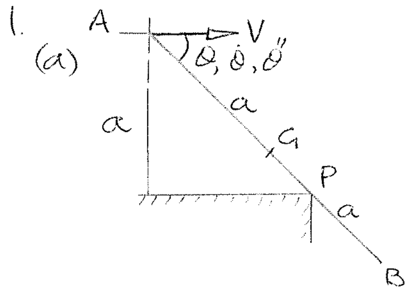
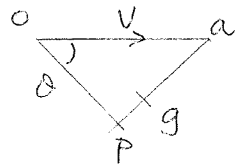
1.

(a)





velocity diagram

from velocity diagram

$ap = V\sin\theta = AP\,\dot\theta$

But $AP\sin\theta = a$

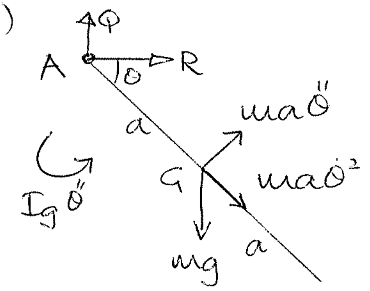
$\therefore \quad V\sin\theta = \dfrac{a\dot\theta}{\sin\theta}$

i.e. $\dot\theta = \dfrac{V}{a}\sin^2\theta$ [8]

Diff. wrt time $\quad \ddot\theta = \dfrac{2V}{a}\sin\theta.\cos\theta.\dot\theta$

i.e. $\ddot\theta = \dfrac{2V^2}{a^2}\sin^3\theta\cos\theta$

(b)



From F.B. Diagram, moments about A:

$mga\cos\theta = ma^2\ddot\theta + I_g\ddot\theta$

if contact just lost at P.

$I_g = \dfrac{ma^2}{3}$

$\cancel{m}ga\cancel{\cos\theta} = \dfrac{4}{3}\cancel{m}a^2 . \dfrac{2V^2}{\cancel{a^2}}\sin^3\theta\cancel{\cos\theta}$

i.e. $\sin\theta = \sqrt[3]{\dfrac{3ga}{8V^2}}$ [8]

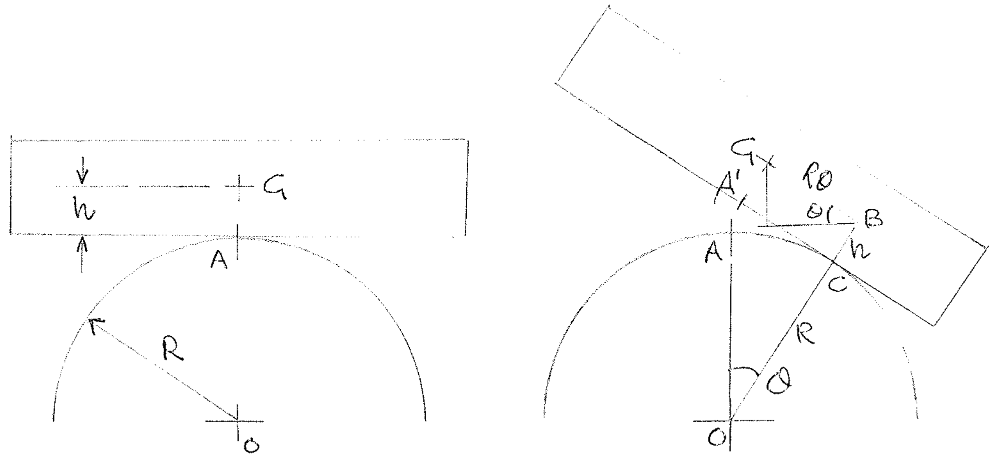
(c)

Resolving, $\quad R + ma\ddot\theta\sin\theta + ma\dot\theta^2\cos\theta = 0$

$R = -ma.\dfrac{2V^2}{a^2}\sin^4\theta\cos\theta - \dfrac{maV^2}{a^2}\sin^4\theta\cos\theta$

$\Rightarrow \quad R = -\dfrac{9}{8}mg\sin\theta\cos\theta$ [4]

2.



