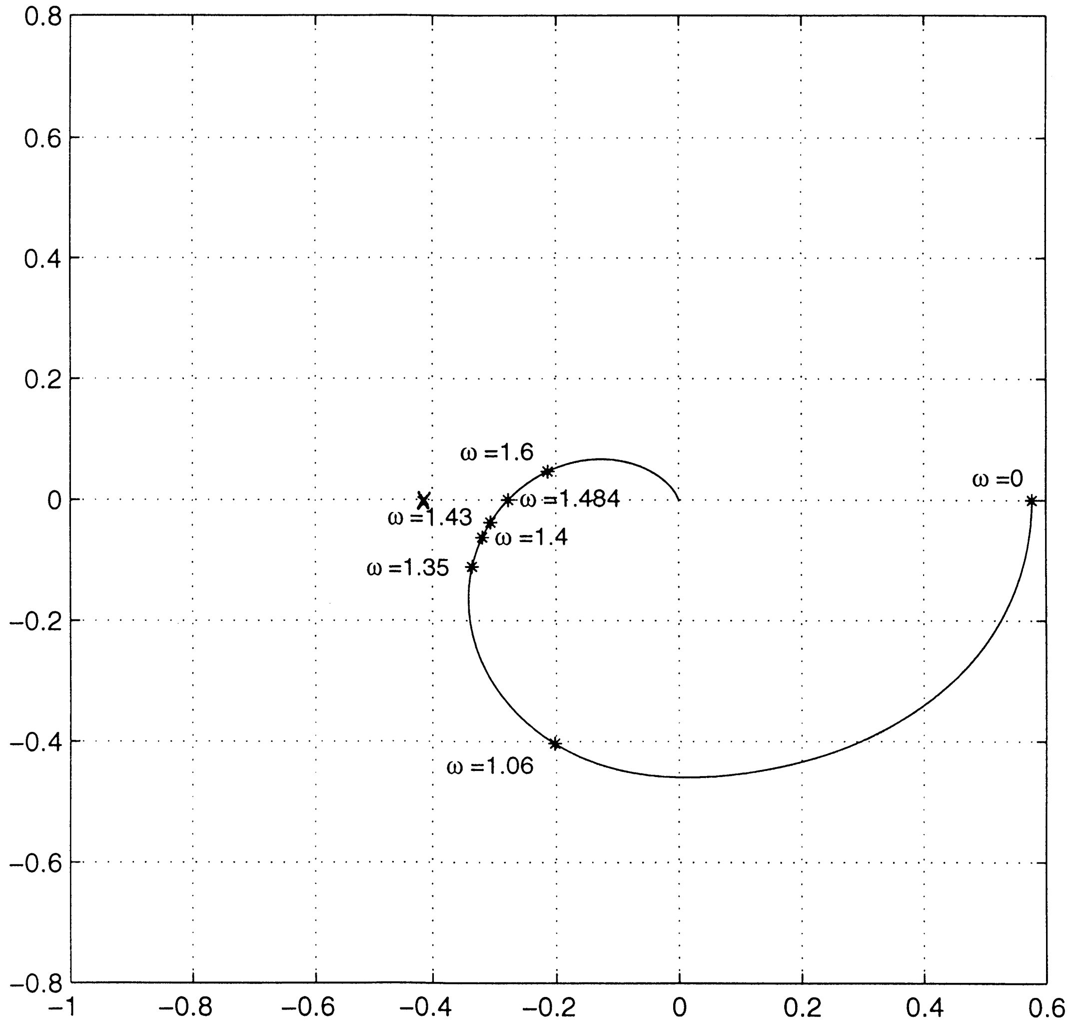
The peak in the magnitude of the frequency response at approx $\omega = 1.4$ rad/s indicated a low Gain margin and hence a decaying oscillatory output in response to a step input. The period of the time domain oscillation will be approximately $\frac{2\pi}{1.4} = 4.5$ s.

