ENGINEERING TRIPOS PART IB

Friday 6 June 2003 9 to 11

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer at least ***one*** *question from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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SECTION A

*Answer at least **one** question from this section.*

1 A long metal plate occupies the region $x > 0$, $-d/2 < y < d/2$, as shown in Fig 1. The top and bottom of the plate are held at zero voltage. A positive voltage distribution is applied to the left-hand side of the plate such that:

$$V = V_0 \exp\left(-\frac{\pi x}{d}\right)\cos\frac{\pi y}{d}$$

(a) The electric field in the plate is related to V by $\mathbf{E} = -\nabla V$. Find $\mathbf{E}$ and sketch the electric field lines and isosurfaces of V. [4]

(b) Calculate the line integral $\int \mathbf{E}\cdot\mathbf{dl}$ from O to Q along the path OPQ and show that it is independent of L. [5]

(c) Show that $\mathbf{E}$ is both conservative and solenoidal and evaluate the closed surface integral $\oint \mathbf{E}\cdot\mathbf{dA}$ for the rectangular surface OPQR. [5]

(d) The vector potential for $\mathbf{E}$ is of the form $A(x,y)\mathbf{k}$ where $\mathbf{k}$ is a unit vector in the z direction. Find $A(x,y)$. [6]

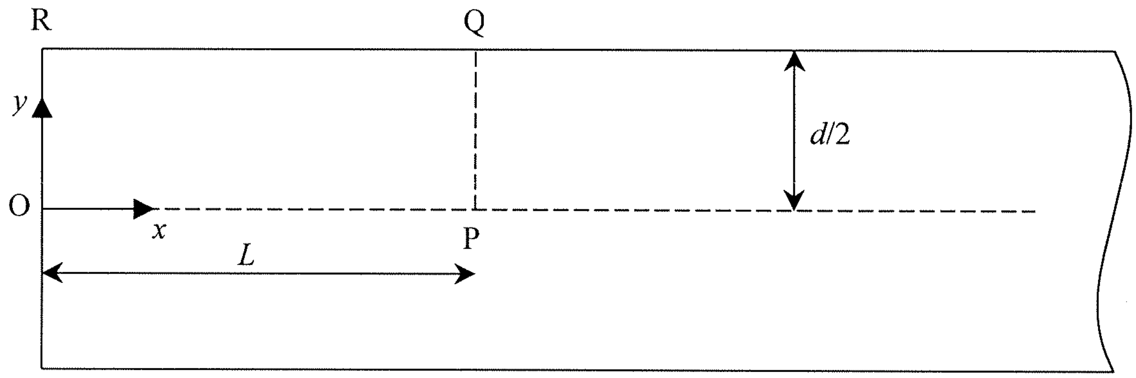

Fig. 1

2 Let $\mathbf{v} = \nabla f$ where $f = ze^{-\left(x^2+y^2\right)}$

(a) Calculate $\mathbf{v}$. [2]

(b) Evaluate $\iint_S \nabla \times \mathbf{v} \cdot \mathbf{dA}$ for the surface S where S is the surface of a closed cylinder of unit radius whose axis runs from $z = -1$ to $z = 1$. [4]

(c) Evaluate $\iiint_V \nabla \cdot \mathbf{v}\, dV$ for the volume enclosed by S. [10]

(d) Show that $\nabla \cdot [f^2 \mathbf{v}] = f^2 \nabla^2 f + 2fv^2$. [4]

3 The pressure distribution in an organ pipe is governed by:

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}$$

where c is the speed of sound. The pipe has length ℓ.

The boundary conditions on p are $\partial p/\partial x = 0$ at $x = 0$ and $p = 0$ at $x = \ell$.

(a) Find the form of $p(x,t)$ corresponding to a single harmonic. [8]

(b) Use the principle of superposition to find the most general form for $p(x,t)$. [5]

(c) Show how you would find $p(x,t)$ corresponding to the initial conditions $p(x,t) = f(x)$ and $\partial p/\partial t = 0$. [7]

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SECTION B

*Answer at least **one** question from this section.*

4 Let $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ where:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & -1 & 2 & 0 \\ 2 & 2 & 0 & 1 & -1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

(a) State the pivots of **A**. [2]

(b) Express **A** as the product of a lower triangular matrix and an upper triangular matrix. [6]

(c) Find the general solution of $\mathbf{A}\,\mathbf{x} = \mathbf{b}$. [6]

(d) Find a basis for each of the four fundamental subspaces of **A**. [6]

5 The following experimental data are measured:

r	s	t	w
2	2	2	0
2	−4	−2	5
2	−1	3	6

(a) Express the relationship of w versus r, s and t in the form of a matrix equation of the kind $\mathbf{A}\,\mathbf{x} = \mathbf{b}$. [2]

(b) Express **A** as the product of an orthonormal matrix and an upper triangular matrix. [10]

(c) Find the least squares solution of $\mathbf{x}$ in terms of r, s and t. [8]

SECTION C

*Answer at least **one** question from this section.*

6 N samples of the function $\sinh t$ are taken at equal intervals over a sampling period T in order to evaluate its discrete Fourier Transform.

(a) State in terms of N and T the frequency of the highest order harmonic of the discrete Fourier transform. [4]

(b) Find a summation expression for the magnitude of each component of the discrete Fourier transform. [6]

(c) Evaluate the discrete Fourier transform giving your answer in terms of trigonometric and hyperbolic functions. [10]

7 (a) A function $f(t)$ has a Fourier transform $F(\omega)$ which equals $1+\cos\omega$ over the range $-\pi < \omega < \pi$ and is zero elsewhere.

(i) Calculate the function $f(t)$. [6]

(ii) Find the Fourier transform of $f(t)$ when it is first delayed by τ and then scaled in time so that it lasts twice as long. [6]

(b) A bag contains five red balls and three green balls. Two balls are removed from the bag.

(i) Calculate the probability that both balls are green. [3]

(ii) Calculate the probability that both balls are the same colour. [5]

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8 A manufacturer produces light bulbs which have a probability of 0.01 that they fail within the first ten hours of use.

(a) Find the probability that, if a customer buys twelve light bulbs, two will fail within the first ten hours of use. [4]

(b) State which distribution can be used to approximate the Binomial Distribution and give the conditions for which this approximation is valid. [3]

(c) If 1000 new light bulbs are fitted in a ship, estimate the probability that five or more will fail in the first ten hours of use. [6]

(d) Find the mean and variance for the number of light bulbs which will fail within the first ten hours of use. [2]

(e) Find the mean and variance for the number of light bulbs which will fail within the first twenty hours of use and explain your reasoning. [5]

END OF PAPER