(a)

displaced position

$CA = CA'$

i.e. $\Rightarrow BG = R\theta$ no slip.

pe, $V = (R+h)mg$

ke $= 0$

pe. $= [(R+h)\cos\theta + R\theta\sin\theta]mg$

ke $= \frac{1}{2}m(CG\dot{\theta})^2 + I_g\dot{\theta}^2$

$$\frac{V(\theta)}{mg} = (R+h)\cos\theta + R\theta\sin\theta$$

$$\therefore \frac{V'(\theta)}{mg} = -(R+h)\sin\theta + R\sin\theta + R\theta\cos\theta$$

$V'(\theta) \Rightarrow 0$ if $\quad -h\sin\theta + R\theta\cos\theta = 0$

i.e. if $\theta = 0$, $V'(0) \Rightarrow 0$

$$\frac{V''(\theta)}{mg} = -(R+h)\cos\theta + R\cos\theta + R\cos\theta - R\theta\sin\theta$$

when $\theta = 0$ $\quad \frac{V''(\theta)}{mg} = -h + R$ $\quad$ so if $V(\theta) > 0$ $R > h$ [8]

(b)

ke. $= \frac{1}{2}m(h^2 + R^2\theta^2)\dot{\theta}^2 + \frac{1}{2}mk^2\dot{\theta}^2$

By conservation of energy

$$[(R+h)\cos\theta + R\theta\sin\theta]mg + \frac{1}{2}m(h^2 + R^2\theta^2)\dot{\theta}^2 + \frac{1}{2}mk^2\dot{\theta}^2 = \text{cons}$$

$\therefore$ Diff. wrt time

$[-(R+h)\sin\theta.\dot\theta + R\dot\theta\sin\theta + R\theta\dot\theta\cos\theta]mg$

$+ m(h^2+R^2\theta^2)\dot\theta\ddot\theta + \frac{1}{2}m.2R^2\theta\dot\theta^3 + mk^2\dot\theta\ddot\theta = 0$

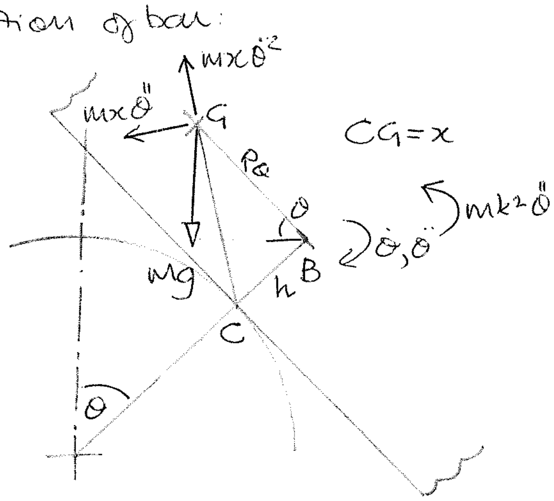
cancelling m, $\dot\theta$, and putting θ small, so $\theta^2 \ll 1$

$[-(R+h)\theta + R\theta + R\theta]g + h^2\ddot\theta + k^2\ddot\theta = 0$

ie. $\ddot\theta = -\frac{(R-h)g}{h^2+k^2}.\theta$

SHM with $\omega_n = \sqrt{\frac{(R-h)g}{h^2+k^2}}$ [12]

OR Consider equation of motion of ball:



Take moments about C

$mg(R\theta\cos\theta - h\sin\theta)$

$+ mx^2\ddot\theta + mk^2\ddot\theta = 0$

But $x^2 = R^2\theta^2 + h^2$
and θ small

ie. $mg(R\theta - h\theta)$

$+ m(R^2\theta^2+h^2)\ddot\theta + mk^2\ddot\theta = 0$

ie. $\ddot\theta = -\frac{(R-h)g}{h^2+k^2}\theta$ as before.

