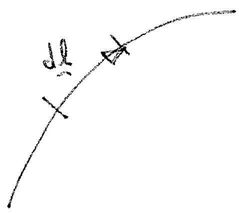
CRIB

1. (a) $\underset{\sim}{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}\right)$

$$\begin{cases} \dfrac{\partial V}{\partial x} = -\dfrac{\pi}{d} V_0 e^{-\frac{\pi x}{d}} \cos\dfrac{\pi y}{d} \\ \dfrac{\partial V}{\partial y} = -\dfrac{\pi}{d} V_0 e^{-\frac{\pi x}{d}} \sin\dfrac{\pi y}{d} \end{cases}$$

$$\underset{\sim}{E} = \frac{\pi}{d} V_0 \exp\left(\frac{-\pi x}{d}\right)\left[\cos\frac{\pi y}{d}\, \underset{\sim}{i} + \sin\frac{\pi y}{d}\, \underset{\sim}{j}\right]$$

To find field lines note: $d\underset{\sim}{l} = \lambda \underset{\sim}{E}$



$\Rightarrow \quad dx = \lambda E_x, \quad dy = \lambda E_y$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{E_y}{E_x}$$

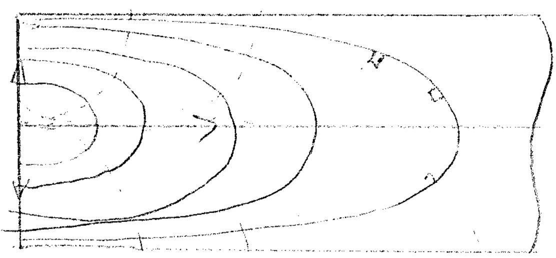
$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin(\pi y/d)}{\cos(\pi y/d)} = \tan\left(\frac{\pi y}{d}\right)$$

So, – flat for $y = 0$
– angle is independent of x, vertical for $y = \pm d/2$
– symmetrical, at 45° for $y = \pm d/4$

On $y = \pm\frac{d}{2}$, $dx = 0$

On $y = 0$, $dy = 0$

On $x = \infty$, $|\underset{\sim}{E}| = 0$



(b) $$\int_O^Q \underset{\frown}{E}\cdot d\underset{\frown}{l} = -\int_O^Q \nabla V\cdot d\underset{\frown}{l}$$

But, $\nabla V\cdot d\underset{\frown}{l} = dV$

$$\Rightarrow \int_O^Q \underset{\frown}{E}\cdot d\underset{\frown}{l} = -\int_O^Q dV = V_O - V_Q = \underline{\underline{V_0}}$$

OR (For those who don't see it ...)

$$\int_O^Q \underset{\frown}{E}\cdot d\underset{\frown}{l} = \int_O^P E_x\,dx + \int_P^Q E_y\,dy$$

$$= \int_0^L E_x(y=0,x)\,dx + \int_0^{d/2} E_y(x=L,y)\,dy$$

$$= \int_0^L \frac{\pi}{d}V_0\exp\left[\frac{-\pi x}{d}\right]dx + \int_0^{d/2}\frac{\pi}{d}V_0\exp\left[\frac{-\pi L}{d}\right]\sin\frac{\pi y}{d}\,dy$$

$$= \left[-V_0\exp\left[\frac{-\pi x}{d}\right]\right]_0^L + \left[-V_0\exp\left[\frac{-\pi L}{d}\right]\cos\left(\frac{\pi y}{d}\right)\right]_0^{d/2}$$

$$= \left[-V_0\exp\left[\frac{-\pi L}{d}\right] + V_0\right] + \left[0 + V_0\exp\left[\frac{-\pi L}{d}\right]\right]$$

$$= \underline{\underline{V_0}} \quad \text{(The hard way!)}$$

(c)

- Conservative if $\nabla\times\underset{\frown}{E} = 0$.

  But $\nabla\times(\nabla V) = 0$, for all functions V.

  Thus $\underset{\frown}{E}$ is conservative. (<u>Alternatively</u>, evaluate $\nabla\times\underset{\frown}{E}$.)

- Solenoidal if $\nabla\cdot\underset{\frown}{E} = 0$.