4.) c)


Gain Magnitude
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4
ω rads⁻¹
0.57
ω = 0
ω = 1.
|G| = 1.06
ω = 1.6
|G| = 1.05

ENGINEERING TRIPOS PART IB, Paper 6, Candidate number:

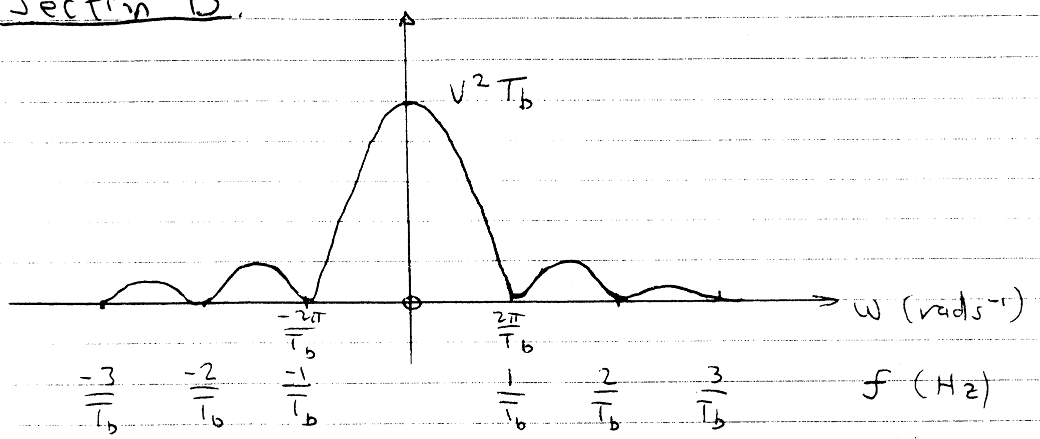
Extra copy of Fig.5 which may be annotated and handed in with your answer to Question 4



ω	MAG
0	51/89 = 0.57
1.06	41/41 = 1.0
1.35	32/12 = 2.6
1.4	29.5/10 = 2.9
1.43	28/10 = 2.8
1.484	25/12.5 = 2
1.6	20/19 = 1.05

Section B

5. a)



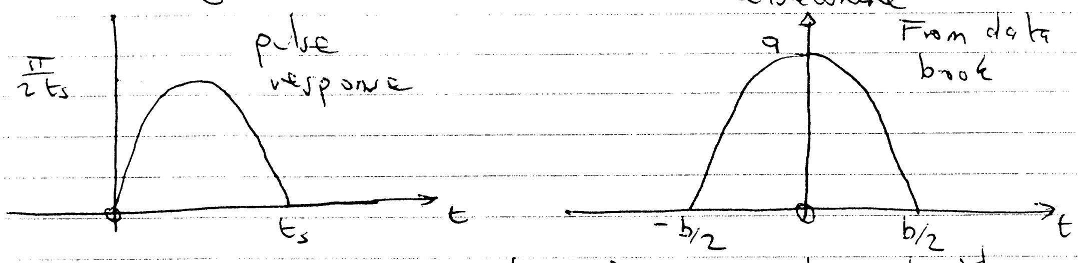
The second zero occurs at $2/T_b$ Hz.

Now $T_b = \frac{1}{2400}$, so,

Required bandwidth, $B = \frac{2}{T_b} = \frac{2}{\left(\frac{1}{2400}\right)} = 4800\,\text{Hz}$.

b) Now,

$$h(t) = \frac{\pi}{2t_s} \sin \frac{\pi t}{t_s} \qquad 0 < t < t_s$$
$$= 0 \qquad \text{elsewhere}$$



This is time shifted (by $t_s/2$) compared with the half-sine pulse in the data book. A time shift however is equivalent to a phase shift of $e^{-j\omega b/2}$ in the frequency domain which has no effect on the magnitude of the frequency response $|H(f)|$.

From the data book and comparing with our pulse,

$$a = \frac{\pi}{2t_s} \qquad \text{and} \qquad b = t_s$$

Consequently we can write,

$$|H(\omega)| = \frac{ab}{2}\left[\operatorname{sinc}\left(\frac{\omega b - \pi}{2}\right) + \operatorname{sinc}\left(\frac{\omega b + \pi}{2}\right)\right]$$

$$|H(\omega)| = \frac{1}{2}\cdot\frac{\pi}{2t_s}\cdot t_s\left[\operatorname{sinc}\left(\frac{\omega t_s}{2} - \pi\right) + \operatorname{sinc}\left(\frac{\omega t_s}{2} + \pi\right)\right]$$

$$= \frac{\pi}{4}\left[\operatorname{sinc}\left(\frac{\omega t_s - \pi}{2}\right) + \operatorname{sinc}\left(\frac{\omega t_s + \pi}{2}\right)\right]$$

c) Look first at

$$\operatorname{sinc}\left(\frac{\omega t_s - \pi}{2}\right)$$

Will have an amplitude of 1 when $\omega_m t_s = \pi$

ie, $\omega_m = \frac{\pi}{t_s}$ as, $\omega_m = 2\pi f_m$ then $f_m = \frac{1}{2t_s}$