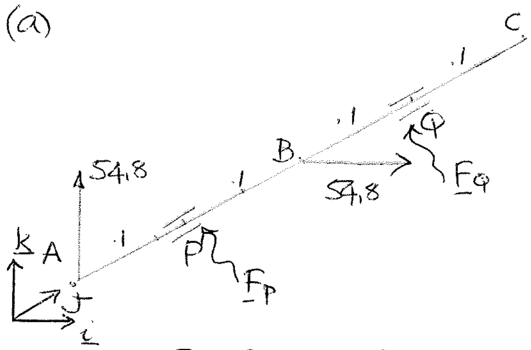
OR Direct application of data book formula

$V''(\theta_0) = (R-h)mg$

$ke = \frac{1}{2}m(h^2+R^2\theta^2+k^2)\dot\theta^2$; $\therefore I(\theta_0) = m(h^2+k^2)$

$\therefore \omega_n^2 = \frac{(R-h)g}{h^2+k^2}$

3(a)



Discs A & B each produce a dynamic force of magnitude

$$m r \omega^2 = 1 \times .05 \times 10^{-3} \times \left(\frac{10^4 \times 2\pi}{60}\right)^2$$

$$= 54.8 \text{ N}$$

To find F_Q take moments about bearing P: using (i, j, k) unit vectors indicated

$$-.1 j \times 54.8 k + .1 j \times 54.8 i + .2 j \times F_Q = 0$$

$$\text{i.e.} \quad -5.48 i - 5.48 k + .2 j \times F_Q = 0$$

$$\text{i.e.} \quad j \times F_Q = 27.4 i + 27.4 k$$

If $F_Q = F_1 i + F_2 k$, $j \times F_Q = F_2 i - F_1 k$

$$\therefore \quad F_Q = -27.4 i + 27.4 k \qquad F_Q = \sqrt{2} \times 27.4 \text{ N}$$

$$= 38.7 \text{ N}$$

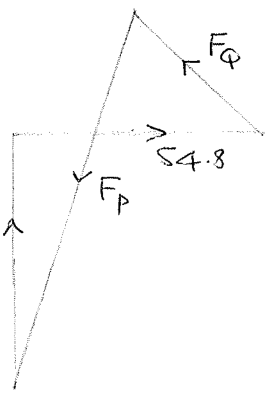
To find F_P we could take moments about bearing Q, or use force equilibrium

$$F_P + F_Q + 54.8 i + 54.8 k = 0$$

$$\therefore \quad F_P = -54.8 i - 54.8 k + 27.4 i - 27.4 k$$

$$F_P = -27.4 i - 82.2 k \qquad F_P = 86.6 \text{ N}$$

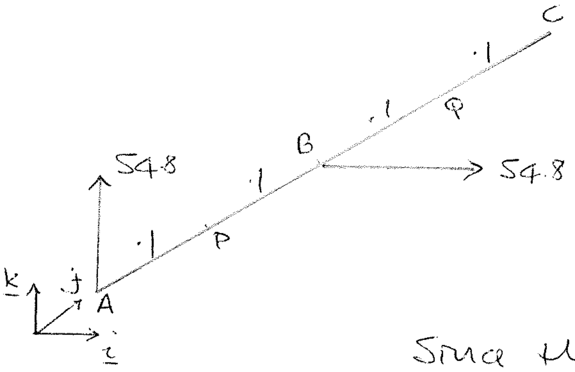
Looking along shaft dynamic force polygon looks like this.



Examiner gave an extra mark to candidates who pointed out that bearings must also support the weight of the shaft + discs which must be of the order of 40 N in total,

54.8

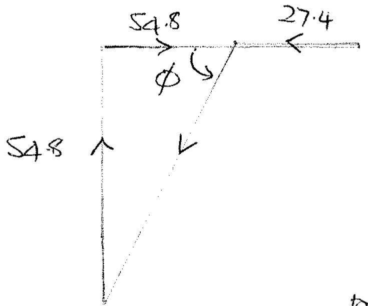
(b) If shaft is both statically & dynamically balanced then there are no dynamic forces generated at the bearings.



By inspection, taking moments about A the balancing mass at C must generate a force to balance the moment produced by $54.8\underline{i}$ acting at B.

Since the balancing mass is added at rim of disc, at radius 50mm, the required mass

is $1/(2 \times 50/.05)$ kg $= 5 \times 10^{-4}$ kg
i.e. ·5 gm.



and generates a force of $-27.4\underline{i}$ N.