  Check: $$\nabla\cdot\underset{\frown}{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \frac{\pi V_0}{d}\exp\left[\frac{-\pi x}{d}\right]\left[-\frac{\pi}{d}\cos\frac{\pi y}{d} + \frac{\pi}{d}\cos\frac{\pi y}{d}\right]$$
  $$= 0$$

  Thus $\underset{\frown}{E}$ is solenoidal.

nb. $\oint \underline{E} \cdot d\underline{A} = \int (\nabla \cdot \underline{E})\, dV$ (from Gauss)

$= 0$ (since $\underline{E}$ solenoidal)

Or (The hard way)

$$\oint \underline{E} \cdot d\underline{A} = -\int_O^P E_y\, dx + \int_P^Q E_x\, dy + \int_Q^R E_y\, dx - \int_R^O E_x\, dy$$

$$= 0 + \int_0^{d/2} \frac{\pi}{d} V_0 \exp\left[\frac{-\pi L}{d}\right] \cos\left(\frac{\pi y}{d}\right) dy + \int_0^L \frac{\pi}{d} V_0 \exp\left[\frac{-\pi x}{d}\right] dx$$

$$- \int_0^{d/2} \frac{\pi}{d} V_0 \cos\frac{\pi y}{d}\, dy$$

$$= \left[V_0 \exp\left[\frac{-\pi L}{d}\right] \sin\left(\frac{\pi y}{d}\right)\right]_0^{d/2} + \left[-V_0 \exp\left[\frac{-\pi x}{d}\right]\right]_0^L$$

$$- \left[\sin\left(\frac{\pi y}{d}\right)\right]_0^{d/2}$$

$$= V_0 \exp\left[\frac{-\pi L}{d}\right] + \left[-V_0 \exp\left[\frac{-\pi L}{d}\right] + 1\right] - [1]$$

$$= \underline{\underline{0}} \quad \text{(as above)}$$

(d) By definition $\underline{E} = \nabla \times \underline{A} = \nabla \times (A(x,y)\underline{k})$

$$\Rightarrow \begin{cases} E_x = \dfrac{\partial A}{\partial y} = \dfrac{\pi}{d} V_0 \exp\left[\dfrac{-\pi x}{d}\right] \cos\dfrac{\pi y}{d} & \text{— ①} \\[2ex] E_y = -\dfrac{\partial A}{\partial x} = \dfrac{\pi}{d} V_0 \exp\left[\dfrac{-\pi x}{d}\right] \sin\dfrac{\pi y}{d} & \text{— ②} \end{cases}$$

Integrate ①: $A = V_0 \exp\left[\frac{-\pi x}{d}\right] \sin\frac{\pi y}{d} + g_1(x)$

Integrate ②: $A = V_0 \exp\left[\frac{-\pi x}{d}\right] \sin\frac{\pi y}{d} + g_2(y)$

Compare: $\underline{A = V_0 \exp[-\pi x/d] \sin(\pi y/d)}$, $g_1 = g_2 = \text{const}$

This question on vector integration and potentials was answered well. Few respondents were able to draw good diagrams of the potential contours in part (a). Most chose to answer (b) by solving the line integral which they did very well. I was surprised that they did not point out the change in potential, and worry that some may not have done so because it seemed too obvious. Part (c) on the definition of "conservative" and "solenoidal" gave no problems. Most solved part (d) on the vector potential of a vector field by inspection rather than rigorous analysis.

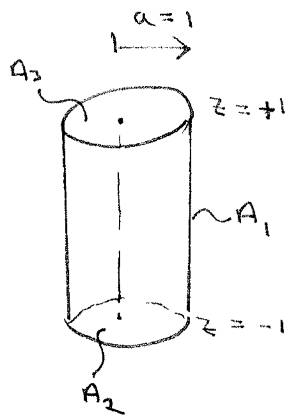
2 (a) $\underset{\sim}{v} = \nabla f = \nabla\left(z\exp(-x^2+y^2)\right)$

$$= \left[-2xz e^{-(x^2+y^2)},\ -2yz e^{-(x^2+y^2)},\ e^{-(x^2+z^2)}\right]$$

$$= \underline{\left[-2xz,\ -2yz,\ 1\right]\exp\left[-x^2-y^2\right]}$$

(b) $\oint \nabla\times\underset{\sim}{v}\cdot d\underset{\sim}{A} = \oint\nabla\times(\nabla f)\cdot d\underset{\sim}{A} = 0$