Zero amplitude will occur when $\sin(\omega t_s - \pi) = 0$

ie, when $\left(\frac{\omega_z t_s - \pi}{2}\right) = n\pi$ $\quad n = \pm 1, \pm 2, \pm 3, \ldots$

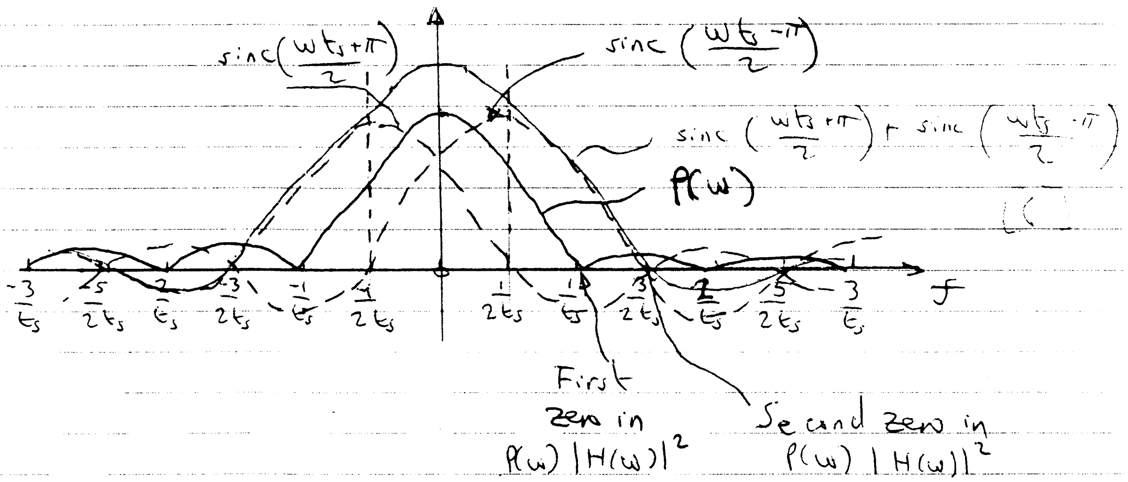
$\therefore f_z = \frac{-5}{2t_s}, \frac{-3}{2t_s}, \frac{-1}{2t_s}, \frac{3}{2t_s}, \frac{5}{2t_s}, \frac{7}{2t_s}$ etc.

Similarly for $\operatorname{sinc}\left(\frac{\omega t_s + \pi}{2}\right)$

amplitude of 1 when $f_m = \frac{-1}{2t_s}$

zero amplitudes when $\left(\frac{\omega_z t_s + \pi}{2}\right) = n\pi$ $\quad n = \pm 1, \pm 2, \ldots$

ie $f_z = \frac{-7}{2t_s}, \frac{-5}{2t_s}, \frac{-3}{2t_s}, \frac{1}{2t_s}, \frac{3}{2t_s}, \frac{5}{2t_s}$ etc.



(21)

See that the second zero in $P(\omega)|H(\omega)|^2$ occurs when

$$f = \frac{3}{2t_s}$$

From a) $f = B = 4800\,Hz$

so

$$t_s = \frac{3}{2f} = \frac{3}{2B}$$

$$\text{Bit rate} = \frac{1}{t_s} = \frac{2B}{3} = \frac{2 \times 4800}{3} = \underline{\underline{3200\ b/s.}}$$

The original rate was 2400 b/s.

4

6. a) $$s(t) = [a_0 + x(t)] \cos w_c t$$

now if $$x(t) = a_x \cos w_m t$$

$$s(t) = [a_0 + a_x \cos w_m t] \cos w_c t$$

$\therefore$ $$s(t) = a_0 \cos w_c t + a_x \cos w_m t \cos w_c t$$

$$s(t) = a_0 \cos w_c t + \frac{a_x}{2}\left[\cos(w_c + w_m)t + \cos(w_c - w_m)t\right]$$

$$s(t) = \underbrace{a_0 \cos w_c t}_{\text{carrier}} + \underbrace{\frac{a_x}{2}\cos(w_c + w_m)t}_{\text{upper sideband}} + \underbrace{\frac{a_x}{2}\cos(w_c - w_m)t}_{\text{lower sideband}}$$

b) power efficiency = $\dfrac{\text{information power}}{\text{total power}}$

Information power = power in both sidebands in $s(t)$.

