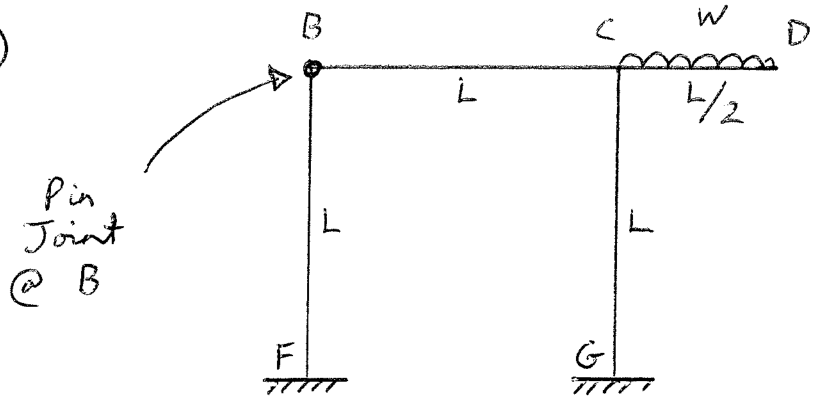
Engineering Tripos Part IB

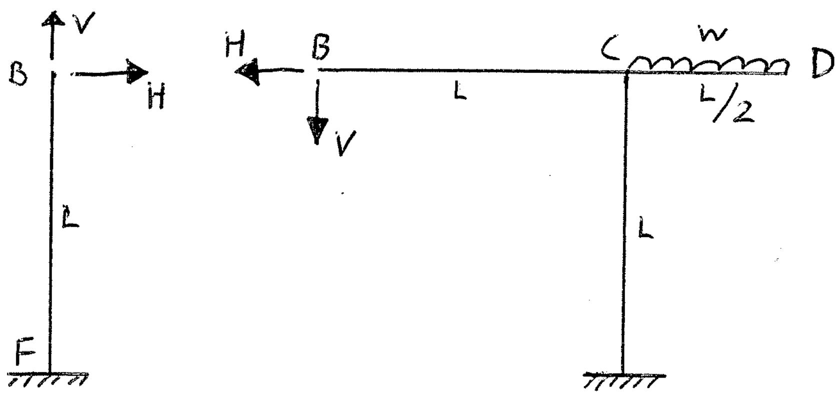
Paper 2 June 2003

STRUCTURES

1) (a)

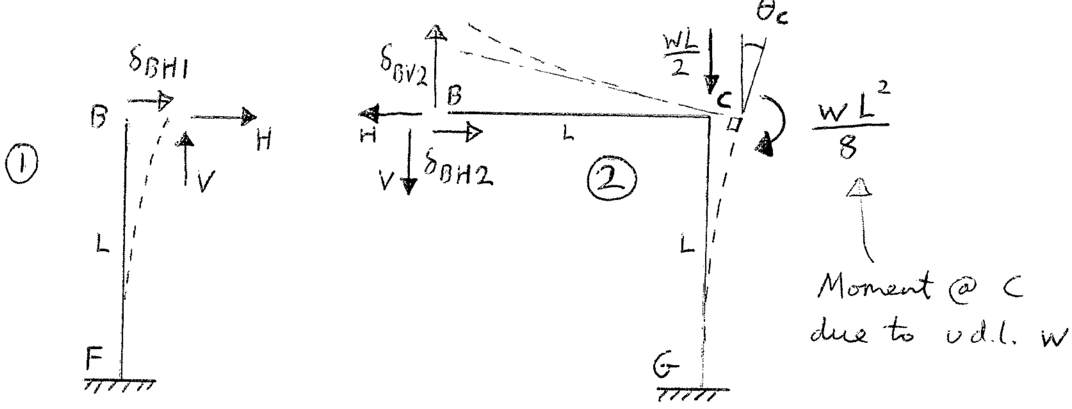


Split structure @ pin joint B into two statically determinate cantilever structures with two unknown forces H and V:



Therefore there are two (2) redundancies.

1) (b) Consider the displacements of point B in each half of the structure.



No vertical deflection

$\therefore \quad \delta_{BVi} = 0$

For ① (FB) $\qquad \delta_{BH1} = \dfrac{HL^3}{3EI}$ $\qquad$ (Structures Data Book)

For ② (BCDG) $\qquad \theta_c = \left(\dfrac{wL^2}{8} - VL\right)\dfrac{L}{EI} - \dfrac{HL^2}{2EI}$

$$\delta_{BH2} = \delta_{CH2} = \left(\frac{wL^2}{8} - VL\right)\frac{L^2}{2EI} - \frac{HL^3}{3EI}$$

$$\delta_{BV2} = \theta_c L - \frac{VL^3}{3EI}$$

Apply compatibility of deflections @ B

$\delta_{BH1} = \delta_{BH2} \qquad \therefore \quad \dfrac{H}{3} = \dfrac{1}{2}\left(\dfrac{wL}{8} - V\right) - \dfrac{H}{3}$

$$\frac{2H}{3} = \frac{wL}{16} - \frac{V}{2}$$

$$V = \frac{wL}{8} - \frac{4H}{3} \qquad \text{Equation I}$$

i) (b) (continued)

$$\delta_{BV1} = \delta_{BV2} = 0$$

$$\therefore \quad \left(\frac{WL}{8} - V - \frac{H}{2}\right)L - \frac{VL}{3} = 0$$

$$\frac{WL}{8} - \frac{H}{2} - \frac{4V}{3} = 0 \qquad \text{Equation II}$$

Substitute Equation I into II

$$\frac{WL}{8} - \frac{H}{2} - \frac{4}{3}\left(\frac{WL}{8} - \frac{4H}{3}\right) = 0$$

$$\frac{WL}{8}\left(1 - \frac{4}{3}\right) + H\left(-\frac{1}{2} + \frac{16}{9}\right) = 0$$

$$\frac{WL}{8} \times -\frac{1}{3} + H \times \frac{23}{18} = 0$$

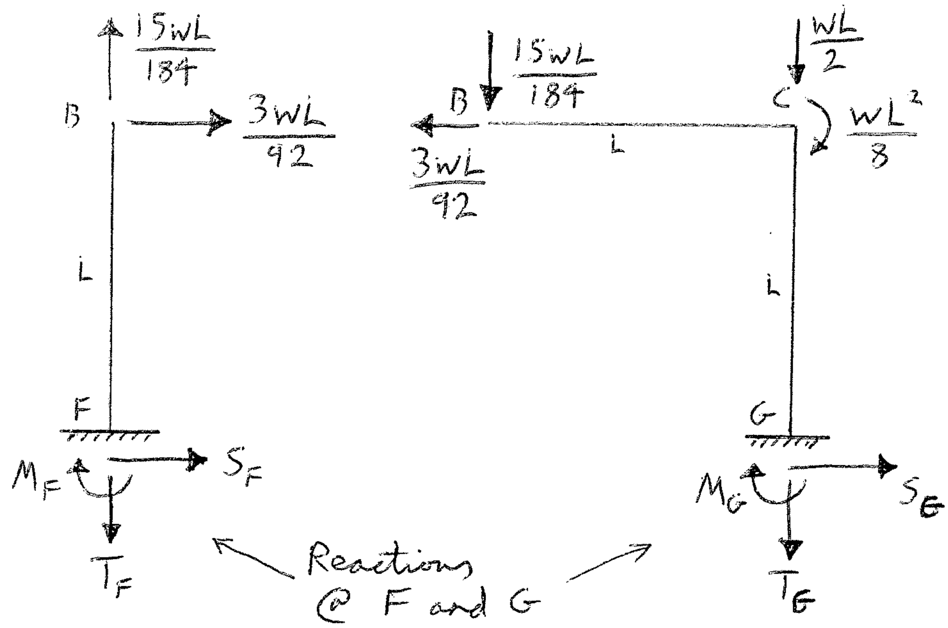
$$\therefore \quad H = \frac{3WL}{92}$$

Equation I $\Rightarrow$ $\quad V = \frac{WL}{8} - \frac{4}{3} \times \frac{3WL}{92}$

$$\therefore \quad V = \frac{15WL}{184}$$

1) (b) (continued)

i)



