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ENGINEERING TRIPOS PART IB

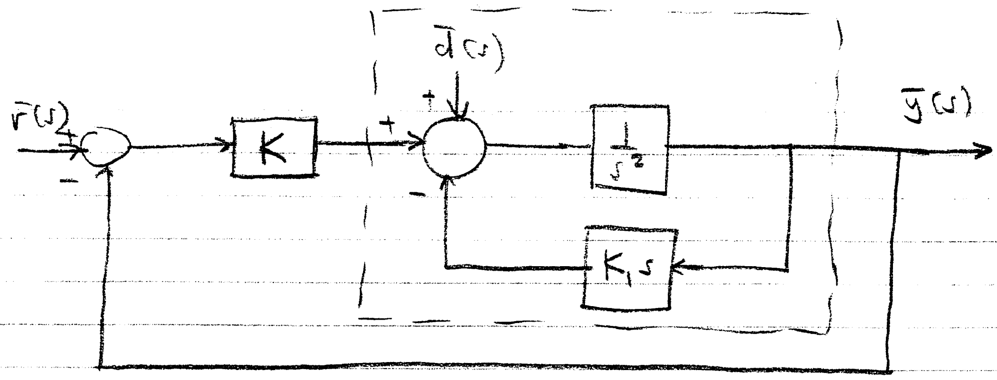
2004

PAPER 6 – INFORMATION ENGINEERING

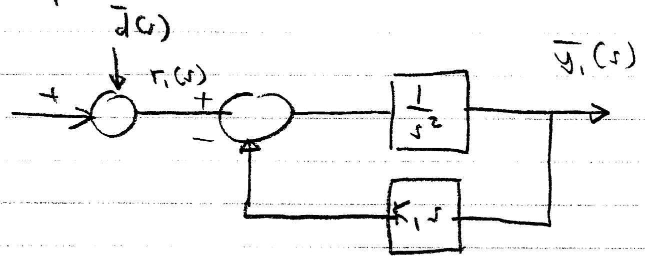
SOLUTIONS

(1)

1.) a)



Redraw part in dashed box,



[1]

For a feedback system we can write

$$\bar{y}_1(s) = \frac{G(s)\,K(s)}{1 + H(s)\,G(s)\,K(s)}\,\bar{r}_1(s)$$

where for the system above, $G(s)\,K(s) = \frac{1}{s^2}$ and $H(s) = K_1 s$

so

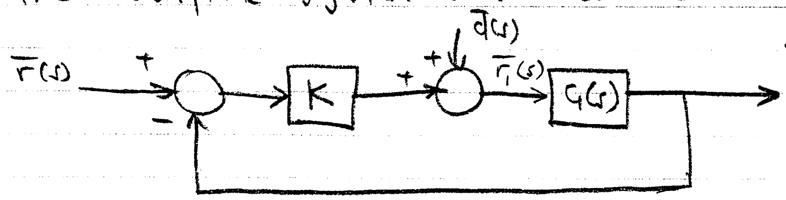
$$\bar{y}_1(s) = \frac{\frac{1}{s^2}}{1 + \frac{1}{s^2}\cdot K_1 s}\,\bar{r}_1(s)$$

So the transfer function is,

$$\frac{\bar{y}_1(s)}{r_1(s)} = G(s) = \frac{1}{s(s+K_1)}$$

[2]

So the complete system can now be redrawn as,



[1]

Now to work out the CLTF,

$$\bar{y}(s) = \bar{r}_1(s)\,G(s)$$

$$\therefore \bar{r}_1(s) = \frac{\bar{y}(s)}{G(s)}$$

Also,

$$\bar{r}_1(s) = K(\bar{r}(s) - \bar{y}(s)) + d(s)$$

$$\frac{\bar{y}(s)}{G(s)} = K(\bar{r}(s) - \bar{y}(s)) + d(s)$$

$$\bar{y}(s) = G(s)K\bar{r}(s) - G(s)K\bar{y}(s) + G(s)\bar{d}(s)$$

$$\bar{y}(s)[1 + KG(s)] = K\,G(s)\bar{r}(s) + G(s)\,\bar{d}(s)$$

(2)

$$\bar{y}(s) = \frac{K\,G(s)}{1+K\,G(s)}\,\bar{r}(s) + \frac{G(s)}{1+K\,G(s)}\,\bar{d}(s)$$ [4]

b) Now,

$$e(t) = r(t) - y(t)$$ so

$$\bar{e}(s) = \bar{r}(s) - \bar{y}(s)$$

$\therefore$

$$\bar{e}(s) = \bar{r}(s) - \frac{K\,G(s)}{1+K\,G(s)}\,\bar{r}(s) - \frac{G(s)}{1+K\,G(s)}\,\bar{d}(s)$$

$$\bar{e}(s) = \left(1 - \frac{K\,G(s)}{1+K\,G(s)}\right)\bar{r}(s) - \frac{G(s)}{1+K\,G(s)}\,\bar{d}(s)$$

(i) For the case where

$\bar{r}(s) = 0$ and $\bar{d}(s) = \frac{1}{s}$,

$$\bar{e}(s) = -\frac{G(s)}{1+K\,G(s)}\cdot\frac{1}{s}$$

Sub for $G(s)$ gives

$$\bar{e}(s) = -\frac{\frac{1}{s(s+K_1)}}{1+\frac{K}{s(s+K_1)}}\cdot\frac{1}{s}$$

$$\bar{e}(s) = -\frac{1}{s(s+K_1)+K}\cdot\frac{1}{s}$$

Using FVT,

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,\bar{e}(s)$$

$$= \lim_{s\to 0}\frac{-1}{s(s+K_1)+K} = -\frac{1}{K}$$ [3]