To keep static balance, the force polygon must close, as shown, so required mass at A provides force of $\sqrt{27.4^2 + 54.8^2}$ is

i.e. 61.3 N and so

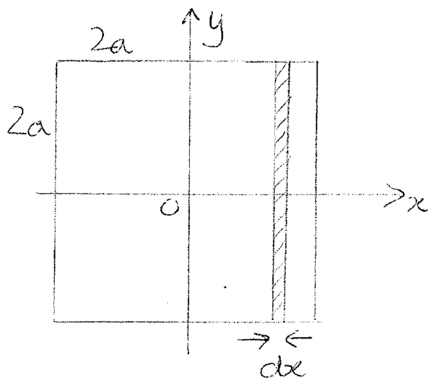
is of magnitude $\frac{61.3}{27.4} \times 5$

$= 1.12$ gm

at angle $\phi = 63.4°$

i.e. $153.4°$ to out of balance at A.

4.



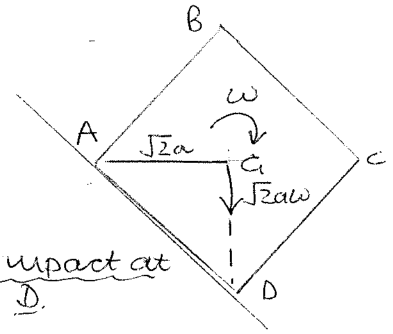
$$dI_{xx} = \frac{1}{12} \cdot \frac{2a\,dx}{4a^2} \cdot m \cdot (2a)^2 = \frac{ma\,dx}{6}$$

$$\therefore I_{xx} = \frac{ma}{6}\int_{-a}^{a} dx = \frac{ma^2}{3}$$

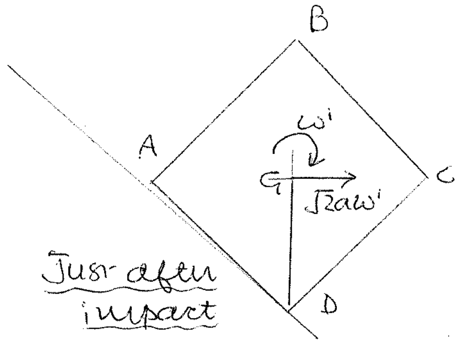
But $I_{zz} = I_{xx} + I_{yy}$ & $I_{xx} = I_{yy}$

$$\therefore I_{zz} = \frac{2ma^2}{3}$$

[6]



Before impact at D.



Just after impact

Moment of mtm conserved about D during impact

$$\therefore \quad I_g \cdot \omega = I_g \omega' + \sqrt{2}a\omega' \cdot m\sqrt{2}a$$

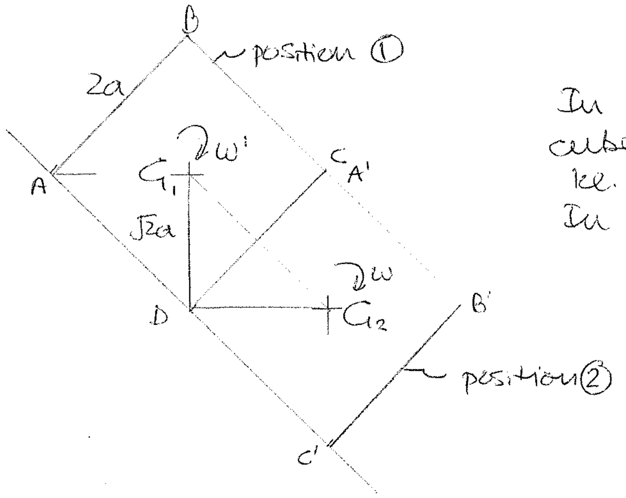
$$\tfrac{2}{3}ma^2\omega = \tfrac{2}{3}ma^2\omega' + 2ma^2\omega'$$

$$\therefore \omega' = \omega/4$$

$$\text{Loss of ke} = \left\{\tfrac{1}{2}I_g\omega^2 + \tfrac{1}{2}m(\sqrt{2}a)^2\omega^2\right\} - \left\{\tfrac{1}{2}I_g\omega' + \tfrac{1}{2}m(\sqrt{2}a)^2\omega'^2\right\}$$

$$= \tfrac{4}{3}ma^2\omega^2 - \tfrac{4}{3}ma^2\omega'^2$$

$$\therefore \text{proportion lost} = 1 - \frac{\omega'^2}{\omega^2} = 15/16$$



In rotating from ① to ② cube loses pe and gains ke.
In particular loss of pe
$= mg\sqrt{2}a$