Resolve horizontally

$$S_F = \frac{-3wL}{92} \qquad S_G = \frac{3wL}{92}$$

Resolve vertically

$$T_F = \frac{15wL}{184} \qquad T_G = -\frac{wL}{2} - \frac{15wL}{184}$$

$$= \frac{-107wL}{184}$$

Resolve moments @ F and G

$$M_F = -\frac{3wL^2}{92}$$

$$M_G = \frac{15wL}{184} \times L + \frac{3wL}{92} \times L - \frac{wL^2}{8}$$

$$= \left(\frac{15}{184} + \frac{3}{92} - \frac{1}{8}\right) wL^2$$

$$= (15 + 6 - 23)\frac{wL^2}{184}$$

$$= \frac{-2wL^2}{184} = -\frac{wL^2}{92}$$

Question 1 Statically-indeterminate frame (ELASTIC)
(109 attempts, average 9.1/20)

A less popular question that many students found difficult.
(a) Many did not realise that there were only two redundancies.
(b) (i) Most students did realise that they needed to split the structure into two determinate structures, use data book coefficients to find deflections and then apply compatibility to find the unknown internal forces. However, only a minority were successful in following this approach without significant errors in their method.

(ii) Many students who were stuck on (b) (i) realised that by using the given internal forces there were some easy marks available for calculating the reactions at the supports although quite a few failed to calculate the moments.

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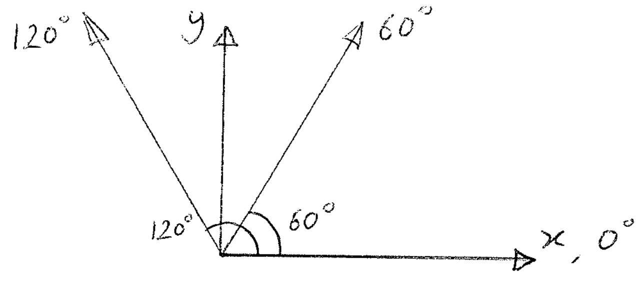
STRUCTURES

2)



Note:

$\cos 60° = \frac{1}{2}$ $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 120° = -\frac{1}{2}$ $\sin 120° = \frac{\sqrt{3}}{2}$

(a) $\varepsilon_{xx} = \varepsilon_0$ ①

$$\varepsilon_{60} = \varepsilon_{xx}\cos^2 60° + \varepsilon_{yy}\sin^2 60° + \gamma_{xy}\sin 60°\cos 60°$$

$$= \frac{1}{4}\varepsilon_{xx} + \frac{3}{4}\varepsilon_{yy} + \frac{\sqrt{3}}{4}\gamma_{xy} \quad ②$$

$$\varepsilon_{120} = \varepsilon_{xx}\cos^2 120° + \varepsilon_{yy}\sin^2 120° + \gamma_{xy}\sin 120°\cos 120°$$

$$= \frac{1}{4}\varepsilon_{xx} + \frac{3}{4}\varepsilon_{yy} - \frac{\sqrt{3}}{4}\gamma_{xy} \quad ③$$

② + ③ $\varepsilon_{60} + \varepsilon_{120} = \frac{1}{2}\varepsilon_{xx} + \frac{3}{2}\varepsilon_{yy}$ ④

① into ④ $\varepsilon_{yy} = \dfrac{-\varepsilon_0 + 2\varepsilon_{60} + 2\varepsilon_{120}}{3}$ ⑤

as required

⑤ & ① into ②

$$\sqrt{3}\gamma_{xy} = 4\varepsilon_{60} - \varepsilon_0 - 3\left(\frac{-\varepsilon_0 + 2\varepsilon_{60} + 2\varepsilon_{120}}{3}\right)$$

$$\therefore \gamma_{xy} = \frac{2}{\sqrt{3}}\left(\varepsilon_{60} - \varepsilon_{120}\right) \quad ⑥$$

2) (b)

(i) If $\varepsilon_0 = -200 \times 10^{-6}$, $\varepsilon_{60} = 250 \times 10^{-6}$ and $\varepsilon_{120} = 75 \times 10^{-6}$

then from (1), (5) and (6)

$\varepsilon_{xx} = -200 \times 10^{-6}$, $\varepsilon_{yy} = 283 \times 10^{-6}$ and $\gamma_{xy} = 202 \times 10^{-6}$

for a pressure of 6MPa

(ii) Assume plane stress, i.e. $\sigma_{zz} = 0$

$$\therefore \quad \sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) = \frac{210 \times 10^9}{1-0.3^2}\left(-200 + 0.3 \times 283\right) \times 10^{-6}$$

$$= -26.6 \times 10^6 \ \mathrm{N/m^2} = -26.6 \ \mathrm{MPa}$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) = \frac{210 \times 10^9}{1-0.3^2}\left(283 + 0.3 \times -200\right) \times 10^{-6}$$

$$= 51.5 \times 10^6 \ \mathrm{N/m^2} = 51.5 \ \mathrm{MPa}$$

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = \frac{210 \times 10^9}{2(1+0.3)} \times 202 \times 10^{-6}$$

$$= 16.3 \times 10^6 \ \mathrm{N/m^2} = 16.3 \ \mathrm{MPa}$$

(iii) Through thickness strain ε_{zz}

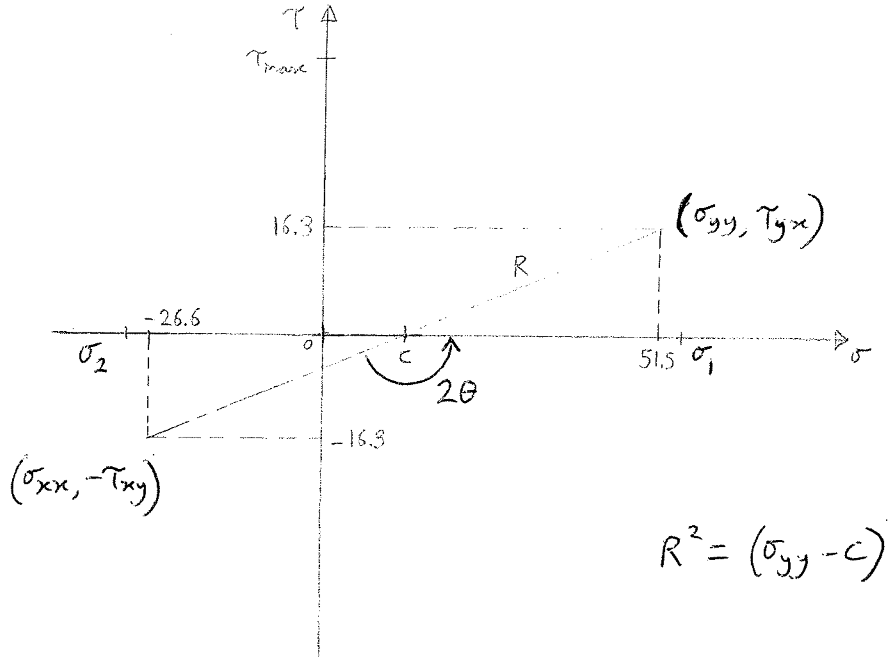
$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\sigma_{yy} - \nu\sigma_{xx}\right)$$

$$= \frac{1}{210 \times 10^9}\left(0 - 0.3 \times 51.5 - 0.3 \times -26.6\right) \times 10^6$$

$$= 35.6 \times 10^{-6}$$

2) (b) (continued)