(ii) For the case where,

$\bar{r}(s) = \frac{b}{s^2}$ and $\bar{d}(s) = 0$

$$\bar{e}(s) = \left(1 - \frac{K\,G(s)}{1+K\,G(s)}\right)\frac{b}{s^2}$$

Sub for $G(s)$ gives,

$$\bar{e}(s) = \left(1 - \frac{\frac{K}{s(s+K_1)}}{1+\frac{K}{s(s+K_1)}}\right)\frac{b}{s^2}$$

$$= \left(1 - \frac{K}{s(s+K_1)+K}\right)\frac{b}{s^2}$$

③

$$\bar{e}(s) = \frac{s(s+K_1)}{s(s+K_1)+K} \cdot \frac{b}{s^2}$$

Using FVT

$$\lim_{s \to 0} s\bar{e}(s) = \lim_{s \to 0} \frac{(s+k_1)b}{s(s+K_1)+K} = \frac{bK_1}{K} \quad [3]$$

c) so,

$$b\frac{K_1}{K} \leqslant 0.16b$$

$$\frac{K_1}{K} \leqslant 0.16$$

ie, $K_1 \leqslant 0.16K$ or $6.25K_1 \leqslant K$ [1]

The characteristic equation is given by,

$$1 + KG(s) = 0$$

$$1 + K\left(\frac{1}{s(s+K_1)}\right) = 0$$

ie, $s^2 + sK_1 + K = 0$ [0]

compare with a 2nd order system CE,

$$s^2 + 2c\omega_n + \omega_n^2 = 0$$

gives,

$\omega_n^2 = K$ ① and $2c\omega_n = K_1$

$\therefore \omega_n = \frac{K_1}{2c}$ ② [2]

so, $\left(\frac{K_1}{2c}\right)^2 = K$, ie $K = \frac{K_1^2}{4c^2}$

for $c = 0.6$,

$$K = \frac{K_1^2}{4(0.6)^2}$$

$$K = \frac{K_1^2}{1.44} \quad \text{or} \quad K_1 = 1.2\sqrt{K} \quad ③$$

ss error for a ramp input (slope b) $= b\frac{K_1}{K}$ ④

This is required to be less than $0.16b$, so

$$b\frac{K_1}{K} \leqslant 0.16b$$

$$\frac{K_1}{K} \leqslant 0.16 \quad ⑤$$

(4)

From (4) it may be tempting to set $K_1 = 0$, since this would reduce the s.s. ramp error to zero. However since K is non-zero this means ω_n is non-zero (see (1)). Hence from (2), if $K_1 = 0$ then this means that c is zero - this in itself is not good, but the question requires that $c = 0.6$.

From (5) $\quad K_1 \leq 0.16K \quad$ or

$$6.25K_1 \leq K$$

Sub for K_1 from (3) gives

$$6.25 \times 1.2\sqrt{K} \leq K$$
$$7.5 \leq \sqrt{K}$$
$$56.25 \leq K$$

so min value for $\underline{K \text{ is } 56.25}$. [2]

From (3),

$$K_1 = 1.2\sqrt{K} = 1.2\sqrt{56.25}$$
$$\underline{K_1 = 9}$$

[1]

(5)

2. a) $\bar{V}_f(s) = R_f \bar{I}_f(s) + L_f s \bar{I}(s)$

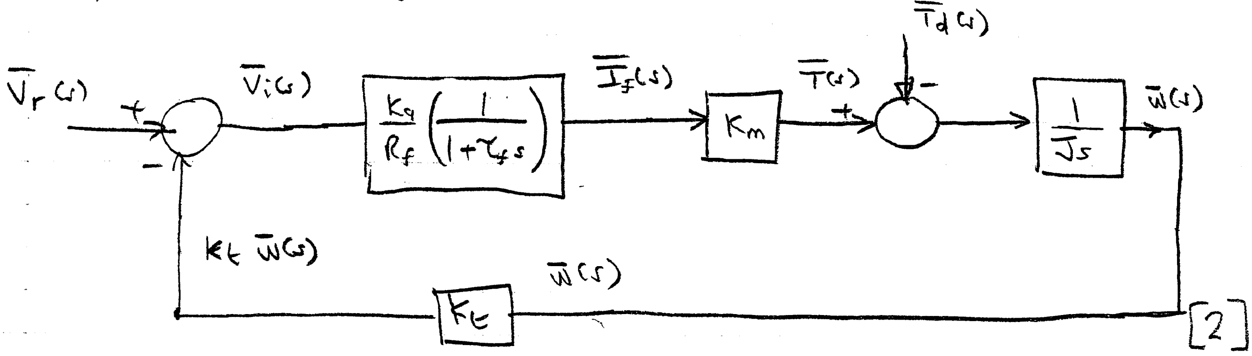
$\bar{V}_f(s) = K_a \bar{V}_i(s)$

so, $K_a \bar{V}_i(s) = R_f \bar{I}_f(s) + L_f s \bar{I}_f(s)$

$K_a \bar{V}_i(s) = (R_f + L_f s)\bar{I}_f(s)$

gives,
$$\bar{I}_f(s) = \frac{K_a \bar{V}_i(s)}{(R_f + L_f s)} = \frac{\frac{K_a}{R_f}\bar{V}_i(s)}{(1 + \frac{L_f}{R_f} s)} = \frac{K_a}{R_f}\left(\frac{\bar{V}_i(s)}{1+\tau_f s}\right),$$
where $\tau_f = L_f / R_f$.

Also $T - T_d = J\ddot{\theta}$
ie, $\bar{T}(s) - \bar{T}_d(s) = J s\, \omega(s)$
and $T = K_m I_f$, ie $\bar{T}(s) = K_m \bar{I}_f(s)$
So, block diagram is,



[2]

So the $OLTF = \frac{K_a}{R_f}\left(\frac{1}{1+\tau_f s}\right) \cdot K_m \cdot \frac{1}{Js} \cdot K_t$

$$= \frac{K_a K_m K_t}{R_f J s (1 + \tau_f s)} = \frac{K_a K_m K_t}{R_f J s (1 + \frac{L_f}{R_f} s)} \quad [2]$$