$(\nabla\times\nabla f = 0)$



(c) $\int \nabla\cdot\underset{\sim}{v}\ dV = \oint \underset{\sim}{v}\cdot d\underset{\sim}{A}$

Divide integral into three parts.

$$I = \int_{A_1}\underset{\sim}{v}\cdot d\underset{\sim}{A} + \int_{A_2}\underset{\sim}{v}\cdot d\underset{\sim}{A} + \int_{A_3}\underset{\sim}{v}\cdot d\underset{\sim}{A}$$

$$= \int_{A_1} v_r\, dA - \int_{A_2} v_z\, dA + \int_{A_3} v_z\, dA \qquad \left[\text{in } (r,\theta,z) \text{ coordinates}\right]$$

But A_2 and A_3 integrals cancel and

$$(v_x, v_y) = -[x, y]\,2z e^{-(x^2+y^2)} = -r\hat{\underset{\sim}{e}}_r\, 2z e^{-r^2}$$

$$\Rightarrow\quad v_r = -2rz e^{-r^2}$$

Thus,

$$I = \int_{A_1} v_r\, dA = -2\int rz e^{-r^2} dA = -2ae^{-a^2}\int z\, dA$$

$$= \underline{0}$$

(because z is anti-symmetric)

$$\nabla\cdot\left[f^2 \underset{\sim}{v} \right] = f^2\, \nabla\cdot\underset{\sim}{v} + (\underset{\sim}{v}\cdot\nabla) f^2$$

$$= f^2\, \nabla\cdot(\nabla f) + 2f(\underset{\sim}{v}\cdot\nabla) f$$

$$= f^2\, \nabla^2 f + 2f\left[\underset{\sim}{v}\cdot\nabla f\right]$$

$$= \underline{\underline{f^2\, \nabla^2 f + 2f\, \underset{\sim}{v}^2}}$$

Many who tried to use Gauss' theorem in part (c) in order to turn the integration into one with respect to vector area ran into problems, it becoming clear that many still have not quite grasped the concept of integration with respect to vector area. Students who ignored Gauss' theorem did better, perhaps because dV is not a vector increment. The other parts of the question were answered well.

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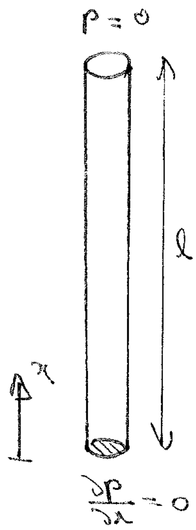
(3) $$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}$$



(a) Use <u>seperation of the variables</u> :

$$p = T(t)\, X(x)$$

$$\Rightarrow \quad T''(t)\, X(x) = c^2 X''(x)\, T(t)$$

$$\Rightarrow \quad \frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)}$$

L.H.S. independent of x, R.H.S. independent of t, so,

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = \text{const}.$$

Need to choose sign of constant.

Note that we want oscillating solutions in time (or space) and so constant must be ~~positive~~ negative:

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = -\lambda^2, \quad \text{for some } \lambda.$$

$$\Rightarrow \begin{cases} T''(t) + \lambda^2 T(t) = 0 \\ X''(x) + \dfrac{\lambda^2}{c^2} X(x) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X = A \cos\dfrac{\lambda x}{c} + B \sin\dfrac{\lambda x}{c} \\ T = D \cos(\lambda t + \phi) \end{cases}$$

A, B, D and ϕ are constants.