(iv)



$$R^2 = (\sigma_{yy} - c)^2 + \tau_{yx}^2$$

$$c = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{-26.6 + 51.5}{2} = 12.45 \text{ MPa}$$

$$R = \sqrt{(51.5 - 12.45)^2 + 16.3^2} = 42.3 \text{ MPa}$$

$$\sigma_1 = c + R = 12.45 + 42.3 = 54.8 \text{ MPa}$$

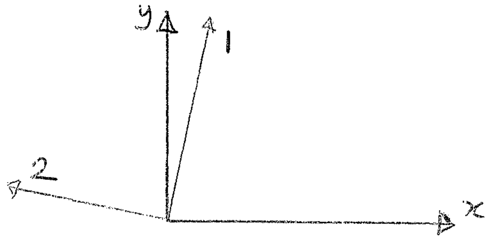
$$\sigma_2 = c - R = 12.45 - 42.3 = -29.9 \text{ MPa}$$

$$(\sigma_3 = \sigma_{zz} = 0)$$

$$2\theta = 180° - \sin^{-1}\left(\frac{\tau_{xy}}{R}\right) = 180° - \sin^{-1}\left(\frac{16.3}{42.3}\right) = 157.3°$$

$\therefore \theta = 78.7°$

$\therefore$ Principal directions are 78.7° and 168.7° (and through-thickness)



2) (b) (continued)

(v) Von Mises' yield criterion:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2$$

$$Y = 230\,MPa$$

$\therefore$ Von Mises' equivalent stress σ_{VM}

$$\sigma_{VM} = \sqrt{\frac{1}{2}\left\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right\}}$$

$$= \sqrt{\frac{1}{2}\left\{(54.8 - -29.9)^2 + (-29.9-0)^2 + (0-54.8)^2\right\}}$$

$$= 74.4\ MPa$$

$\therefore$ For a pressure of 6 MPa the factor of safety is $\dfrac{Y}{\sigma_{VM}} = \dfrac{230}{74.4} = 3.09$

Question 2 Mohr's circles of stress and strain and yield criterion (ELASTIC)
(Attempts: 148. Average mark: 12.6/20)

A fairly popular and straightforward question.
(a) Most were broadly successful in applying the data book formula for strain transformations and solving some basic algebra. Some students simply skipped the second part which required them to find the shear strain as a function of the strain gauge readings.
(b) (i) A straightforward conversion from the strain gauge readings to strains in the *x,y* coordinate system. Most scored well here – but a surprising number of students lost a minus sign and were unable to use the given formula correctly.
(ii) Application of Hooke's law from the databook. Generally well done, just a few surprising numerical errors.
(iii) Drawing of Mohr's circle of stress, calculation of principal stresses and their directions. Most made good attempts but the full 5 marks were only awarded to students who adopted the correct data book sign convention, used calculations in addition to a careful diagram and specified both principal directions.
(iv) Most students quoted von Mise's criterion for yield correctly. However, many did not correctly understand the concept of a "factor of safety" and lost marks.

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3)

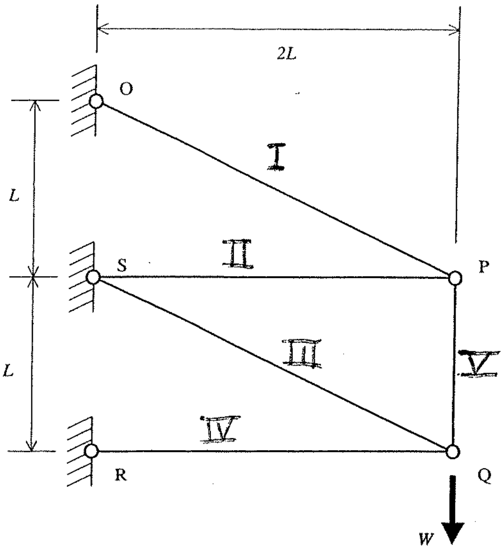


Fig. 3

Mechanisms $m = 0$

Bars $b = 5$

Restraints (3 nodes held in X and Y) $\therefore r = 3 \times 2 = 6$

Two-dimensional problem $D = 2$

Joints $j = 5$

(a) Maxwell's rule: $s - m = b + r - Dj$

$\therefore \quad s - 0 = 5 + 6 - 2 \times 5$

$\therefore \quad s = 1$

One redundancy

(b) Length of bars $L_{I} = L_{III} = \sqrt{5}L$

$L_{II} = L_{IV} = 2L$

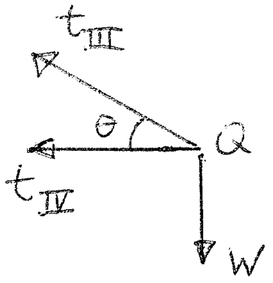
$L_{V} = L$

(i) To obtain a particular equilibrium solution assume t_{V} as a redundant tension

$\therefore$ set $t_{V} = 0$

3) (b) (i) (continued)

Consider equilibrium of node Q



$$\sin\theta = \frac{L}{\sqrt{5}\,L} = \frac{1}{\sqrt{5}}$$

$$\cos\theta = \frac{2}{\sqrt{5}}$$

Resolve $\uparrow$ $\quad t_{III}\sin\theta - W = 0 \quad \therefore t_{III} = \sqrt{5}\,W$

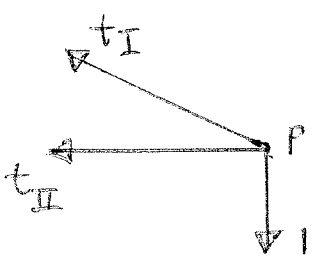
Resolve $\rightarrow$ $\quad t_{III}\cos\theta + t_{IV} = 0 \quad \therefore t_{IV} = -2W$

$\therefore$ Particular equilibrium solution:

$$\underset{\sim}{t}_o = \begin{bmatrix} t_I \\ t_{II} \\ t_{III} \\ t_{IV} \\ t_V \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{5} \\ -2 \\ 0 \end{bmatrix} W$$

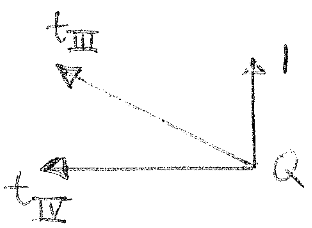
(ii) State of self stress $\underset{\sim}{s}$ obtained when $t_V = 1$ and $W = 0$ (no external load)

Consider equilibrium of nodes P and Q:



$\Rightarrow$ $\quad t_I = \sqrt{5}$

$t_{II} = -2$



$\Rightarrow$ $\quad t_{III} = -\sqrt{5}$

$t_{IV} = 2$

$$\therefore \underset{\sim}{s} = \begin{bmatrix} \sqrt{5} \\ -2 \\ -\sqrt{5} \\ 2 \\ 1 \end{bmatrix}$$

3) (b) (continued)

(iii) Actual loads due to W are: $\underset{\sim}{t} = \underset{\sim}{t}_0 + x \underset{\sim}{s}$

Flexibility matrix:

$$\underset{\sim}{F} = \frac{L}{EA}\begin{bmatrix} \sqrt{5} & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Bar extensions $\underset{\sim}{e}$

$$\underset{\sim}{e} = \underset{\sim}{F}\,\underset{\sim}{t} = \frac{WL}{EA}\begin{bmatrix} 0 \\ 0 \\ 5 \\ -4 \\ 0 \end{bmatrix} + \frac{xL}{EA}\begin{bmatrix} 5 \\ -4 \\ -5 \\ 4 \\ 1 \end{bmatrix}$$

Use virtual work $\underset{\sim}{F}^* \cdot \underset{\sim}{\delta} = \underset{\sim}{s}^* \cdot \underset{\sim}{e}$ with a virtual set of zero external forces $\underset{\sim}{F}^* = 0$ in equilibrium with $\underset{\sim}{s}$ (state of self stress)

$$\underset{\sim}{s} \cdot \underset{\sim}{e} \Rightarrow 0 = \frac{WL}{EA}\left(-5\sqrt{5} - 8\right)$$
$$+ \frac{xL}{EA}\left(5\sqrt{5} + 8 + 5\sqrt{5} + 8 + 1\right)$$
$$\therefore 0 = \left(-5\sqrt{5} - 8\right) W + \left(10\sqrt{5} + 17\right) x$$

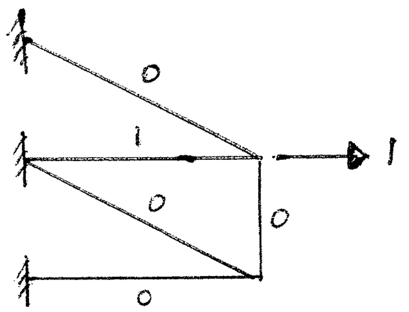
$$x = \frac{\left(5\sqrt{5} + 8\right) W}{10\sqrt{5} + 17} = 0.4873\, W$$

$$\underset{\sim}{t} = \left\{ \begin{bmatrix} 0 \\ 0 \\ \sqrt{5} \\ -2 \\ 0 \end{bmatrix} + 0.4873 \begin{bmatrix} \sqrt{5} \\ -2 \\ -\sqrt{5} \\ 2 \\ 1 \end{bmatrix} \right\} W = \begin{bmatrix} 1.0896 \\ -0.9746 \\ 1.1464 \\ -1.0254 \\ 0.4873 \end{bmatrix} W$$

3) (b) (continued)

(iv)
$$\underset{\sim}{e} = \underset{\sim}{F}\,\underset{\sim}{t} = \begin{bmatrix} 2.4365 \\ -1.9492 \\ 2.5635 \\ -2.0508 \\ 0.4873 \end{bmatrix} \frac{WL}{EA}$$

To find horizontal displacement of node P consider virtual force system:



$$\underset{\sim}{t}^{*} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The horizontal displacement of node P δ_H may be obtained by virtual work:

$$1.\,\delta_H = \underset{\sim}{t}^{*}\cdot\underset{\sim}{e}$$

$$\therefore\ \delta_H = 1 \times -1.9492\,\frac{WL}{EA} = -1.9492\,\frac{WL}{EA}$$

(Alternatively: observe directly that the horizontal displacement of P is equal to the extension of bar SP (bar II)

$$\therefore\ \delta_H = e_{II}$$

$$\therefore\ \delta_H = -1.9492\,\frac{WL}{EA}$$
)

Question 3 Statically-indeterminate pin-jointed truss (ELASTIC)
(Attempts: 208. Average mark: 13.4/20)

The most popular question in Section A and quite straightforward.

(a) Although quite a few students thought there was more than one redundancy, in general this did not prevent them having a reasonable attempt at the remainder of the question. (A few, however, did go on to try and find more than one state of self stress and wasted a lot of time).

(b) (i) Mostly well done. Correct equilibrium solutions used either only SQ and RQ or, alternatively, only OP, SP & PQ to resist the load. Marks were deducted for inaccuracy of signs or numerical inaccuracies.

(ii) Again generally well done. Some students set bar SQ to a load of 1, others chose PQ. The examiner preferred exact answers (i.e. including square roots) but also gave full marks to correct numerical approximations (e.g. 3 s.f.).

(iii) Even those students who had completely failed on the first parts of the question generally outlined and attempted the correct expected method (i.e. $\mathbf{t} = \mathbf{t_0} + x\mathbf{s}$, $\mathbf{e} = \mathbf{Ft}$, $\mathbf{s.e} = 0$ to find x). However, one student successfully solved the problem using a displacement diagram to determine the compatibility condition.

(iv) Finding the horizontal displacement of node P could be found by using the formal method of the virtual work equation with a virtual force and equilibrium system together with the real bar extensions. However many students correctly realised that the horizontal displacement of P was simply the extension of bar SP and therefore wrote the answer down directly for an easy full 3 marks. A few students attempted to calculate the vertical displacement, or even chose the wrong node.

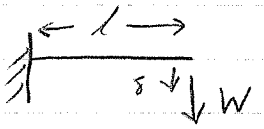
Q4 (a) $F = \frac{eAE}{L}$ $\quad e = 27.36\,mm$ $\quad A = \frac{\pi \times 25}{4} = 19.63\,mm^2$

$E = 210\,GPa$ $\quad L = 3\,m$

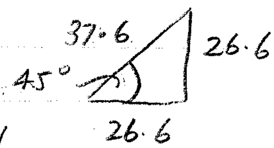
Tension in wire

$$\therefore F = 27.36 \times 19.63 \times \frac{210 \times 10^3}{3 \times 10^3} = 37600\,N = \underline{37.6\,kN}$$

(NB: If $E_{wire} = 210\,GPa$ then $\sigma_{wire} = 1915\,MPa$ & wire would have to be very high tensile wire to avoid yielding. This was not part of this question).

(b) 

Databook $\delta = \frac{WL^3}{3EI}$



$T_x = T_y = 26.6\,kN$

$$I_{xx} = \frac{200 \times 400^3}{12} - \frac{190 \times 380^3}{12} = 197.9 \times 10^6\,mm^4 \simeq 200 \times 10^6\,mm^4$$

$$\delta_y = \frac{26.6 \times 10^3 \times 8000^3}{3 \times 210 \times 10^3 \times 197.9 \times 10^6} = \underline{109\,mm} \text{ in } y\text{-direction}$$

$$I_{yx} = \frac{400 \times 200^3}{12} - \frac{380 \times 190^3}{12} = 49.47 \times 10^6\,mm^4 \simeq 50 \times 10^6\,mm^4.$$

$$\delta_x = 109 \times \frac{197.9}{49.5} = \underline{437\,mm} \text{ in } x\text{-direction}$$

(c) Rotation $T = \frac{G \cdot 4 A_e^2 \phi}{\oint \frac{ds}{t}}$

$$\therefore \phi = \frac{T \oint \frac{ds}{t}}{G \cdot 4 \cdot A_e^2}$$

where $T = 26.6(0.2 + 0.1) = 7.98\,kNm$ (8 kNm)

$$= \frac{8 \times 10^3 \times 200}{81 \times 10^9 \times 4 \times (76050 \times 10^{-6})^2}$$

$G = 81\,GPa.$

$A_e = 390 \times 195 = 76\,050\,mm^2$

$\oint \frac{ds}{t} = 2 \times \frac{200}{10} + 2 \times \frac{400}{5} = 200.$

$$= 854 \times 10^{-6}\,\text{rads.}$$

$$\therefore \theta = \phi L = 854 \times 10^{-6} \times 8 = \underline{6.83 \times 10^{-3}}\,\text{rads}$$

(0.39 degrees)

Q4 (cont.)