But angular vely must go from ω' back to ω.

ie. $\cancel{m}g\sqrt{2}a = \frac{15}{16} \times \frac{4}{3}\cancel{m}a^2\omega^2$

$$\omega = \frac{2\sqrt[4]{2}}{\sqrt{5}}\sqrt{\frac{g}{a}}$$

Examiner's comments

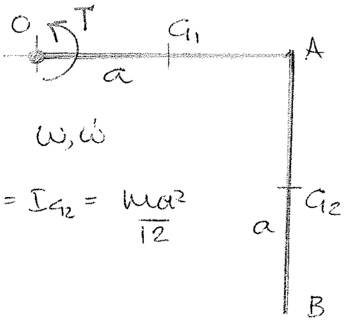
Question 1: rigid body dynamics
Attempts: 161, average mark 11.1
Having demonstrated part (a) satisfactorily many candidates failed to obtain the correct expression for $\ddot{\theta}$; $\frac{2V}{a}\sin\theta\cos\theta$ was a popular choice. A sizable minority for some reason changed from polars to cartesians in going from part (a) to part (b).

Question 2: small oscillations
Attempts: 23, average mark 11.1
An unpopular question. More attempts with a sensible method, usually energy, would have been successful if candidates had drawn a diagram with a ruler and thought about the (not difficult) geometry.

Question 3: rotary balancing
Attempts: 197, average mark 10.5
A popular question. Many candidates assumed that the initial configuration was statically balanced so that the bearing forces formed a couple and the balancing masses must be equal and at 180° to one another: it isn't and they don't.

Question 4: tumbling cube
Attempts: , average mark 10.6,
Very popular, attempted by approx. 85% of all candidates. There were several 20/20s and almost all who attempted the problem were successful in solving part (a). Candidates were less successful on part (c) and a few did not take up on the hints of 'average' acceleration = 0 and the calculation in part (b) to apply energy conservation over the cycle between impacts..

5



$I_{G_1} = I_{G_2} = \frac{ma^2}{12}$

$I_O = I_{G_1} + m(OG_1)^2$ (||el axis theorem)

$+ I_{G_2} + m(OG_2)^2$

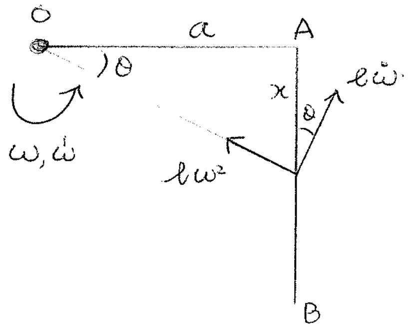
ie. $I_O = \frac{ma^2}{12} + m\left(\frac{a}{2}\right)^2$

$+ \frac{ma^2}{12} + m\left\{\left(\frac{a}{2}\right)^2 + a^2\right\}$

ie. $I_O = \frac{ma^2}{12} + \frac{ma^2}{4} + \frac{ma^2}{12} + ma^2 \cdot \frac{5}{4}$

$= \frac{5}{3}ma^2$

Moments about O: $T = I_O \cdot \dot\omega = \frac{5}{3}ma^2\dot\omega$ [5]

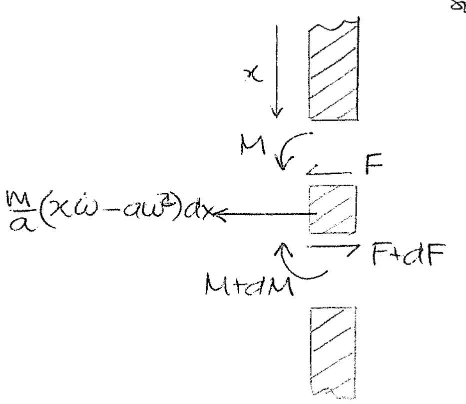


component of acceleration normal to bar is

$l\dot\omega \sin\theta - l\omega^2\cos\theta$

But $\sin\theta = \frac{x}{l}$; $\cos\theta = \frac{a}{l}$

So accln normal bar $\Rightarrow x\dot\omega - a\omega^2$



Thinking of AB as a 'beam'