$$= \left(\frac{a_x}{2}\cdot\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{a_x}{2}\cdot\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{a_x^2}{8} + \frac{a_x^2}{8}$$

$$= \frac{a_x^2}{4} \quad \text{(assuming unit impedance)}$$

carrier power $= \left(\dfrac{a_0}{\sqrt{2}}\right)^2 = \dfrac{a_0^2}{2}$

$\therefore$ $$\text{Efficiency} = \frac{\frac{a_x^2}{4}}{\frac{a_0^2}{2} + \frac{a_x^2}{4}} = \frac{a_x^2}{2a_0^2 + a_x^2}$$

$\div$ through by a_0^2,

$$\text{efficiency} = \frac{a_x^2/a_0^2}{\frac{2a_0^2}{a_0^2} + \frac{a_x^2}{a_0^2}}$$

now, $m_A = a_x/a_0$ so,

$$\text{efficiency} = \frac{m_A^2}{2 + m_A^2}$$

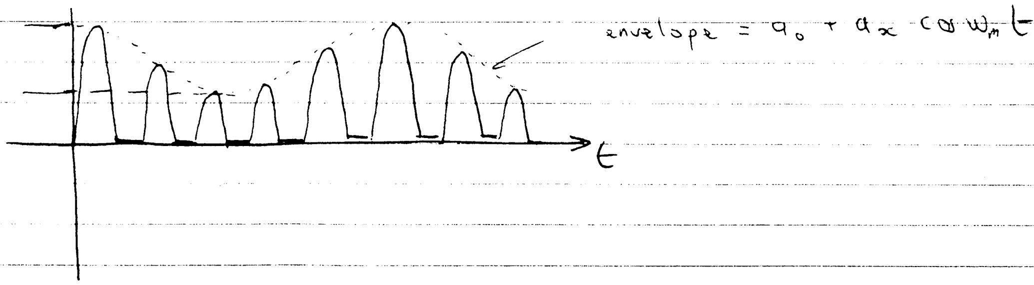
For envelope demodulated DSB-AM, the max possible modulation index, $m_A = 1$.

So the maximum efficiency in this case is,

$$\text{efficiency}_{MAX} = \frac{1}{2+1} = \frac{1}{3}$$

ie. not very good.

c) The signal $u(t)$ has the form



ie, a ½ wave rectified version of $s(t)$.

We can create a signal which looks like this by multiplying a ½ wave rectified sinusoid $g(t)$ by $a_0 + a_x \cos w_m t$

ie

$$u(t) = g(t)\left[a_0 + a_x \cos w_m t\right]$$

From E + I data book,

$$g(t) = \frac{1}{\pi} + \frac{1}{2}\cos w_c t + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n w_c t)}{4n^2-1}$$

So,

$$u(t) = \left[a_0 + a_x \cos w_m t\right]\left[\frac{1}{\pi} + \frac{1}{2}\cos w_c t + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n w_c t)}{4n^2-1}\right]$$

(24)

Clearly terms multiplied by a_0 have no information component.

The term $a_x \cos w_m t \times \frac{1}{2} \cos w_c t$ can be expressed as

$$\frac{a_x}{2}\left[\cos(w_c + w_m)t + \cos(w_c - w_m)t\right]$$

These components are near w_c ($w_c >> w_m$) and will be eliminated by the following R-C filter.

Similar comments apply for terms of form

$$a_x \cos w_m t \times \cos(2n\, w_c t).$$

Thus the only information bearing component in $y(t)$

is $\underline{\dfrac{a_x}{\pi} \cos w_m t}$

which is clearly $\propto \cos w_m t$ (the information).

As we have seen the RC filter must eliminate high frequency components in excess of $f_m = 5\,kHz$

For an RC 1st order filter, the corner frequency is given by

$$f_1 = \frac{1}{2\pi RC}$$

$$\therefore \quad RC = \frac{1}{2\pi f_1}$$

let $f_1 = f_m = 5\,kHz$

$$RC = \frac{1}{2\pi \times 5 \times 10^3} \leq \underline{31 \cdot 8\,\mu s}.$$