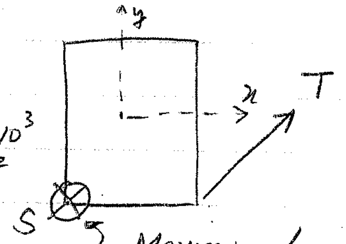
(d) Maximum bending stress will be at base plate at A.

$\sigma = \pm \frac{My}{I} \pm \frac{P}{A}$

Axial load upwards reduces compression & increases tension so maximum will be in corner as shown below.

$M = 26.6 \times 8 = 212.8$ kNm in x & y direction.

$A_{cross\ section} = 400 \times 200 - 190 \times 380 = 7.8 \times 10^3$ mm^2



Maximum longitudinal stress.

$$\sigma_{max} = \frac{212.8 \times 200 \times 10^6}{197.9 \times 10^6} + \frac{212.8 \times 100 \times 10^6}{49.5 \times 10^6} + \frac{1000 \times 10^3}{7.8 \times 10^3}$$

$= 215.2 + 430.1 + 128.2$

$= 773.5$ MPa at location in bottom LHS corner as shown above.

NOTE: 1) Clearly this exceeds the yield strength of most normal mild steels. Thus either the steel is a very high yield steel (> 800 MPa) or else the steel would yield & deform plastically, thus invalidating the calculations above which assume elastic behaviour.)

2) Some sol$^{n's}$ assumed bending stress to refer to longitudinal stresses from M_x + M_y only & did not include the $\frac{P}{A}$ term. No marks were deducted for this assumption if it was clearly defined.

(e) Shear stress at R.

Due to torque. $\tau_1 = \frac{T}{2A_e t} = \frac{8}{2 \times 76050 \times 10^{-6} \times 10 \times 10^{-3}} = 5.25 \times 10^6$ N/m^2 (5.25 MPa)

Due to shear forces for T_x component. (component due to T_y is 0.)

$$\tau_2 = \frac{SA\bar{y}}{I_{yy} b} = \frac{26.6 \times 10^3 \times 3900 \times 10^{-6} \times 73.14 \times 10^{-3}}{49.5 \times 10^6 \times 10^{-12} \times 2 \times 10 \times 10^{-3}} = 7.66 \times 10^6 \text{ N/m}^2$$ (7.66 MPa)

Since $A_c = 100 \times 10 \times 2 + 380 \times 5 = 3900$ mm^2

$\bar{x} = \frac{2 \times 10 \times 100 \times 50 + 380 \times 5 \times 97.5}{3900} = 73.14$ mm.

$\therefore \tau_2 - \tau_1 = 2.41$ MPa

Question 4 Thin walled structure (ELASTIC)
(128 attempts, average 11.2/20)

This was a standard problem on the response of a thin walled structure to bending, torsion and axial load. It was very similar in principle to questions on this topic in several exams in recent years although a slight complication was introduced by having bending applied about two axes due to the orientation of the wire applying the tension force to the structure. A disappointingly large number of students failed to determine the tensile force in a steel wire whose extension was given. Many also failed to recognise that this was a biaxial bending problem and only looked at flexure about one axis. As in past years a large number of students had difficulty determining the second moment of area for this very simple section. Of those that could, many took quite some time working through the parallel axes formulae when a trivial calculation could have been made. A common mistake was to assume that the torsion about the z-axis was the wire tension multiplied by the distance from the corner Q to the centre of the column, rather than force times the perpendicular distance from the line of action to the axis of rotation.

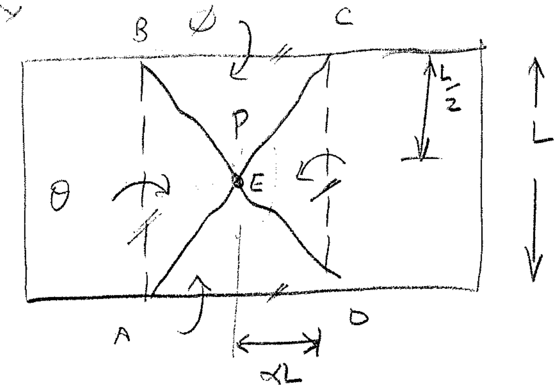
The calculation of rotation in part (c) threw up the same mistakes seen every year with some confused over the term for the enclosed area, A_e and others confusing J with I. One complication in this question was the fact that the applied load would cause yield in the steel section and thus elastic solutions would not in fact be valid. The original intention had been to select a tension force that would ensure the section remained fully elastic however in the final version a larger force resulted. Nevertheless it provided an excellent opportunity to test whether or not students understood the significance of the results they obtained. It had been hoped that the better students would pick up on this point however only one student recognised that yield would occur, and even this student only referred to the possibility of the wire itself yielding and not the column section.

Relatively few students recognised that the shear stress on the line of symmetry at R due to the component of load in the y-direction was zero although most were able to obtain the uniform shear stress resulting from the torsion T.

Q3 IB PAPER 2 12/1/03 CTM.

(a) DUCTILITY

(b)



$\phi \simeq \tan\phi = \frac{\delta}{L/2} = \frac{2\delta}{L}$

$\theta = \tan\theta = \frac{\delta}{\alpha L}$.

$\phi = \frac{2\delta}{L} = \frac{2}{L}$ $\theta = \frac{1}{\alpha L} = \frac{\delta}{\alpha L}$

$\delta = \frac{L\phi}{2} = \alpha L \theta$

$\therefore \phi = 2\alpha\theta$ OR $\theta = \frac{\phi}{2\alpha}$

$\delta = 1$

WD $= P$.

Using projection method:-

ED $= \Sigma m l \theta. = m' \cancel{L} . \frac{1}{\alpha \cancel{L}} \times 2 + m' 2\alpha \cancel{L} . \frac{2}{\cancel{L}} \times 2$

$+ m . \cancel{L} . \frac{1}{\alpha \cancel{L}} . 2. + m\, 2\alpha \cancel{L} . \frac{2}{\cancel{L}} \times 2.$

$= \frac{2m'}{\alpha} + 8m'\alpha + \frac{2m}{\alpha} + 8m\alpha.$

$= 2m'\left(\frac{1}{\alpha} + 4\alpha\right) + 2m\left(\frac{1}{\alpha} + 4\alpha\right)$