$\frac{dF}{dx} = (x\dot\omega - a\omega^2)\frac{m}{a}$

$\therefore F = \frac{m\dot\omega}{a}\frac{x^2}{2} - m\omega^2 x + C_1$

But $F = 0$, at free end, when $x = a$

$\therefore F = \frac{m\dot\omega}{2a}x^2 - m\omega^2 x - \frac{m\dot\omega a}{2} + m\omega^2 a$

But also $\frac{dM}{dx} = F$

$\therefore M = \frac{m\dot{\omega}}{2a}\frac{x^3}{3} - \frac{m\omega^2 x^2}{2} - \frac{m\dot{\omega} ax}{2} + m\omega^2 ax + C_2$

again $M=0$ at free end $x=a$

So $M = \frac{m\dot{\omega}}{6a}x^3 - \frac{m\omega^2}{2}x^2 - \frac{m\dot{\omega}ax}{2} + m\omega^2 ax$

$- \frac{m\dot{\omega}}{6}a^2 + \frac{m\omega^2}{2}a^2 + \frac{m\dot{\omega}}{2}a^2 - m\omega^2 a^2$

[9]

M will have a max. when $F \Rightarrow 0$

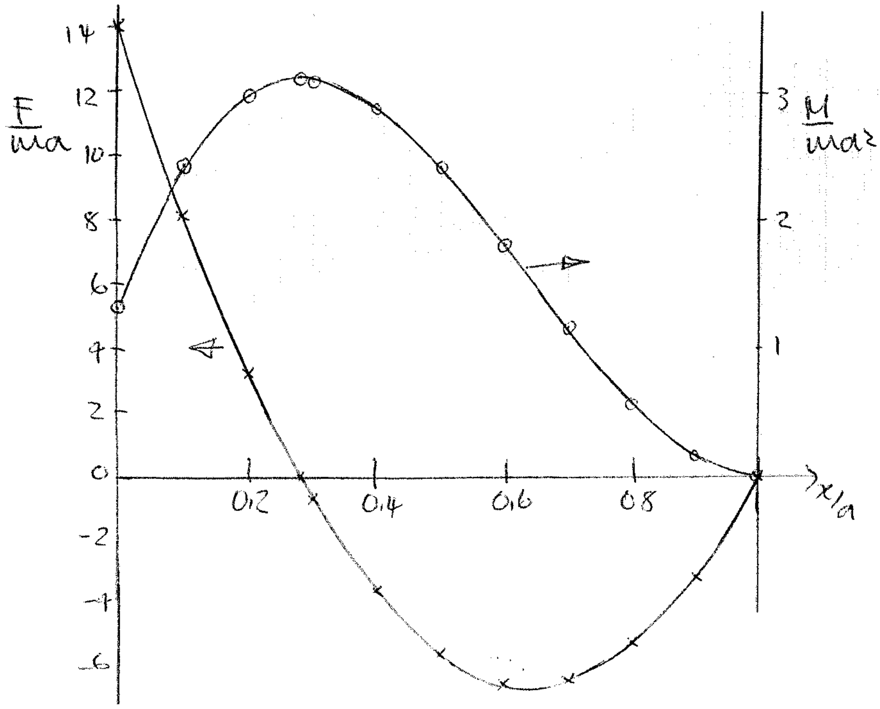
If $F=0$, and $\omega = 8s^{-1}$, $\dot{\omega} = 100s^{-1}$ then

$0 = 50z^2 - 64z - 50 + 64$ where $z = x/a$

ie. $25z^2 - 32z + 7 = 0$

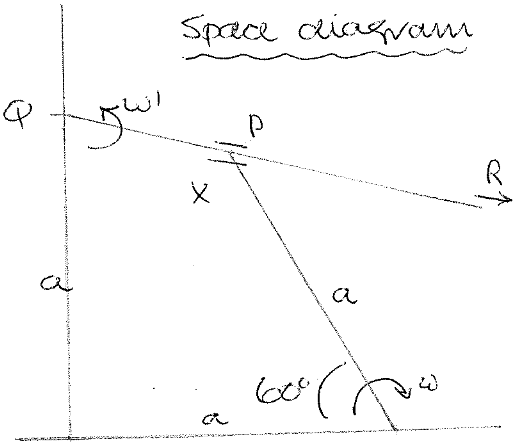
$z = \frac{x}{a} = \frac{32 \pm \sqrt{32^2 - 700}}{50} = 0.28$ or 1.0