But we can absorb D into defn. of A, B, so,

$$\begin{cases} X(x) = A \cos\dfrac{\lambda x}{c} + B \sin\dfrac{\lambda x}{c} \\ T(t) = \cos(\lambda t + \phi) \end{cases}$$

Boundary conditions require: $X'(0) = 0$, $X(l) = 0$

$$X'(x) = \frac{\lambda}{c}\left(-A \sin\frac{\lambda x}{c} + B \cos\frac{\lambda x}{c}\right)$$

$\Rightarrow$ $X'(0) = 0$ demands $B = 0$

$$\Rightarrow \quad X = A \cos\frac{\lambda x}{c}$$

But $X(l) = 0 \Rightarrow \dfrac{\lambda l}{c} = \dfrac{\pi}{2}, \dfrac{3\pi}{3}, \dfrac{5\pi}{2} + \ldots$

$$\Rightarrow \underline{\underline{\frac{\lambda}{c} = \frac{n\pi}{2l}}}, \quad \underline{\underline{n \text{ odd}}}$$

$$\Rightarrow \begin{cases} \underline{\underline{X(x) = A \cos\dfrac{n\pi x}{2l}}} \\ \underline{\underline{T(t) = \cos(\lambda t + \phi)}}, \quad \underline{\underline{\lambda = \dfrac{n\pi c}{2l}}} \quad (\underline{\underline{n \text{ odd}}}) \end{cases}$$

This is the solution for a single harmonic.

(b) We have an infinite number of solutions

$$p_n(x,t) = X_n(x)\,T_n(t)$$

$$= A_n \cos\left(\frac{n\pi x}{2l}\right)\cos(\lambda_n t + \phi_n)$$

$$\left(\lambda_n = \frac{n\pi c}{2l}\right)$$

By superposition the sum of these is also a solution

$$p = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{2l}\right)\cos(\lambda_n t + \phi_n)$$

(c)

$$\frac{\partial p}{\partial t} = -\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{2l}\right)\lambda_n \sin(\lambda_n t + \phi_n)$$

$$\left.\frac{\partial p}{\partial t}\right)_{t=0} = -\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{2l}\right)\lambda_n \sin\phi_n = 0$$

Only possibility is $\phi_n = 0$. Thus,

$$\underline{\underline{p(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{2l}\right)\cos(\lambda_n t)}}$$

To find A_n use

$$p(x,0) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{2l}\right) = f(x)$$

Now use Fourier analysis to find coefficients A_n.

This question on the wave equation should have been quite straightforward but marks were low and it seems partly that candidates were confused by what was expected of them in part (c), and partly by the nature of travelling versus standing waves. Most candidates attempted a solution of the wave equation from first principles, and some seemed to confuse themsevels by looking for the exceptional case when this is in fact a standard solution.

1)

subtract $1\times$ row 1
subtract $2\times$ row 1

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & -1 & 2 & 0 \\ 2 & 2 & 0 & 1 & -1 \end{bmatrix} \underline{x} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

subtract $2\times$ row 2

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 2 & -4 & -1 & -1 \end{bmatrix} \underline{x} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 2 & -3 & -1 \end{bmatrix} \underline{x} = \begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix}$$

upper triangular matrix, U

pivots are 1, 1, 2

obtain, $\underline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 2 & -3 & -1 \end{bmatrix}$

$A\underline{x} = \underline{b} \Rightarrow LU\underline{x} = \underline{b}$

$U\underline{x} = \underline{c} \Rightarrow L\underline{c} = \underline{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \underline{c} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

hence, by inspection, $\underline{c} = \begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix}$

particular solution:

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 2 & -3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix} \Rightarrow \text{Particular solution is } \begin{bmatrix} 5 \\ -9/2 \\ -5/2 \\ 0 \\ 0 \end{bmatrix}$$

zero, because there are only 3 pivots.

homogeneous solutions:

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 2 & -3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{x} = \begin{bmatrix} -4 \\ 7/2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 2 & -3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{x} = \begin{bmatrix} -1 \\ 3/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$$

General solution

$$\underline{x} = \begin{bmatrix} 5 \\ -9/2 \\ -5/2 \\ 0 \\ 0 \end{bmatrix} + \lambda\begin{bmatrix} -4 \\ 7/2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} + \mu\begin{bmatrix} -1 \\ 3/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$$

1)(1) A basis of the <u>column space</u> is formed by the columns of A where a pivot is found during the elimination, i.e.:

$$\begin{bmatrix}1\\1\\2\end{bmatrix} \quad \begin{bmatrix}0\\1\\2\end{bmatrix} \quad \begin{bmatrix}2\\-1\\0\end{bmatrix}$$

