ENGINEERING TRIPOS PART IB

Friday 7 June 2002 9 to 11

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer at least **one** *question from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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SECTION A

Answer at least ***one*** *question from this section*

1 (a) Consider the volume integral $\int yz\,dV$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. The tetrahedron is shown in Fig 1. Show that the integral can be written as

$$\int_{z=0}^{1}\left[\int_{y=0}^{1-z}\left(\int_{x=0}^{1-y-z} yz\,dx\right)dy\right]dz$$

and hence evaluate the integral. [10]

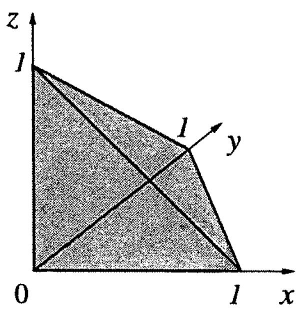

Fig. 1

(b) Consider the flux integral $\int_S \mathbf{J}.\mathbf{dA}$, where $\mathbf{J} = xy(\mathbf{i}+\mathbf{k})$ and S is the quarter circle in the x-y plane shown in Fig. 2. By changing to a polar coordinate system centred at $(x, y) = (1, 1)$, show that the integral can be written as

$$\int_{\theta=\theta_1}^{\theta_2}\int_{r=0}^{1}(1 + r\cos\theta)(1 + r\sin\theta)\,r\,dr\,d\theta$$

for suitable values of θ_1 and θ_2. Hence evaluate the integral. [10]

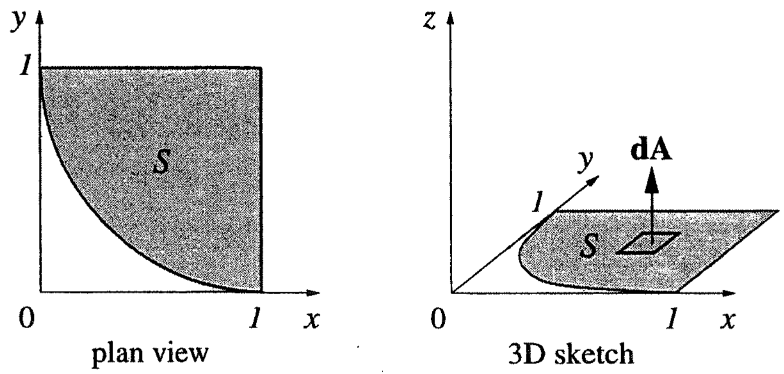

Fig. 2

2 (a) Explain what is meant by the *scalar potential* of an *irrotational* vector field. By expanding in Cartesian coordinates, prove the identity $\nabla \times (\nabla\phi) = \mathbf{0}$ for any differentiable scalar field ϕ. [6]

(b) Consider the vector field

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2} .$$

(i) Show that $\mathbf{F}$ can be written in cylindrical polar coordinates as $\mathbf{F} = r^n\mathbf{e}_r$, where $\mathbf{e}_r$ is the unit vector in the radial direction (see Fig. 3). Find n. [2]

(ii) Show that $\mathbf{F}$ is irrotational. Find the scalar potential of $\mathbf{F}$ and sketch the family of equipotential surfaces. [5]

(iii) Evaluate the line integral $\int \mathbf{F}.\mathbf{dl}$ along the curve $x = 1 + t^2$, $y = t$, $z = \sin t$, starting at the point $t = 0$ and finishing at the point $t = 1$. [2]

(c) In cylindrical polar coordinates, show that there is only one value of n for which the field $r^n\mathbf{e}_r$ is solenoidal (except for a singularity at the z-axis). Use this result to evaluate the flux integral $\oint_S \mathbf{F}.\mathbf{dA}$, where $\mathbf{F}$ is given in (b) and S is the cube bounded by the planes $|x| = 1$, $|y| = 1$ and $|z| = 1$. [5]

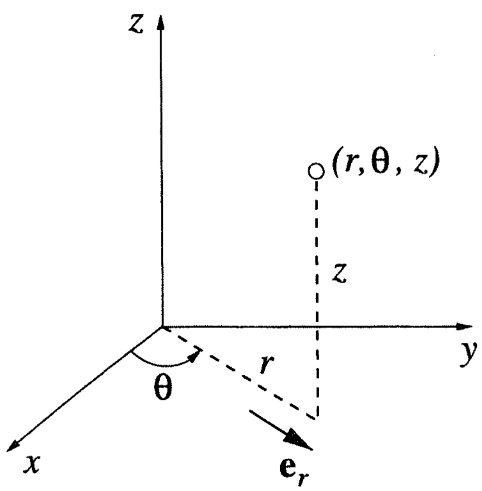

Fig. 3

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3 Consider a long, thin bar of constant cross-section and homogeneous composition. The bar is oriented along the x-axis and is perfectly insulated laterally. The temperature $T(x,t)$ along such a bar satisfies the one-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} ,$$

where α is the thermal diffusivity. The bar is insulated at its ends $x = 0$ and $x = L$.