[6]



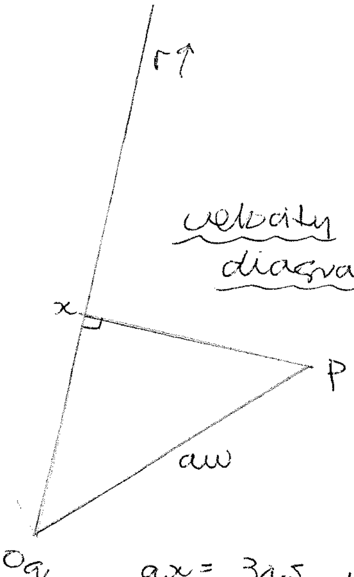
drawing

Space diagram



R

velocity diagram



$QX = \frac{26}{50} . a = .52a$

$\omega' = \frac{qx}{QX} = \frac{34.5/50\ a\omega}{.52\ a} = 1.33\omega$

$qx = \frac{34.5}{50} \times a\omega$

$px = \frac{36}{50} a\omega = .72 a\omega$

accelerations $\quad P \to O$ at $a\omega^2$

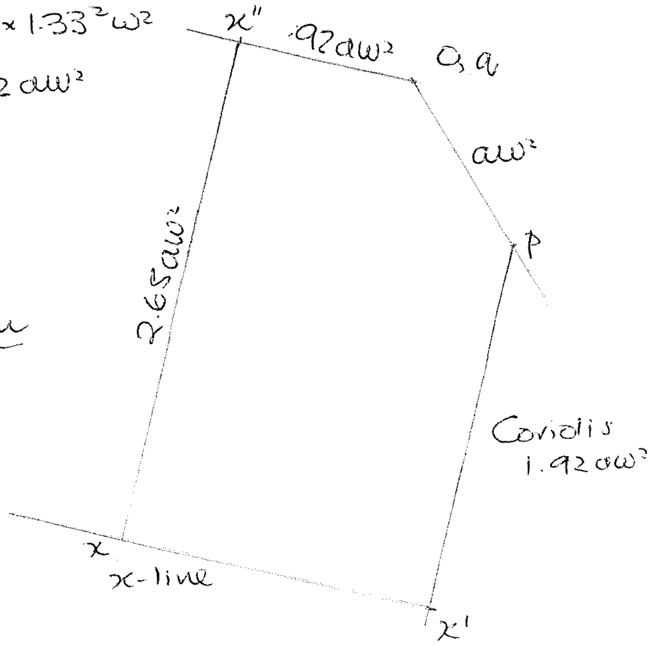
at P Coriolis $: 2\underline{\omega'} \times \underline{px} = 2 \times 1.33\omega \times .72 a\omega$

$= 1.92 a\omega^2 \swarrow \perp r\ QR$

$X \to Q$ at $QX.\omega'^2$

i.e. $.52a \times 1.33^2\omega^2$

$= 0.92 a\omega^2$



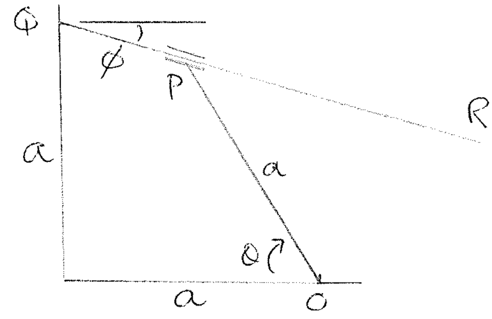
accn diagram

$x''x = QX . \dot{\omega}'$

$\dot{\omega}' = \frac{2.65 a\omega^2}{.52a}$

$= 5.10\omega^2$

By calculation



$$\left.\begin{aligned} a\sin\theta + QP\sin\phi &= a \\ a\cos\theta + QP\cos\phi &= a \end{aligned}\right\}$$

$$\therefore\ QP\sin\phi = a(1-\sin\theta)$$

$$QP\cos\phi = a(1-\cos\theta)$$

$$\therefore\ \tan\phi = \frac{1-\sin\theta}{1-\cos\theta}$$

Diff. wrt time

$$\sec^2\phi.\dot\phi = \frac{-(1-\cos\theta)\cos\theta.\dot\theta - (1-\sin\theta).\sin\theta.\dot\theta}{(1-\cos\theta)^2}$$