CLTF from $\bar{V}_r(s)$ to $\bar{\omega}(s)$

$$\bar{\omega}(s) = \frac{1}{Js}\left(\bar{T}(s) - \bar{T}_d(s)\right) = \frac{1}{Js}\left(K_m \bar{I}_f(s) - \bar{T}_d(s)\right)$$

$$\bar{\omega}(s) = \frac{1}{Js}\left(\frac{K_m K_a}{R_f}\left(\frac{1}{1+\tau_f s}\right)\bar{V}_i(s) - \bar{T}_d(s)\right)$$

$$\bar{\omega}(s) = \frac{1}{Js}\left(\frac{K_m K_a}{R_f}\left(\frac{1}{1+\tau_f s}\right)(\bar{V}_r(s) - K_t \bar{\omega}(s)) - \bar{T}_d(s)\right)$$

$$\bar{\omega}(s)\left(1 + \frac{K_m K_a K_t}{J s R_f (1 + \tau_f s)}\right) = \frac{1}{Js}\left(\frac{K_m K_a}{R_f(1+\tau_f s)}\bar{V}_r(s) - \bar{T}_d(s)\right)$$

(6)

$$\bar{\omega}(s)\left(1+\frac{K_m K_a K_t}{J_s R_f(1+\tau_f s)}\right) = \frac{K_m K_a}{R_f J_s(1+\tau_f s)}\bar{V}_r(s) - \frac{1}{J_s}\bar{T}_d(s)$$

so

$$\bar{H}_1(s) = \frac{\bar{\omega}(s)}{\bar{V}_r(s)} = \frac{\dfrac{K_m K_a}{R_f J_s(1+\tau_f s)}}{\left(1+\dfrac{K_m K_a K_t}{J_s R_f(1+\tau_f s)}\right)}$$

$$= \frac{\dfrac{K_m K_a}{R_f J_s(1+\tau_f s)}}{\dfrac{R_f J_s(1+\tau_f s)+K_m K_a K_t}{R_f J_s(1+\tau_f s)}} = \frac{K_m K_a}{R_f J_s(1+\tau_f s)+K_m K_a K_t}$$

[3]

CLTF from $\bar{T}_d(s) \rightarrow \bar{\omega}(s)$

$$\bar{H}_2(s) = \frac{\bar{\omega}(s)}{\bar{T}_d(s)} = \frac{-\dfrac{1}{J_s}}{\left(1+\dfrac{K_m K_a K_t}{J_s R_f(1+\tau_f s)}\right)} = \frac{-1/J_s}{\dfrac{R_f J_s(1+\tau_f s)+K_m K_a K_t}{R_f J_s(1+\tau_f s)}}$$

$$= \frac{-R_f J_s(1+\tau_f s)}{J_s\left[R_f J_s(1+\tau_f s)+K_m K_a K_t\right]}$$

$$= \frac{-R_f(1+\tau_f s)}{R_f J_s(1+\tau_f s)+K_m K_a K_t}$$

[3]

b) The closed-loop characteristic equation is

$$R_f J_s(1+\tau_f s)+K_m K_a K_t$$

$$R_f J_s + R_f \tau_f J_s^2 + K_m K_a K_t$$

$$R_f J_s + \frac{R_f L_f}{R_f} J_s^2 + K_m K_a K_t$$

$$L_f J s^2 + R_f J_s + K_m K_a K_t$$

$$\propto \quad s^2 + \frac{R_f J_s}{L_f J} + \frac{K_m K_a K_t}{L_f J}$$

$$s^2 + \frac{R_f}{L_f}s + \frac{K_m K_a K_t}{L_f J}$$

[1]

Compare with 2nd-order mechanical system,

(7)

i.e, $\frac{R_f}{L_f} = 2\omega_n c$ and $\omega_n^2 = \frac{K_m K_a K_t}{L_f J}$ [2]

so, $\omega_n = \frac{R_f}{2cL_f}$ so $\omega_n^2 = \frac{R_f^2}{4c^2L_f^2}$

$$\therefore \quad \frac{R_f^2}{4c^2L_f^2} = \frac{K_m K_a K_t}{L_f J}$$

or $$\frac{4c^2L_f^2}{R_f^2} = \frac{J L_f}{K_m K_a K_t}$$

$$c^2 = \frac{J L_f}{K_m K_a K_t} \cdot \frac{R_f^2}{4L_f^2}$$

$$c^2 = \frac{J R_f^2}{4 K_m K_a K_t L_f}$$

$$\therefore \quad K_a = \frac{J R_f^2}{4 K_m K_t L_f c^2}$$, sub in values gives [2]

$$K_a = \frac{J}{4 K_m K_t \times 0.1 \left(\frac{1}{\sqrt{2}}\right)^2} = \underline{\underline{\frac{5J}{K_m K_t}}}$$ [1]

c) From final value theorem.
- For case where $T_d = 0$,

$$\lim_{t\to\infty} \omega(t) = \lim_{s\to 0} s\,\bar{\omega}(s)$$

$$\lim_{s\to 0} s\,\bar{\omega}(s) = \lim_{s\to 0} s \frac{K_m K_a}{R_f J s(1+\tau_f s) + K_m K_a K_t} \cdot \overset{\downarrow \text{step}}{\frac{\alpha}{s}}$$

$$= \lim_{s\to 0} \frac{K_m K_a}{R_f J s + R_f \tau_f J s^2 + K_m K_a K_t} \cdot \alpha$$

$$= \frac{K_m K_a \alpha}{K_m K_a K_t} = \underline{\underline{\frac{\alpha}{K_t}}}$$ [2]

For case where $T_d \neq 0$,

$$\lim_{t\to\infty} \omega(t) = \lim_{s\to 0} s\,\bar{\omega}(s)$$

$$= \lim_{s\to 0} \frac{-s R_f (1+\tau_f s)}{R_f J s (1+\tau_f s) + K_m K_a K_t} \cdot \frac{1}{s}$$

$$= \frac{-R_f}{K_m K_a K_t}$$ [2]

so final angular velocity $= \frac{\alpha}{K_t} - \frac{R_f}{K_m K_a K_t} = \underline{\underline{\frac{1}{K_t}\left(\alpha - \frac{R_f}{K_m K_a}\right)}}$

(8)

3) a) Gain margin – Is a measure of how much the gain of the return ratio ($L(s) = K(s)\,G(s)$) can be increased before the closed-loop system becomes unstable. [3]

Phase margin – Is a measure of how much phase-lag can be added to the return ratio before the closed-loop system becomes unstable. [3]

b) The use of $k_p = 4$ multiplies the gain curve by 12dB, ie the Bode plot is raised by $20 \log_{10}(4) = 12$ dB. Equivalently the 0dB gain for $G(s)$ is now at -12 dB.