(a) Explain why $\partial T/\partial x = 0$ at $x = 0$ and $x = L$. [2]

(b) Use the method of separation of variables to show that

$$T_n = A_n \cos \frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

is a solution of the diffusion equation for any integer n. Deduce an expression for λ_n in terms of n, α and L, and explain why the general solution for T is

$$T(x,t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-\lambda_n^2 t} .$$ [10]

(c) If the initial temperature distribution $T(x,0)$ is as in Fig. 4, find the coefficients A_0, A_1 and A_2 (you may wish to differentiate the Fourier series for a triangular wave in the electrical data book). Sketch $T(x,t)$ for the case $t \to \infty$ and also for a representative, intermediate value of t between 0 and ∞. [8]

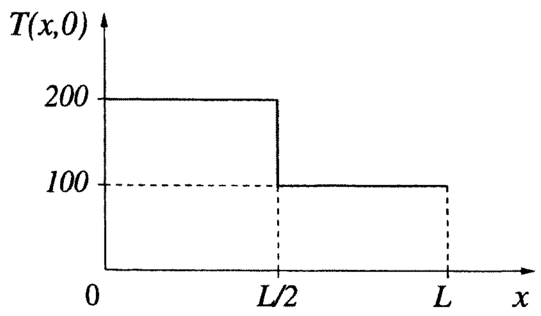

Fig. 4

SECTION B

*Answer at least **one** question from this section*

4 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 4 \\ 6 & 2 & 1 \end{bmatrix} .$$

(a) Compute the $\mathbf{LU}$ decomposition of $\mathbf{A}$. What is the rank of $\mathbf{A}$? [4]

(b) How are the dimensions of the four fundamental subspaces related to a matrix's rank, number of rows and number of columns? Write down the dimensions of the four fundamental subspaces of $\mathbf{A}$. [4]

(c) Find a basis for each of the four fundamental subspaces of $\mathbf{A}$. [6]

(d) Consider the system of equations

$$\mathbf{A}^T\mathbf{x} = \begin{bmatrix} -3 \\ 7 \\ 4 \end{bmatrix} .$$

If the least squares solution is $\bar{\mathbf{x}}$, find $\mathbf{A}^T\bar{\mathbf{x}}$. Hence find $\bar{\mathbf{x}}$. [6]

5 (a) An $n \times n$ matrix $\mathbf{A}$ has eigenvalues $\lambda_1 \ldots \lambda_n$ and corresponding unit eigenvectors $\mathbf{u}_1 \ldots \mathbf{u}_n$. Show that $\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^{-1}$, where Λ is the diagonal matrix whose elements are $\lambda_1 \ldots \lambda_n$, and $\mathbf{U}$ is the matrix whose columns are $\mathbf{u}_1 \ldots \mathbf{u}_n$. [4]

(b) Consider the following system of linear difference equations:

$$y_{k+1} = 0.7y_k + 0.1z_k \qquad y_0 = 0$$

$$z_{k+1} = 0.3y_k + 0.9z_k \qquad z_0 = 3$$

(i) Write the equations in the form $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$, where $\mathbf{x}_k = [\ y_k \ \ z_k\ ]^T$. [2]

(ii) Show that the solution of the equations is $\mathbf{x}_k = \mathbf{U}\Lambda^k\mathbf{U}^{-1}\mathbf{x}_0$, where the matrices $\mathbf{U}$ and Λ are defined in (a) above. [2]

(iii) Calculate the eigenvalues and unit eigenvectors of $\mathbf{A}$. Hence deduce the limiting value of $\mathbf{x}_k$ as $k \to \infty$. [8]

(c) Now consider the difference equation $\mathbf{x}_{k+1} = \mathbf{A}^{-1}\mathbf{x}_k$, where $\mathbf{A}$ is the matrix derived in (b) above. Describe qualitatively what happens to $\mathbf{x}_k$ as $k \to \infty$. [4]

SECTION C

*Answer at least **one** question from this section*

6 A bandlimited signal $x(t)$ has Fourier transform $X(\omega)$, where $X(\omega) = 0$ for $|\omega| \geq \omega_m$. $x(t)$ is sampled by the impulse train $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$.