(2) A basis of the <u>row space</u> is given by the first three rows of U:

$$\begin{bmatrix}1\\0\\2\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\-3\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\2\\-3\\-1\end{bmatrix}$$

(3) A basis of the <u>null space</u> of A is the n-r homogeneous solutions $Ux=0$ where n = number of rows $=5$ and r = rank $=3$:

$$\begin{bmatrix}-4\\7/2\\3/2\\1\\0\end{bmatrix}, \quad \begin{bmatrix}-1\\3/2\\1/2\\0\\1\end{bmatrix}$$

(4) A basis of the <u>left null space</u> of A has dimension $m-r$ where m = number of columns $=3$, so the basis is the zero vector:

$$\begin{bmatrix}0\\0\\0\end{bmatrix}$$

A popular question - few candidates did not attempt it. The most difficult part seemed to be part (a), the naming of the pivots. It really is worth getting this right: if one thinks there are only two pivots, then it becomes tougher to get marks in the rest of the question. Most candidates managed the LU decomposition of part (b) correctly, then there was the usual spread of performance on parts (c) (solution of the matrix equation) and part (d).

Solutions to Part 1B Paper 7 Section B (Q5) and Section C, 2003

5. *QR-decomposition*

(a) Relationship may be written as

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & -4 & -2 \\ 2 & -1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}$$

[2]

(b) Need to do QR decomposition

$$\mathbf{q}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{q}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

The overall decomposition is

$$\mathbf{A} = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & -\sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 6\sqrt{2} & -3\sqrt{2} & 3\sqrt{2} \\ 0 & 6\sqrt{3} & 4\sqrt{3} \\ 0 & 0 & -6 \end{bmatrix}$$

[10]

(c) Since $\mathbf{A}$ is square and of full rank, there is a unique, exact, solution.

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{QRx} &= \mathbf{b} \\ \mathbf{Rx} &= \mathbf{Q}^T\mathbf{b} \end{aligned}$$

Hence

$$\begin{bmatrix} 6\sqrt{2} & -3\sqrt{2} & 3\sqrt{2} \\ 0 & 6\sqrt{3} & 4\sqrt{3} \\ 0 & 0 & -6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 11\sqrt{2} \\ -5\sqrt{3} \\ -7 \end{bmatrix}$$

So

$$\mathbf{x} = \frac{1}{18} \begin{bmatrix} 8 \\ -29 \\ 21 \end{bmatrix}$$

Check by substituting into original relationship.

Relationship from data is

$$w = \frac{1}{18}(8r - 29s + 21t)$$

[8]

[**Examiners comments:**
This question tested the students knowledge of QR decomposition and how it may be used. The results on this question were disappointing, given that the question was very straight-forward. Many candidates appeared to use an algorithm that they memorised to do the QR decomposition with little understanding of what they were expecting as a result. Very few checked that the matrix derived was orthonormal, nor that the relationship derived was appropriate for the data given in the question.
]

6. *Discrete Fourier Transforms*

[For this question due to the wording it was also acceptable to consider the period as T/N]

(a) The frequency of the highest order harmonic is $\frac{(N-1)}{NT}$ Hz. [4]

(b) Summation expression for components of the Fourier transform (from data book)

$$\begin{aligned} F_k &= \sum_{n=0}^{N-1} f_n e^{-jnk2\pi/N} \\ &= \sum_{n=0}^{N-1} \sinh(nT) e^{-jnk2\pi/N} \end{aligned}$$

Question asks for magnitude

$$|F_k| = \left| \sum_{n=0}^{N-1} \sinh(nT) e^{-jnk2\pi/N} \right|$$

[6]

(c) Expressing in trignometric functions

$$\begin{aligned} F_k &= \sum_{n=0}^{N-1} \sinh(nT) e^{-jnk2\pi/N} \\ &= \sum_{n=0}^{N-1} \frac{1}{2} \left(e^{nT} - e^{-nT}\right) e^{-jnk2\pi/N} \\ &= \frac{1}{2} \sum_{n=0}^{N-1} \left(e^{n(T-jk2\pi/N)} - e^{-n(T+jk2\pi/N)}\right) \end{aligned}$$