GM = 5.3dB [3]

PM = 180 − 160 = 20° [3]

c) See Bode plot. [4]

GM = 17.3 dB [1]

PM = 63° [1]

For the proportional controller the PM and GM are quite low so we can expect some overshoot and oscillation in response to a step change on the input.

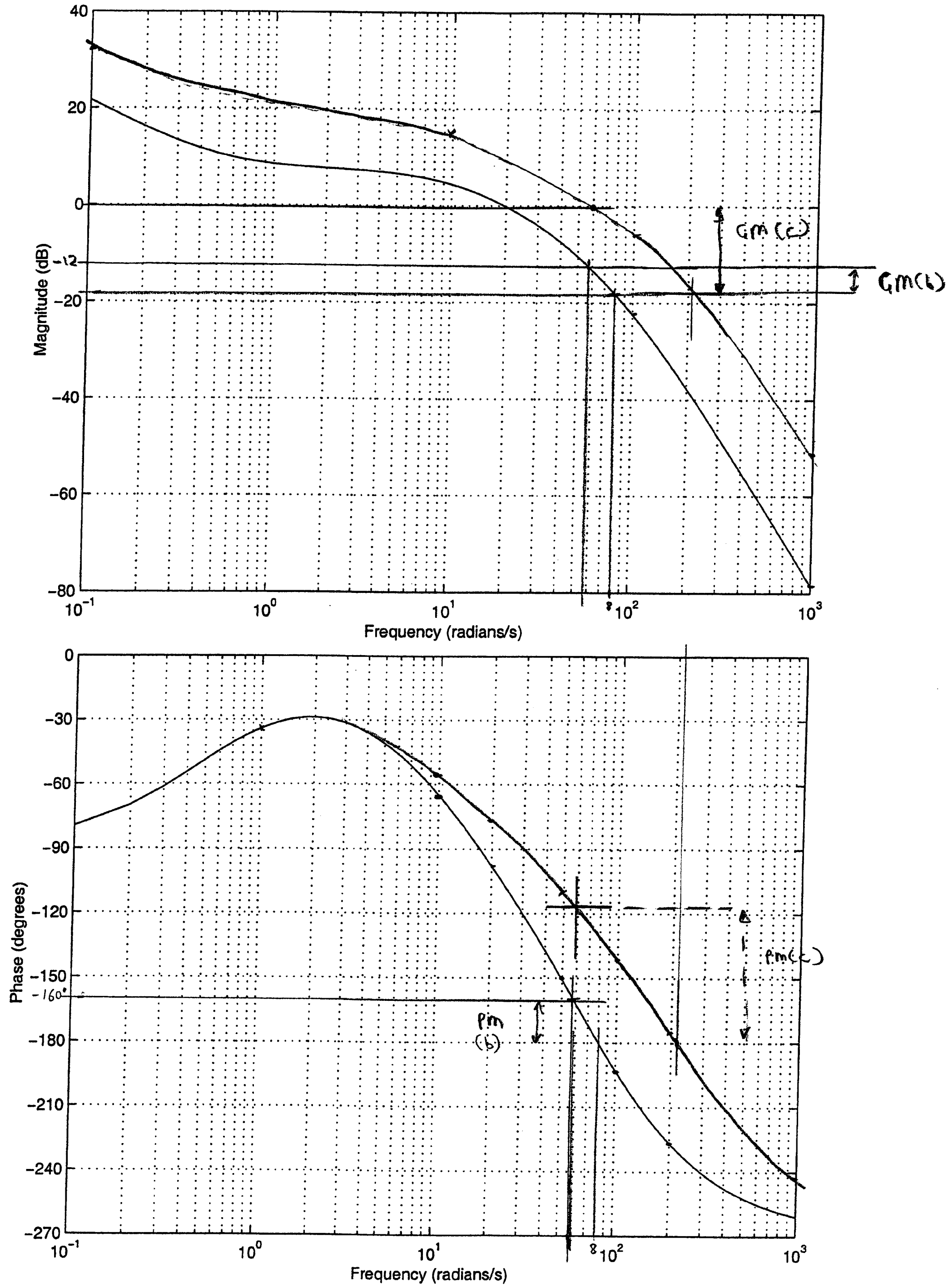
This will be reduced when the compensator is included owing to the raised GM and PM.

[2]

9

ENGINEERING TRIPOS PART IB, Paper 6, 3rd June 2004. Candidate number:

Extra copy of Fig. 4 which may be annotated and handed in with your answer to Question 3.