But $\sec^2\phi = 1+\tan^2\phi = \dfrac{(1-\cos\theta)^2+(1-\sin\theta)^2}{(1-\cos\theta)^2}$

$$= \frac{1-2\cos\theta+\cos^2\theta+1-2\sin\theta+\sin^2\theta}{(1-\cos\theta)^2}$$

i.e. $(3-2\cos\theta-2\sin\theta)\dot\phi = (\cos^2\theta-\cos\theta-\sin\theta+\sin^2\theta)\dot\theta$

$$\text{So}\quad \dot\phi = \frac{1-\cos\theta-\sin\theta}{3-2\cos\theta-2\sin\theta}.\dot\theta \qquad (1)$$

So if $\theta=60^\circ$ $\cos\theta=1/2$ $\sin\theta=\sqrt3/2$, $\dot\theta=\omega$

$$\therefore\ \dot\phi = \frac{1-\frac12-\sqrt3/2}{3-1-\sqrt3}.\omega = \frac{1-\sqrt3}{2(2-\sqrt3)}\omega = -1.36\omega$$

i.e. in opposite sense.

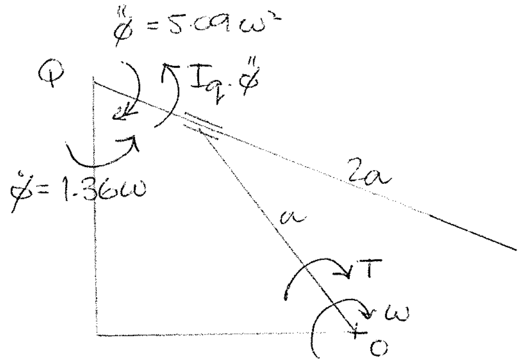
Diff. (1) wrt time, remember $\dot\theta=\omega=$ const.

$$\ddot\phi = \frac{[(3-2\cos\theta-2\sin\theta)(\sin\theta-\cos\theta)-(1-\cos\theta-\sin\theta)(2\sin\theta-2\cos\theta)]\omega^2}{(3-2\cos\theta-2\sin\theta)^2}$$

So if $\theta=60^\circ$,

$$\ddot\phi = \frac{[(3-1-\sqrt3)(\sqrt3-1).5-(1-\frac12-\sqrt3/2)(\sqrt3-1)]\omega^2}{(3-1-\sqrt3)^2}$$

i.e. $\ddot{\phi} = 5.09\,\omega^2$



By V. work

$$I_q \cdot \ddot{\phi} \cdot \dot{\phi} + T \cdot \omega = 0$$

$$I_q = \frac{8}{3} m a^3$$

$$\therefore \quad \frac{8}{3} m a^3 \times 5.09\,\omega^2 \times 1.36\,\omega + T \cdot \omega = 0$$

$$\therefore \quad T = -18.5\, m a^3 \omega^2$$

$$I_q = \frac{1}{12}(2ma)(2a)^2 + (m \cdot 2a) \cdot a^2$$

$$= \frac{2}{3} m a^3 + 2 m a^3$$

$$= \frac{8}{3} m a^3$$

Examiner's comments

Question 5: inertial stresses
Attempts: , average mark 10.2,
Less popular, but still attempted by approx. 75% of all candidates. Calculation of moment of inertia of the bar about one end was suprisingly hard for a number of candidates. This turned out to be problem with the lowest average mark though students were exposed to a similar problem in one of the examples papers.

Question 6: kinematics
Attempts: , average mark 11.3,
Very popular, again attempted by approx. 85% of all candidates. Overwhelming preference for the use of a graphical solution to find the angular acceleration and velocities. Construction of velocity diagram seen as straightforward, but acceleration diagram less so – many were confused as to the directions of the acceleration vectors. Some left out the Coriolis term. For part (c), none came up with the right value for moment of inertia of the bar QR rotating about Q!

JAW
June 03