This is the sum of two geometric expressions

$$\begin{aligned} \sum_{n=0}^{N-1} e^{n(T-jk2\pi/N)} &= \frac{1-e^{N(T-jk2\pi/N)}}{1-e^{(T-jk2\pi/N)}} \\ &= \frac{1-e^{NT}e^{jk2\pi}}{1-e^{(T-jk2\pi/N)}} \\ &= \frac{1-e^{NT}}{1-e^{(T-jk2\pi/N)}} \end{aligned}$$

Similarly for the second term gives

$$\begin{aligned} F_k &= \frac{1}{2}\left(\frac{1-e^{NT}}{1-e^{(T-jk2\pi/N)}} - \frac{1-e^{-NT}}{1-e^{(-T-jk2\pi/N)}}\right) \\ &= \frac{1}{2}\left(\frac{(1-e^{NT})(1-e^{(-T-jk2\pi/N)}) - (1-e^{-NT})(1-e^{(T-jk2\pi/N)})}{e^{-jk2\pi/N}(e^{jk2\pi/N} + e^{-jk2\pi/N} - e^{T} - e^{-T})}\right) \end{aligned}$$

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$$= \frac{1}{2}\left(\frac{(1-e^{NT})(e^{jk2\pi/N}-e^{-T})-(1-e^{-NT})(e^{jk2\pi/N}-e^{T})}{e^{jk2\pi/N}+e^{-jk2\pi/N}-e^{T}-e^{-T}}\right)$$

$$= \frac{1}{2}\left(\frac{e^{T}-e^{-T}+e^{(N-1)T}-e^{-(N-1)T}-e^{jk2\pi/N}(e^{NT}-e^{-NT})}{\cos(k2\pi/N)-\cosh(T)}\right)$$

$$= \frac{1}{2}\left(\frac{\sinh(T)+\sinh((N-1)T)-(\cos(k2\pi/N)+j\sin(k2\pi/N))\sinh(NT)}{\cos(k2\pi/N)-\cosh(T)}\right)$$

[10]

[**Examiners comments:**
A very unpopular question on discrete fourier transforms. Some students moved the magnoitude sign inside the summation. Only a few got the geometric progression, despite this being similar to an example paper question.]

7. *Fourier Transforms and Probability*

(a)(i) Taking the inverse Fourier transform directly

$$\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}(1+\cos(\omega))e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}(2e^{j\omega t}+e^{j\omega(1+t)}+e^{j\omega(t-1)})d\omega \\
&= \frac{1}{4\pi}\left[\frac{2}{jt}e^{j\omega t}+\frac{1}{j(1+t)}e^{j\omega(1+t)}+\frac{1}{j(t-1)}e^{j\omega(t-1)}\right]_{-\pi}^{\pi} \\
&= \frac{1}{4\pi}\left(4\frac{\sin(\pi t)}{t}+2\frac{\sin(\pi(1+t))}{1+t}+2\frac{\sin(\pi(t-1))}{t-1}\right) \\
&= \frac{1}{2}(2\mathrm{sinc}(\pi t)+\mathrm{sinc}(\pi(1+t))+\mathrm{sinc}(\pi(t-1)))
\end{aligned}$$

[If desired this may be further re-expressed using

$$\begin{aligned}
\sin(\pi t+\pi) &= \sin(\pi t)\cos(\pi)+\cos(\pi t)\sin(\pi) \\
&= -\sin(\pi t)
\end{aligned}$$

Similarly

$$\begin{aligned}
\sin(\pi t-\pi) &= \sin(\pi t)\cos(\pi)-\cos(\pi t)\sin(\pi) \\
&= -\sin(\pi t)
\end{aligned}$$

So

$$\begin{aligned}
f(t) &= \frac{1}{2\pi}\left(\sin(\pi t)\left[\frac{2}{t}-\frac{1}{1+t}-\frac{1}{t-1}\right]\right) \\
&= \frac{1}{2\pi}\sin(\pi t)\left(\frac{-2}{t(t^2-1)}\right) \\
&= \left(\frac{1}{1-t^2}\right)\mathrm{sinc}(\pi t)
\end{aligned}$$

] [8]