(b)(i) $\boxed{P = 2(m'+m)\left(\frac{1}{\alpha} + 4\alpha\right)}$

$\therefore \frac{dP}{d\alpha} = 2(m'+m).4 + 2(m'+m)(-1)\alpha^{-2}$

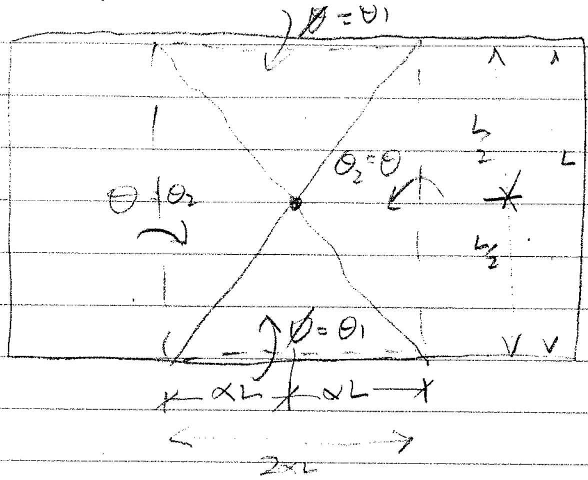
$= 8(m'+m) - \frac{2(m'+m)}{\alpha^2} = 0$

when $8 = \frac{2}{\alpha^2}$ $\therefore \alpha^2 = \frac{1}{4}$ $\alpha = \frac{1}{2}$

i.e. square geometry

(b)(ii) $\therefore P_1 = 2(m'+m)(2+2) = 8(m'+m)$

5(b) Alternative solution using hodograph method.



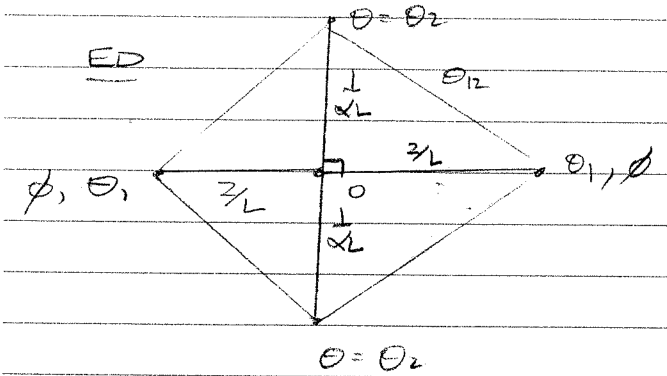
$\phi = \frac{2\delta}{L} = \frac{2}{L}$

$\theta = \frac{\delta}{\alpha L} = \frac{1}{\alpha L}$

$\delta = \frac{\phi L}{2} = \alpha L \theta$

$WD = P\delta$

ED



$$\theta_{12} = \sqrt{\left(\frac{1}{\alpha L}\right)^2 + \left(\frac{2}{L}\right)^2} = \frac{1}{L}\sqrt{\frac{1}{\alpha^2} + 4}$$

$$= \sqrt{\frac{1 + 4\alpha^2}{(\alpha L)^2}}$$

$$= \frac{1}{\alpha L}\sqrt{1 + (2\alpha)^2}$$

$$L_{sagging} = \sqrt{(\alpha L)^2 + \left(\frac{L}{2}\right)^2} = L\sqrt{\alpha^2 + \frac{1}{4}}$$

$$= L\sqrt{\frac{4\alpha^2 + 1}{4}}$$

$$= \frac{L}{2}\sqrt{1 + 4\alpha^2}$$

$$\therefore ED = m'.L.\theta_2.2 + m'.2\alpha L.\theta_1.2 + m.\cancel{\tfrac{L}{2}}(1+4\alpha^2)^{\frac{1}{2}}.\frac{1}{\alpha \cancel{L}}(1+4\alpha^2)^{\frac{1}{2}}.\cancel{4}^{\,2}$$

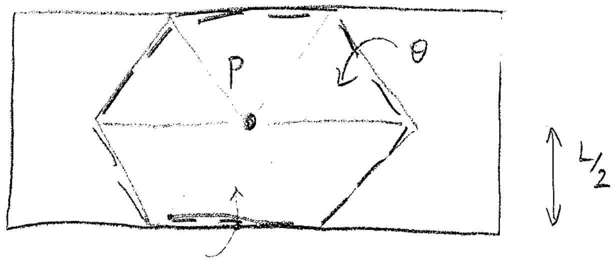
$$= m'\left(2\cancel{L}.\frac{1}{\alpha \cancel{L}} + 4\alpha \cancel{L}.\frac{2}{\cancel{L}}\right) + m.2.\frac{1}{\alpha}(1 + 4\alpha^2)$$

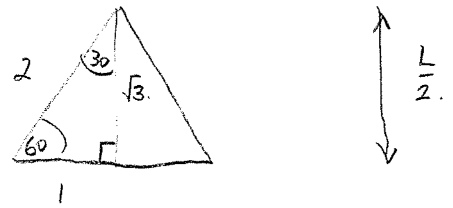
$$= m'\left(\frac{2}{\alpha} + 8\alpha\right) + \frac{2m}{\alpha}(1 + 4\alpha^2)$$

$$= 2m'\left(\frac{1}{\alpha} + 4\alpha\right) + 2m\left(\frac{1}{\alpha} + 4\alpha\right)$$

$$= 2(m' + m)\left(\frac{1}{\alpha} + 4\alpha\right) \qquad \underline{QED}$$

5(c)



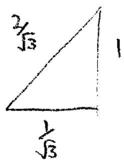


$\delta = r\theta$

$\therefore \theta = \frac{\delta}{r} = \frac{1}{r} = \frac{1}{L/2} = \frac{2}{L}.$

$\delta = r\theta$

$= \frac{L}{2}\theta$



WD by load $\quad WD = P.\delta = P.1 = P.$

Using projection method;-

ED in yieldlines $\quad ED = \Sigma m l \theta.$

$= 6m' . \frac{\cancel{L}}{\sqrt{3}} . \frac{2}{\cancel{L}} + 6m . \frac{\cancel{L}}{\sqrt{3}} . \frac{2}{\cancel{L}}$

$= \frac{12}{\sqrt{3}}(m'+m) = \frac{\sqrt{3}}{\sqrt{3}}\frac{12}{\sqrt{3}}(m'+m) = 4\sqrt{3}(m'+m)$

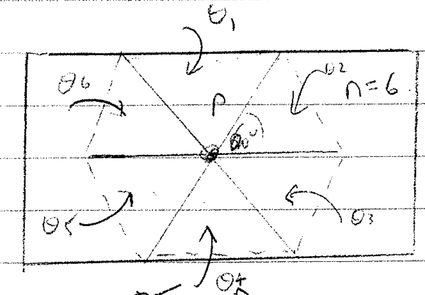
$= 6.93(m'+m)$

$\therefore \quad WD = ED$

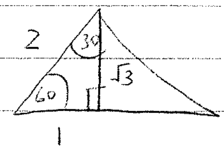
$P_2 = 6.93(m'+m)$

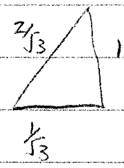
$= 4\sqrt{3}(m+m')$

5(c) Alternative sol^n using hodograph method.