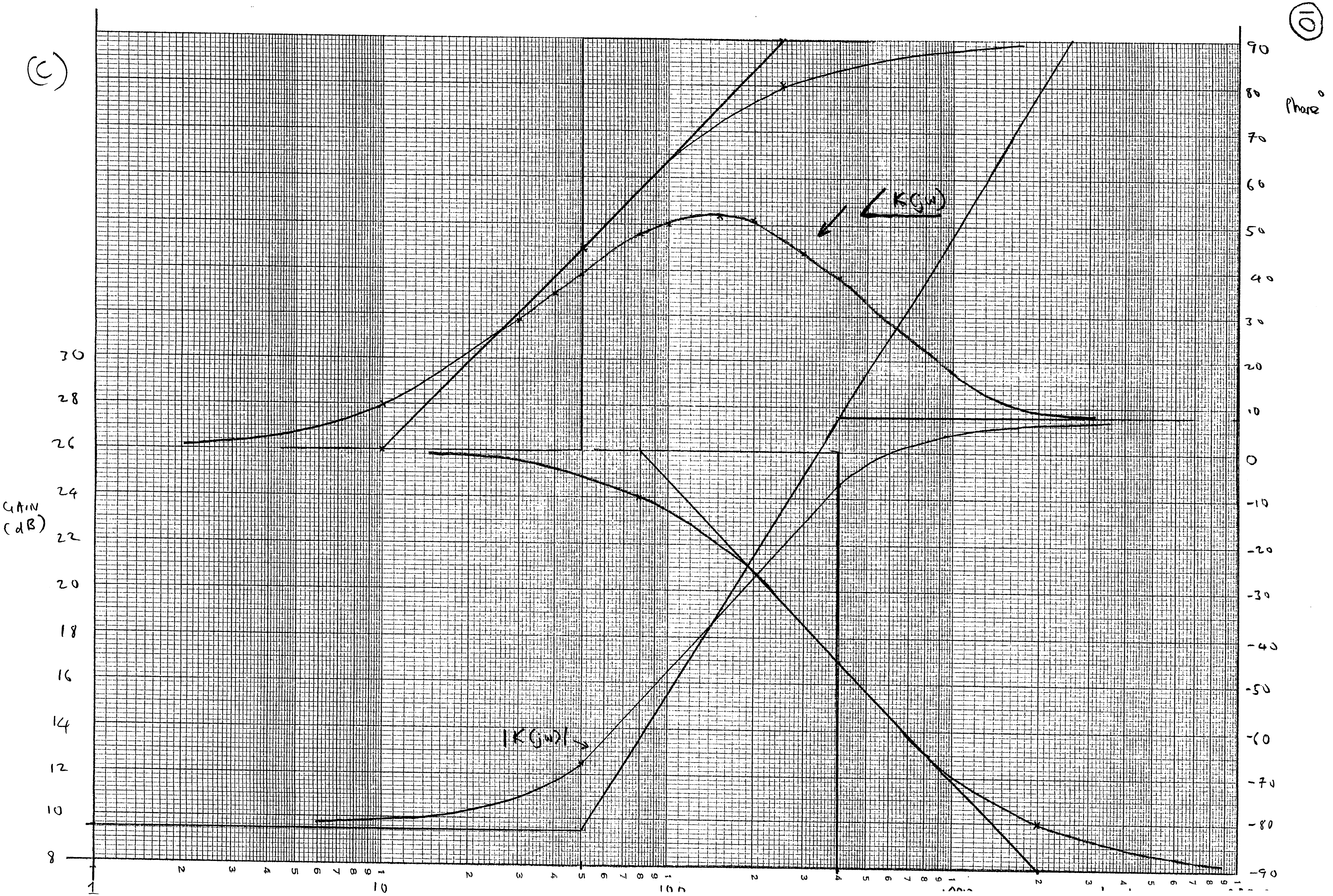
(b) PM = 180 - 160 = 20°
GM = 5.3 dB

(c) PM = 63°
GM = 17.3dB

Chartwell Graph Data Ref. 5541 Log 4 Cycles x mm, ½ and 1 cm

(10)

(c)



4.) a) To construct Nyquist diagram -

Take the <u>open loop</u> system $G(s)$

- Input sinusoidal signals over a wide range of frequencies to $G(s)$
- Once any transients have decayed, measure gain and phase shift from input to output of $G(s)$
- Plot gain and phase shift values on an Argand diagram to yield the Nyquist diagram. [3]

Practical problems -

- Unstable systems never reach steady state
- Integral action gives rise to d.c. drift $\therefore$ system hits 'end stops'
- Noise, non-linearities and variation with time. [3]

Determination of closed-loop stability -

The Nyquist diagram should not encircle the point $-1/K_p$ for the system to be stable (or equivalently leaves the point $-1/K_p$ to the left of the curve). [2]

b) (i) For a PM of 45°, from plots,

$\frac{1}{K_p} = 1.95$, $\therefore$ $K_p = 0.514$ [2]

$GM = \frac{1.95}{0.354} = 5.51 = (14.8\,dB)$ [2]

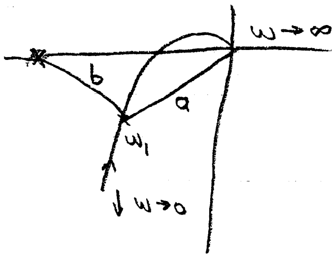
(ii) From the plot the max value of the sensitivity function is given by,

$c = 1.12$

$\therefore \quad |S(j\omega)|_{max} = \frac{1.95}{c} = \frac{1.95}{1.12}$

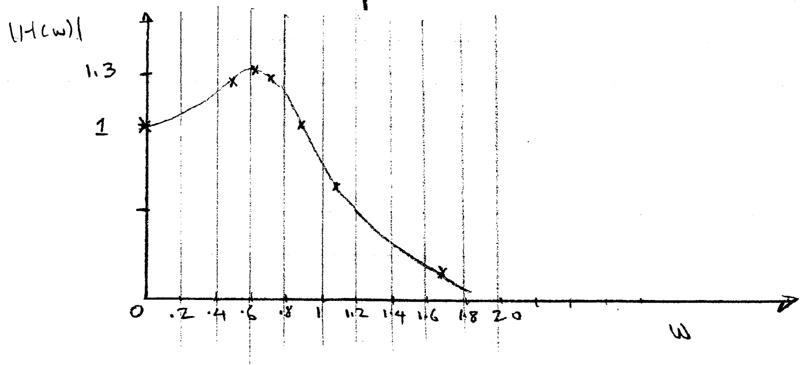
$= \underline{\underline{1.74}}$ [2]

c) To calculate closed-loop freq response (at a particular frequency ω_1) need to take ratio of length a to b (ie, a/b) as shown in diagram below



From the plot,

ω	a	b	a/b
0.52	141	113	1.25
0.64	107	82	1.31
0.75	86	68	1.26
0.89	66	63	1.05
1.1	47	68	0.69
1.7	20	90	0.22



[3]

∴ Peak value of CLTF is about 1.3, (2.3 dB) [1]

The CLTF does not rise significantly above unity, hence the step-response should not be too oscillatory, ie will be quite well damped.