(ii) From Lecture Notes

$$f(\alpha t) \leftrightarrow \frac{1}{\alpha}F\left(\frac{\omega}{\alpha}\right)$$

and

$$f(t-\tau) \leftrightarrow F(\omega)e^{-j\omega\tau}$$

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Initially delay and then scale in time so that the function lasts twice as long ($\alpha = \frac{1}{2}$)

$$F(\omega) = \begin{cases} 2\left(1+\cos\left(2\omega\right)\right)e^{-j\omega 2\tau} & \frac{-\pi}{2} < \omega < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

[Question is not clear as to whether the time delay is scaled as well. If time delay not scaled then

$$F(\omega) = 2\left(1+\cos\left(2\omega\right)\right)e^{-j\omega\tau}$$

]

[6]

(b) (i) Probability both balls greeen

$$P(G,G) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28} = 0.107$$

[3]

(ii) Probability both balls are red

$$P(R,R) = \frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$$

So probability that both balls the same

$$P(G,G) + P(R,R) = \frac{13}{28} = 0.464$$

[3]

[**Examiners comments:**
A popular and generally well done question. The question tested the students knowledge of the inverse Fourier transform and the effects of time delay and scaling on the spectrum. A few candidates had difficulty with the straight-forward (and standard) integration. A large number could not correctly derive the spectrum after a delay and scaling. In particular few could correctly interprete the effects of scaling in time. The simple probabilty questions in the second section were well answered.]

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8. *Binomial and Poisson Distribution*

(a) Probability that two bulbs fail

$$P(2 \text{ fail}) = \binom{12}{2}(0.01)^2(1-0.01)^{10} = 0.0060$$

[Alternative answer, depending on reading of question, is that two or more bulbs fail hence

$$P(\geq 2 \text{ fail}) = 1 - \left((1-0.01)^{12} + \binom{12}{1}(0.01)(1-0.01)^{11}\right) = 0.0062$$

]

[4]

(b) The Poisson distribution can be used to approximate the Binomial distribution provided that

- number of events n is large (normally $n > 50$)
- probability of event, p, occuring is small (normally $p < 0.1$)

[3]

(c) Probability that 5 or more light bulbs out of 1000 fail - using Poisson distribution,

$$P(r \text{ fail}) = e^{-\lambda}\frac{\lambda^r}{r!}$$

with mean $\lambda = 1000 \times 0.01 = 10$ is

$$\begin{aligned} P(\geq 5 \text{ fail}) &= 1 - (P(0 \text{ fail}) + P(1 \text{ fail}) + P(2 \text{ fail}) + P(3 \text{ fail}) + P(4 \text{ fail})) \\ &= 1 - e^{-10}\left(1 + \frac{10}{1} + \frac{10^2}{2} + \frac{10^3}{6} + \frac{10^4}{24}\right) \\ &= 0.9707 \end{aligned}$$

[Alternative approach is to use the Binomial expansion -

$$\begin{aligned} P(\geq 5 \text{ fail}) &= 1 - (P(0 \text{ fail}) + P(1 \text{ fail}) + P(2 \text{ fail}) + P(3 \text{ fail}) + P(4 \text{ fail})) \\ &= 1 - (0.0000 + 0.0004 + 0.0022 + 0.0074 + 0.0186) = 0.9713 \end{aligned}$$

]

[6]

(d) Using the Poisson approximation the mean and the variance are the same

$$\mu = \sigma^2 = 10$$

The exact solution from the binomial distribution

$$\begin{aligned} \mu &= np = 1000 \times 0.01 = 10 \\ \sigma^2 &= npq = 1000 \times 0.01 \times 0.99 = 9.9 \end{aligned}$$

[2]

(e) Assume that the light bulb will have the same probability of failing in the second block of ten hours as the first. Using the poisson distribution approximation, double the time period will double the value of λ. Hence [5]

$$\mu = \sigma^2 = 20$$

[**Examiners comments:**
A popular and generally well done question. This question examined the candidates knowledge of the Binomial and Posson distribution. A number of candidates proposed a normal distribution as a good approximation without considering that the binomial distribution is discrete and that for DeMoivre-Laplace theorem $npq >> 1$ which is not the case here.]