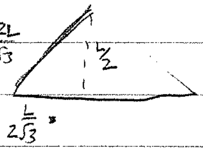


$\frac{L}{2}$





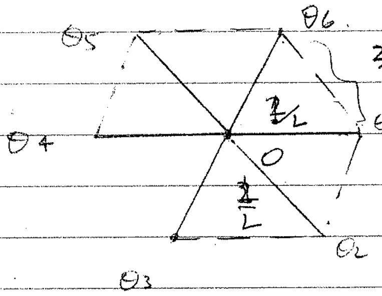
$\frac{L}{\sqrt{3}} = \frac{2L}{2\sqrt{3}}$



$\frac{L}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}L}{3}$
$= 0.577L$

<u>WD</u> $= P\delta = P$

<u>ED</u> $\theta_1 = \theta_2 = \dots = \theta_6 = \frac{\delta}{L/2} = \frac{2\delta}{L} = \frac{2}{L} = \theta_{12}$ i.e. $\delta = \frac{\theta L}{2}$



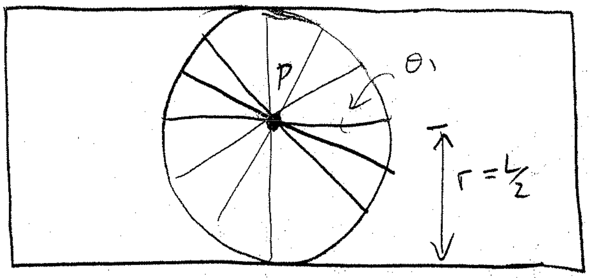
Relative rotations are also $2/L = \theta_{12}$ etc.

Hogging

$\therefore ED = \left(\frac{L}{\sqrt{3}} \cdot \frac{2}{L} \cdot m' + \frac{L}{\sqrt{3}} \cdot \frac{2}{L} \cdot m\right) 6$

$= \frac{12}{\sqrt{3}}(m'+m) = \frac{\sqrt{3}}{\sqrt{3}} \cdot \frac{12}{\sqrt{3}}(m'+m) = \underline{4\sqrt{3}(m'+m)}$

5 (d)



$$\theta_1 = \frac{\delta}{L/2} = \frac{2\delta}{L} = \frac{2}{L}$$

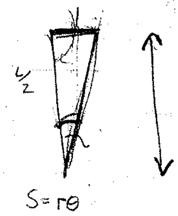
WD = P.1

Using projection method:-

$$ED = \frac{2\pi \cancel{r}}{\cancel{n}} . m' . \frac{1}{\cancel{r}} . \cancel{n}$$

$$+ \frac{2\pi \cancel{r}}{\cancel{n}} . m . \frac{1}{\cancel{r}} . \cancel{n}$$

$$= 2\pi(m' + m)$$



$$ED_{hog} = 2\pi . \frac{L}{2} m' . \theta_1$$

$$= 2\pi . \frac{\cancel{L}}{\cancel{2}} m' . \frac{\cancel{2}}{\cancel{L}} = 2\pi m'$$

$$ED_{sag} = \frac{L}{\cancel{2}} . \frac{2\pi}{\cancel{n}} . \cancel{n} . \theta_1 . m .$$

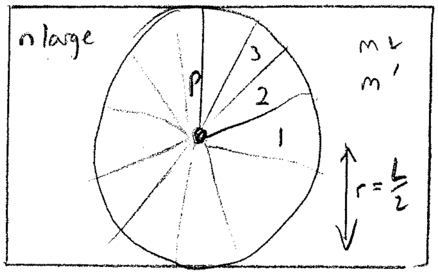
$$= \frac{\cancel{L}}{\cancel{2}} \frac{2\pi}{\cancel{n}} \cancel{n} . \frac{\cancel{2}}{\cancel{L}} m$$

$$= 2\pi m .$$

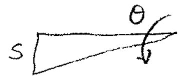
$$ED = (m + m')2\pi .$$

$$\therefore P_3 = 2\pi(m' + m) = \underline{6.28(m' + m)}$$

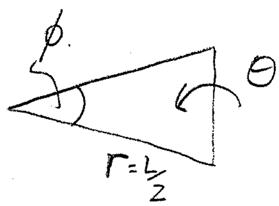
Q5(d) Alternative Sol^n using hodograph method.



Assume $\delta = 1$.



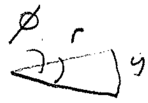
<u>WD</u> $= P.\delta = P.$



<u>ED</u> $(= \Sigma m l \theta)$

For hogging yieldline $s = r\theta = 1$

$\theta = \frac{1}{r} = \frac{2}{L}.$



Length of yieldline $y \simeq r\phi$

$= r.\frac{2\pi}{n}.$

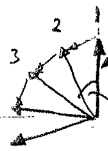
∴ Due to hogging YL

$ED = \cancel{n}.m'.\cancel{r}.\frac{2\pi}{\cancel{n}}.\frac{1}{\cancel{r}} = 2\pi m'$

For sagging yieldline draw hodograph.

$ED = n.\frac{L}{2}\theta_{12}\,m.$

$= n\frac{L}{2}\,\frac{2\pi}{n}.\frac{2}{L}.m$



$\theta_1 = \frac{1}{r}. = \theta_2 = \theta_3 \ldots$

$\phi = \frac{2\pi}{n}.$ | $\theta_{12} = \phi\frac{1}{r} = \frac{2\pi}{n}\frac{1}{r}. = \theta_{23} = \theta_{34} \ldots$

$\therefore\ ED = \cancel{n}.m.\cancel{r}.\frac{2\pi}{\cancel{n}}\frac{1}{\cancel{r}} = 2\pi m.$ $\quad = \frac{2\pi}{n}.\frac{2}{L}. = \frac{4\pi}{nL}$

$\Rightarrow$ Total $ED = 2\pi(m' + m)$

$ED = WD$

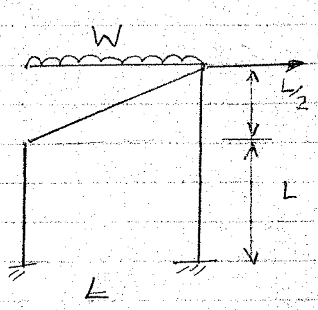
$\therefore\ \underline{P = 2\pi(m' + m)}$

5(e) $P_3 = 6.28(m' + m)$ is least upper bound since $P_3 < P_2 < P_1$.

Question 5 Yield-line analysis of a slab (PLASTIC)
(217 attempts, average 12.1/20)

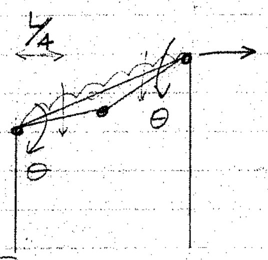
This question was very popular being attempted by 85% of candidates and was generally well done. The derivation in part (b) was straightforward. The use of vector rotation diagrams to calculate the relative rotation of yield-lines confused many students. The most common mistake was in the calculation of the length of the sagging yield lines in the hexagonal mechanism, which many gave as $L/2$ rather than the correct value of $L/\sqrt{3}$. The examiner wishes to emphasise the advantages of using the projection method, as is taught almost universally in the textbooks and is widely adopted in practice, purely because it is so much simpler and quicker when applied to this common type of problem. It is recommended that it be taught in future years, even if only as an alternative computational technique to the use of the vector rotation diagram.

Q6.