[2]

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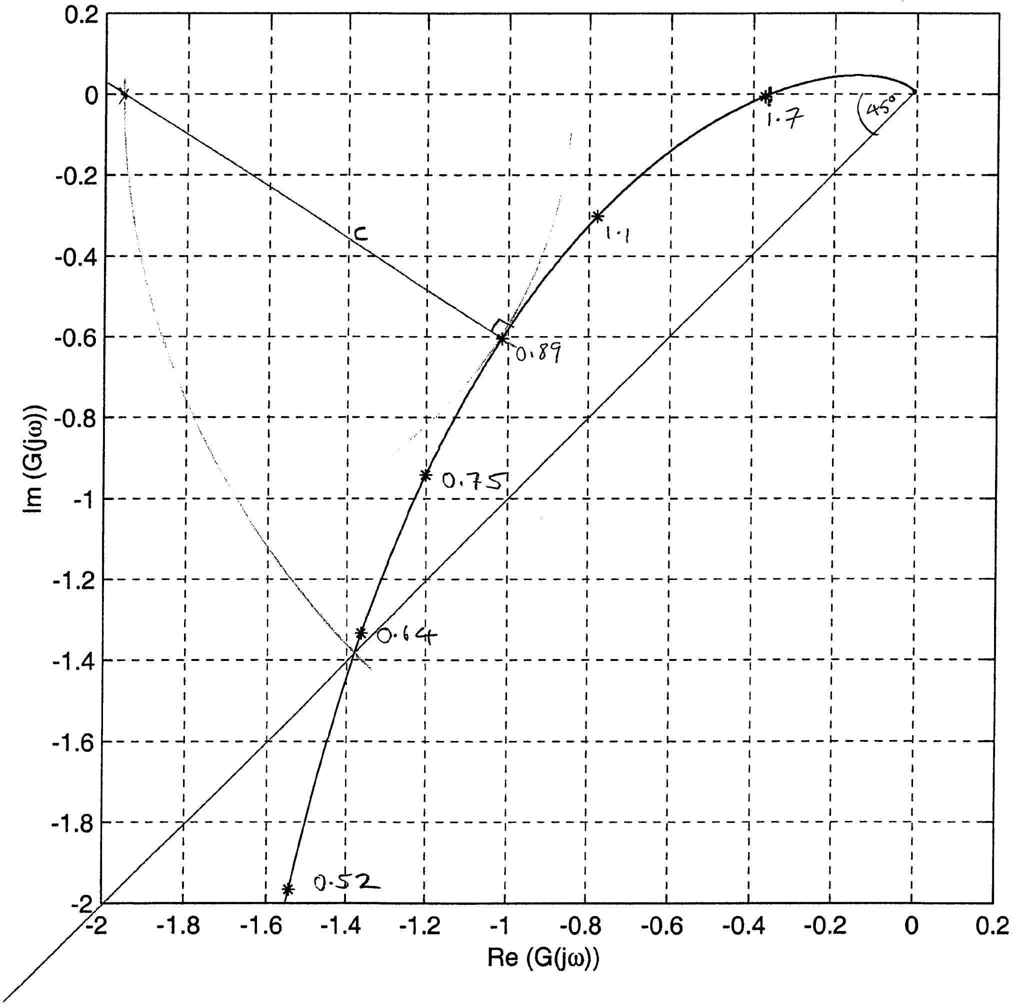


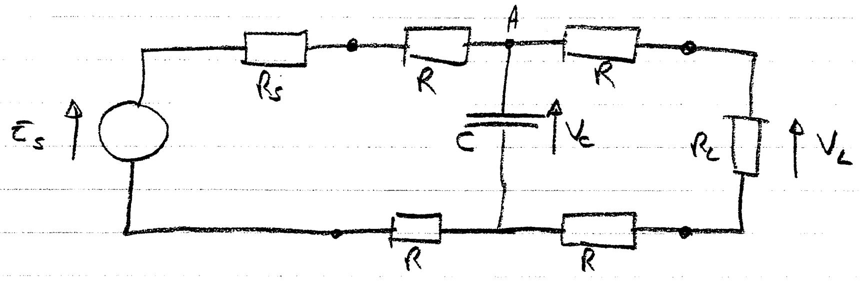
for PM $\quad 110\text{mm} \quad 2 = 113\text{mm} \quad \therefore \frac{1}{k_p} = \frac{110}{113} \times 2 = 1.95 \quad \therefore k_p = \frac{1}{1.95} = 0.514$

for GM. $\quad$ 180° crossing occurs at 20mm which $= \frac{20}{113} \times 2 = 0.354$

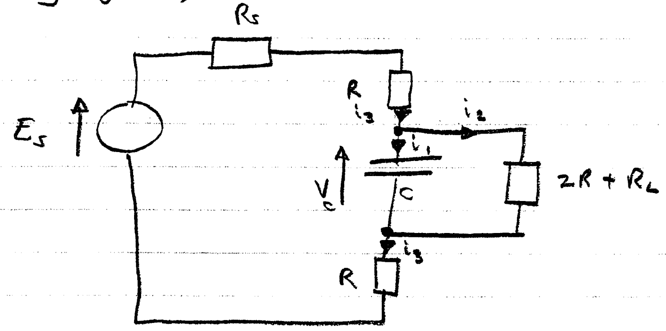
$\therefore GM = \frac{1.95}{0.354} = 5.51 \quad (14.8\text{ dB})$

$c = 63\text{mm} = \frac{63}{113} \times 2 = 1.12$

5.) (a)



Redrawing gives,



Sum currents at A,

$$i_1 + i_2 = i_3$$

$$C\frac{dV_c}{dt} + \frac{V_c}{2R+R_L} = \frac{E_s - V_c}{2R+R_s}$$

Taking Laplace transform,

$$sC\bar{V}_c(s) + \frac{\bar{V}_c(s)}{2R+R_L} = \frac{\bar{E}_s(s) - \bar{V}_c(s)}{2R+R_s}$$

$$\bar{V}_c(s)\left[sC + \frac{1}{2R+R_L} + \frac{1}{2R+R_s}\right] = \frac{\bar{E}(s)}{2R+R_s}$$

$$\bar{V}_c(s)\left[sC + \frac{1}{R_p}\right] = \frac{\bar{E}(s)}{2R+R_s}$$

where, $R_p = 2R + R_L \,||\, 2R + R_s$

so, the transfer function from $\bar{E}_s$ to $\bar{V}_c$ is,

$$\frac{\bar{V}_c(s)}{\bar{E}(s)} = \frac{\frac{1}{2R+R_s}}{\left[sC + \frac{1}{R_p}\right]} = \frac{\frac{1}{2R+R_s}}{\frac{sCR_p+1}{R_p}} = \frac{R_p}{2R+R_s}\cdot\left(\frac{1}{sCR_p+1}\right)$$

$$= \frac{(2R+R_L)(2R+R_s)}{(2R+R_L+2R+R_s)}\cdot\frac{1}{2R+R_s}\cdot\left(\frac{1}{sCR_p+1}\right)$$

$$= \frac{2R+R_L}{4R+R_L+R_s}\cdot\left(\frac{1}{sCR_p+1}\right)$$

(15)