(a) What is the maximum value of T if aliasing is to be avoided? For this limiting value of T, sketch the spectrum of the sampled signal $x_s(t) = s(t)x(t)$. Assume an arbitrary, bandlimited form for $X(\omega)$. [4]

(b) Show that the Fourier transform of $x_s(t)$ can be written as

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-i\omega nT} .$$

Assuming the limiting value of T in (a), write down the frequency response $H(\omega)$ of an ideal filter that can be used to recover $x(t)$ from $x_s(t)$. Now take the inverse Fourier transform of $H(\omega)X_s(\omega)$ to show that $x(t)$ can be reconstructed as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\frac{\pi(t - nT)}{T} .$$ [8]

(c) Explain why the perfect signal reconstruction technique in (b) is impractical for real-time digital-to-analogue conversion applications. Consider the two approximate reconstructions $x_0(t)$ and $x_1(t)$ in Fig. 5. Sketch the impulse and frequency responses of filters that can be used to recover $x_0(t)$ and $x_1(t)$ from $x_s(t)$. [8]

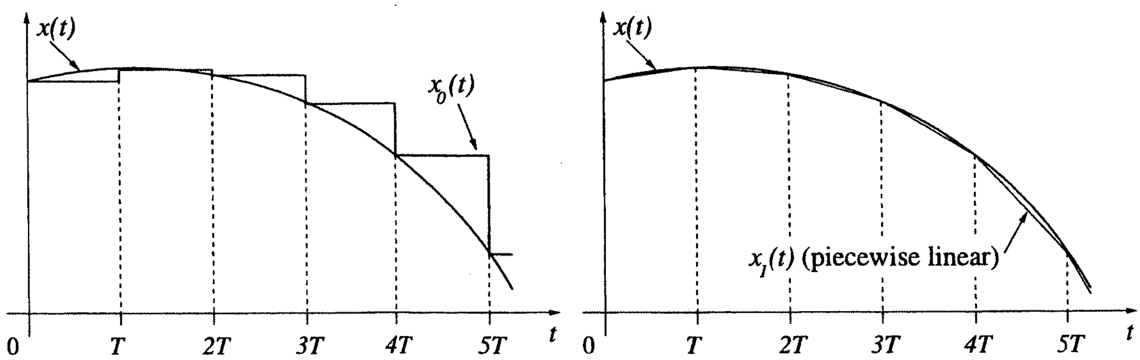

Fig. 5

7 (a) Consider the finite duration signal

$$x(t) = \begin{cases} \cos \omega_0 t & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} .$$

(i) Calculate $X(\omega)$, the Fourier transform of $x(t)$. [7]

(ii) Sketch $|X(\omega)|$ for the two cases T large and T small. [3]

(b) Two independent, continuous random variables X and Y are uniformly distributed over the ranges $0 \le x \le 1$ and $0 \le y \le 2$ respectively. A third random variable Z is defined by $Z = X - Y$.

(i) Calculate and sketch the probability density function of Z. Verify that you have arrived at a valid probability density function. [8]

(ii) Hence, or otherwise, calculate the expected value of Z. [2]

8 (a) There are 5 faulty items in a batch of 500. The distributor checks a random sample of 10 items and ships the batch only if none of the 10 are faulty.

(i) Discuss whether the number of faulty items in a sample of 10 is well modelled by the Binomial distribution. [4]

(ii) Find the exact probability that the batch is shipped. [4]

(iii) Repeat (ii) using a suitable Binomial approximation. [2]

(b) My doctor makes appointments at regular 15 minute intervals from 9am to 6pm, with no breaks. In practice, the time taken for each consultation is uniformly distributed between 12 and 19 minutes. I am trying to decide how long I might have to spend in the waiting room if I make an appointment at different times of the day.

(i) State the *central limit theorem* and explain its relevance to this problem. [3]

(ii) Find the probability that I need to wait for more than two minutes if I have the 9.15am appointment. [2]

(iii) Repeat (ii) for the 5.30pm appointment. State any assumptions you make. [5]

END OF PAPER