(a)(i) Equilibrium + Material Law (LB)
(ii) Compatibility + Material Law (UB)

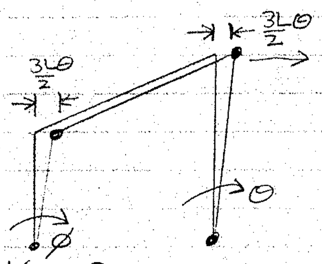
(b)
(i) Feasible collapse mechanisms



$$2 \cdot \frac{W_1}{2} \cdot \frac{L}{4} \cdot \cancel{\theta} = M_p \cdot (\theta + 2\theta + \theta) = 4M_p\cancel{\theta}$$

$$\frac{W_1 L}{M_p} = 16.$$

(1) Beam.



$$H_2 \cdot \frac{3L\theta}{2} = M_p\left(\frac{3}{2}\theta + \frac{3}{2}\theta + \theta + \theta\right) = 5M_p\theta$$

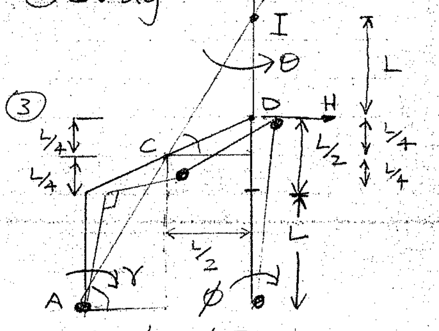
$$\frac{H_2 L}{M_p} = \frac{10}{3} = 3.33$$

$$L\phi = \frac{3L\theta}{2}$$

$$\therefore \phi = \frac{3}{2}\theta$$

(2) Sway

(3)



Combined I

Length $AC = IC \Rightarrow \gamma = \theta$

" $ID = L \Rightarrow L\theta = \frac{3L}{2}\phi \Rightarrow \phi = \frac{2}{3}\theta$

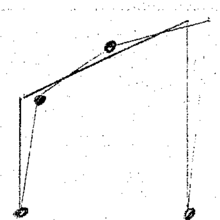
$$2 \cdot \frac{W}{2} \cdot \frac{L}{4} \cdot \phi + H \cdot L\theta = M_p[\gamma + (\gamma+\theta) + (\phi+\theta) + \phi]$$

$$\left(\frac{WL}{4M_p} + \frac{HL}{M_p}\right)\theta = \left(\theta + 2\theta + \frac{2}{3}\theta + \theta + \frac{2\theta}{3}\right) = \frac{16}{3}\theta$$

$$\frac{WL}{4M_p} + \frac{HL}{M_p} = \frac{16}{3} = 5.33$$

If $H = 0$ $\frac{WL}{M_p} = 21.33$ If $W = 0$ $\frac{HL}{M_p} = 5.33$

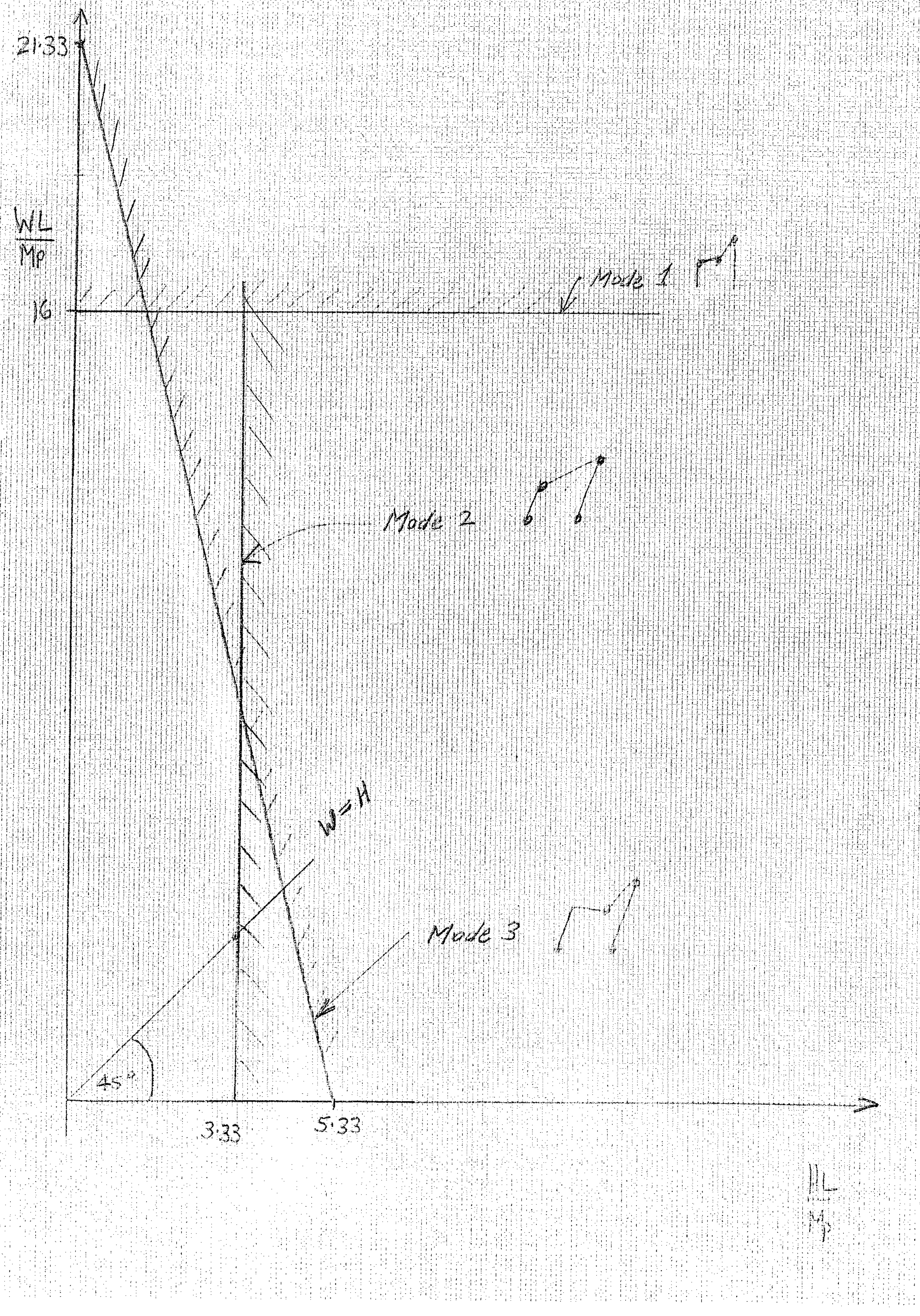
(4)



Combined II

Unlikely to be critical since −ve work must be done by UDL live load so ignore this mechanism.

Q6(ii)



(iii)

If $W = H$, from interaction diagram Mode 2 (Sway) governs.
$L = 5$.

$$\frac{HL}{M_p} = \frac{10}{3} \qquad M_p = \sigma_y Z_p = 350\times10^6 \times 84.2\times10^{-6} = 29.5\times10^3\,Nm$$

$$\Rightarrow H = \frac{10 M_p}{3L} = \frac{10\times 29.5}{3\times 5} = 19.7\,kN = W.$$

$$\left(\Rightarrow w = \frac{W}{L} = 3.93\,kN/m.\right)$$

Note: Graphically, can see that critical load is $\frac{HL}{M_p} = \frac{10}{3} = 3.33$

Question 6 Plastic collapse of a frame (PLASTIC)
(196 attempts, average 11.2/20)

This question was also quite popular but rather poorly answered. As a result the original marking scheme was adjusted to give more weight to the correct identification of compatible failure mechanisms than had originally been intended. It was very disappointing to see how many students drew "collapse mechanisms" which were in fact either statically determinate, or in some cases statically indeterminate rigid frames. The other common mistake was to postulate collapse mechanisms with a full beam mechanism with 3 hinges plus an additional hinge at one (or even two) of the supports. The derivation of the work equation for the combined mechanism also proved problematic for many candidates.