Since,

$$\frac{\bar{V}_L(s)}{\bar{V}_c(s)} = \frac{R_L}{2R + R_L}$$

the transfer function from $\bar{E}(s)$ to $\bar{V}_L(s)$ is

$$H(s) = \frac{\bar{V}_L(s)}{\bar{E}(s)} = \frac{R_L}{2R + R_L} \cdot \frac{2R + R_L}{4R + R_L + R_s} \cdot \left(\frac{1}{sCR_p + 1}\right)$$

$$H(s) = \frac{R_L}{4R + R_L + R_s} \cdot \left(\frac{1}{sCR_p + 1}\right)$$

[10]

(16)

5. b)

$$H(s) = \frac{R_L}{4R + R_s + R_L} \cdot \frac{1}{(sCR_p + 1)}$$ (From part (a))

where

$$R_p = (2R + R_L) \,||\, (2R + R_s)$$

If $R_L = R_s$, then

$$H(s) = \frac{R_L}{4R + 2R_L} \cdot \frac{1}{\left[sC\left(\frac{2R+R_L}{2}\right) + 1\right]}$$

$$H(s) = \frac{1}{2} \cdot \frac{R_L}{(R+R_L)} \cdot \frac{1}{\left[sC\left(R + \frac{R_L}{2}\right) + 1\right]}$$

For a cable of length x km, shunt capacitance

$$C = xc.$$

Total loop resistance $= 4R$ so,

$$4R = xr$$

$$\therefore \quad R = \frac{xr}{4}$$

Freq response given by substituting $s = j\omega$

$$H(j\omega) = \frac{1}{2} \cdot \frac{R_L}{R+R_L} \cdot \frac{1}{1 + j\omega T_c}$$

where,

$$T_c = C\left(R + \frac{R_L}{2}\right)$$

At -3dB point, $\omega T_c = 1$

also $\omega = 2\pi f$

So, $2\pi f \, T_c = 1$ $\therefore$ $T_c = \frac{1}{2\pi f}$

Sub for C and R gives,

$$\frac{1}{2\pi f} = xc\left(\frac{xr}{4} + \frac{R_L}{2}\right)$$

$$\frac{1}{2\pi f} = \frac{x^2 cr}{4} + \frac{xcR_L}{2}$$

$$\frac{cr}{4}x^2 + \frac{cR_L}{2}x - \frac{1}{2\pi f} = 0$$

$$x = \frac{-\frac{cR_L}{2} \pm \sqrt{\left(\frac{cR_L}{2}\right)^2 + \frac{4cr}{4 \times 2\pi f}}}{\frac{cr}{2}}$$

(17)

$$x = \frac{-\frac{cR_L}{2} \pm \sqrt{\left(\frac{cR_L}{2}\right)^2 + \frac{cr}{2\pi f}}}{\frac{cr}{2}}$$

$$x = -\frac{\cancel{c}R_L}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{c}r} \pm \frac{2}{cr}\sqrt{\left(\frac{cR_L}{2}\right)^2 + \frac{cr}{2\pi f}}$$

$$x = -\frac{R_L}{r} \pm \frac{2}{cr}\sqrt{\left(\frac{cR_L}{2}\right)^2 + \frac{cr}{2\pi f}}$$

3.098×10^{-6}

$$x = -\frac{600}{24} \pm \frac{2}{10\times10^{-9}\times24}\sqrt{\left(\frac{10\times10^{-9}\times600}{2}\right)^2 + \frac{10\times10^{-9}\times24}{2\pi\times64\times10^3}}$$

$$x = -25 \pm 25.8156$$

$$x = 0.8156\ \text{km}$$

[6]

c) We have,

$$\frac{1}{2\pi f} = xc\left(\frac{xr}{4} + \frac{R_L}{2}\right)$$

$$\frac{1}{2\pi f} = xc\left(\frac{xr + 2R_L}{4}\right)$$

$$2\pi f = \frac{4}{xc(xr + 2R_L)}$$

$$f = \frac{4}{2\pi xc(xr + 2R_L)}$$

$$f = \frac{2}{\pi xc(xr + 2R_L)} = \frac{2}{\pi\times10\times10^{-9}(24 + 2\times600)}$$

$$f = 52011.4\ \text{Hz} \approx \underline{52\text{ kHz}}$$

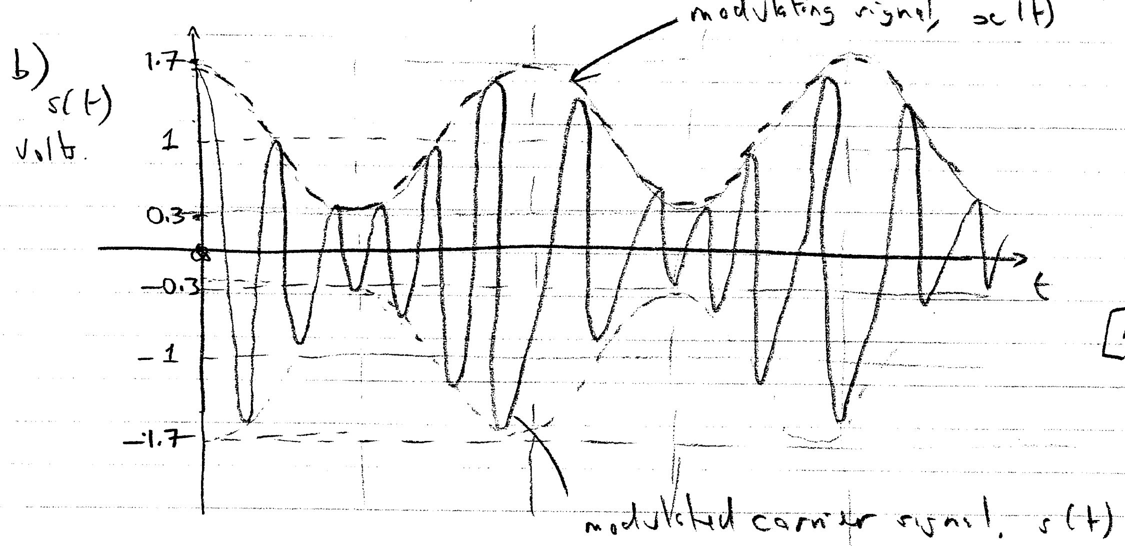
[4]

(18)

6) a) • Audio has too low a frequency to be emitted efficiently. [1]

• Need to have many separate channels, eg 'radio stations'. [1]

b)



[4]

Max value of $b = 1V$. [1]
Otherwise significant distortion will result - since the envelope cannot become -ve - critical if envelope detection is used. [1]

c) $x(t) = 2 + a_1(t) + a_2(t)$

Now,

$y(t) = x(t)^2$ so,

$y(t) = (2 + a_1(t) + a_2(t))(2 + a_1(t) + a_2(t))$

$$y(t) = 4 + 2a_1(t) + 2a_2(t) + 2a_1(t) + a_1(t)^2 + a_1(t)a_2(t) + 2a_2(t) + a_1(t)a_2(t) + a_2(t)^2$$

$$y(t) = 4 + 4a_1(t) + 4a_2(t) + 2a_1(t)a_2(t) + a_1(t)^2 + a_2(t)^2$$

Looking at the terms we have.

$$2a_1(t)a_2(t) = 2b\cos\omega_m t \times a\cos\omega_c t$$
$$= 2ab\cos\omega_m t\cos\omega_c t$$
$$= 2ab \times \frac{1}{2}[\cos(\omega_c+\omega_m)t + \cos(\omega_c-\omega_m)t]$$
$$= ab[\cos(\omega_c+\omega_m t) + \cos(\omega_c-\omega_m t)]$$

$$a_1(t)^2 = (b\cos\omega_m t)^2 = \frac{b^2}{2}(1 + \cos 2\omega_m t)$$

$$a_2(t)^2 = (a \cos \omega_c t)^2 = \frac{a^2}{2}(1 + \cos 2\omega_c t)$$

So writing out in full we have

$$y(t) = 4 + 4b\cos\omega_m t + 4a\cos\omega_c t + ab[\cos(\omega_c+\omega_m)t + \cos(\omega_c-\omega_m)t] + \frac{b^2}{2}(1+\cos 2\omega_m t) + \frac{a^2}{2}(1+\cos 2\omega_c t)$$

Now bandpass filter $y(t)$ with a filter centred on $\omega_c t$.

The required terms for an AM signal are

$$4a\cos\omega_c t + ab[\cos(\omega_c+\omega_m)t + \cos(\omega_c-\omega_m)t] \quad (1)$$

[3]

This can be rewritten as

$$4a\cos\omega_c t + 2ab\cos\omega_c t\cos\omega_m t$$

$$= [4a + 2ab\cos\omega_m t]\cos\omega_c t$$

$$= \left[1 + \frac{2ab}{4a}\right]\cos\omega_c t$$

$$= \left[1 + \frac{b}{2}\right]\cos\omega_c t$$

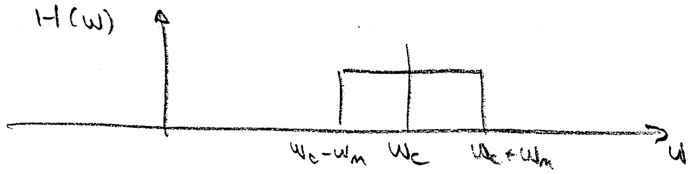
[1]

modulation index, $m_A = \dfrac{\text{message amplitude}}{\text{carrier amplitude}}$

$$= \frac{\frac{b}{2}}{1} = \underline{\underline{\frac{b}{2}}}$$

[1]

The filter bandwidth is



ie, the passband is from $\omega_c - \omega_m$ to $\omega_c + \omega_m$. [1]

d) From part (c) or otherwise, we see that DSB-AM has a carrier and 2 sidebands. Only the sidebands carry useful power. - See eqn (1).

(20)

$$\eta = \frac{\text{useful power}}{\text{total power}}$$

$$\eta = \frac{2\left(\frac{ab}{\sqrt{2}}\right)^2}{2\left(\frac{ab}{\sqrt{2}}\right)^2 + \left(\frac{4a}{\sqrt{2}}\right)^2}$$

$$\eta = \frac{2 \times \frac{a^2b^2}{2}}{\frac{2a^2b^2}{2} + \frac{16a^2}{2}} = \frac{a^2b^2}{a^2b^2 + 8a^2} =$$

$$\eta = \frac{a^2b^2}{a^2b^2+8a^2} = \frac{b^2}{b^2+8}$$

with $m_A = 1$, $b = 2$

$\therefore \eta_{max} = \frac{4}{4+8} = \frac{4}{12} = \frac{1}{3}$ [4]

OR For DSB-AM max η occurs when modulation index, $m_A = 1$,

ie, when $4a = 2ab$

ie, $b = \frac{4a}{2a} = 2$ or $m_A = \frac{b}{2}$, $b = 2m_A$

$b = 2$

$$\eta_{max} = \frac{4a^2}{4a^2 + 8a^2} = \frac{4a^2}{12a^2}$$

$$\eta_{max} = \frac{4}{12} = \frac{1}{3}$$

DSB-Suppressed Carrier (DSB-SC) modulation [1]

- No power transmitted in carrier $\therefore$ η now 100%.
- Same bandwidth as DSB-AM - more complex demod ½

Single Sideband AM (SSB) modulation

- 100% efficiency
- ½ bandwidth of DSB-AM or DSB-SC [1]
- more complex demodulation