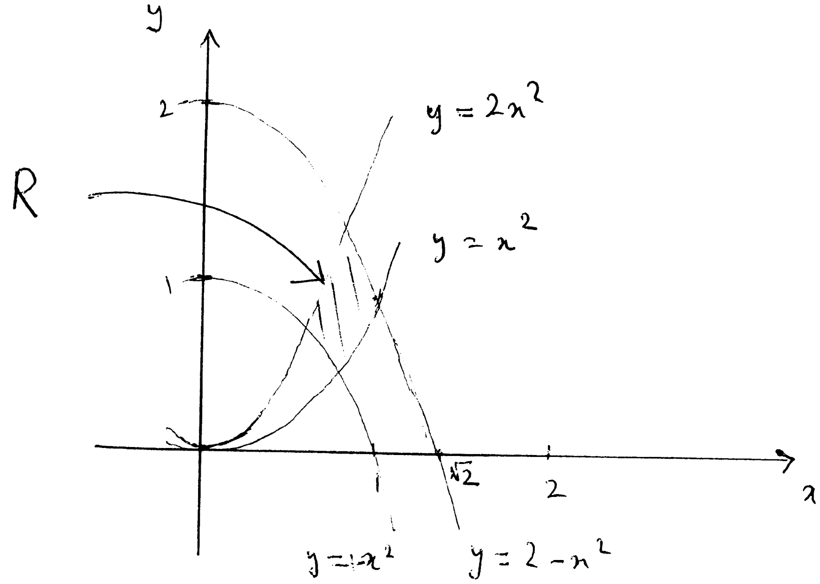
IB Paper 7 – Mathematical Methods – 2004
Solution

1 (a)



(b)

$$I_1 = \iint_R x \, dx\, dy = \iint_{R'} x \frac{\partial(x,y)}{\partial(u,v)} \, du\, dv$$

$$u = y/x^2, \quad v = x^2 + y$$

Easiest to calculate $\dfrac{\partial(u,v)}{\partial(x,y)} = 1\Big/\dfrac{\partial(x,y)}{\partial(u,v)}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left|\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\right| = \left|\det\begin{pmatrix} -2y/x^3 & 1/x^2 \\ 2x & 1\end{pmatrix}\right|$$

$$= \left| -2y/x^3 - 2/x \right| = \frac{2}{x^3}(y + x^2) = \frac{2}{x^3} v$$

Note: $v = x^2 + ux^2 \Rightarrow x^2 = v/(1+u)$

Hence

$$I_1 = \int_1^2\int_1^2 x\left(\frac{x^3}{2v}\right) du\, dv = \int_1^2\int_1^2 \frac{v^{\cancel{2}}}{(1+u)^2}\frac{1}{2\cancel{v}} \, du\, dv$$

$$= \frac{1}{2}\left[\frac{v^2}{2}\right]_1^2 \left[\frac{-1}{1+u}\right]_1^2 = \frac{1}{2}\cdot\frac{3}{2}\cdot\left(-\frac{1}{3}+\frac{1}{2}\right) = \frac{1}{8}$$

(b) (alter.) Calculating $\frac{\partial(x,y)}{\partial(u,v)}$ directly.

$$x^2 = \frac{v}{1+u}, \qquad y = \frac{uv}{1+u}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right|$$

$$2x \frac{\partial x}{\partial u} = \frac{-v}{(1+u)^2}, \qquad 2x \frac{\partial x}{\partial v} = \frac{1}{1+u}$$

$$\frac{\partial y}{\partial u} = \frac{v}{1+u} - \frac{uv}{(1+u)^2} = \frac{v(1+u-u)}{(1+u)^2} = \frac{v}{(1+u)^2}$$

$$\frac{\partial y}{\partial v} = \frac{u}{1+u}$$

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \left| \det \begin{pmatrix} \frac{-v}{2x(1+u)^2} & \frac{1}{2x(1+u)} \\ \frac{v}{(1+u)^2} & \frac{u}{1+u} \end{pmatrix} \right|$$

$$= \frac{uv + u}{2x(1+u)^3} = \frac{v}{2x(1+u)^2}$$

$$\text{Hence } I_1 = \int_1^2 \int_1^2 \cancel{x} \frac{v}{2\cancel{x}(1+u)^2} \, du \, dv = \ldots\ldots$$

(as before)

(c)

$$I_2 = \int_C \underline{V} \cdot d\underline{r} = \iint_R \nabla \times \underline{V} \cdot d\underline{A}$$

by Stoke's theorem, where $d\underline{A} = \underline{k}\, dA$

$$\nabla \times \underline{V} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & -x^2 & 0 \end{pmatrix} = \underbrace{\dots + \dots}_{\text{not necessary to calculate (equal to zero)}} + \underline{k}\left(-2x - x\right)$$

Hence $$I_2 = \iint_R -3x\, dA = -3 I_1 = -\frac{3}{8}$$

Question 1. The sketch in part (a) and the application of Stokes' theorem in part (c) were well done by most candidates. The calculation of the Jacobian in part (b) led to many errors. Unfortunately many candidates did not understand that $\frac{\partial u}{\partial x}$ requires y to be held constant and $\frac{\partial x}{\partial u}$ requires v to be held constant etc. Many candidates incorrectly wrote things like: $y = v - x^2$ so $\frac{\partial y}{\partial v} = 1$.

2 (a) (i)

Mass of fluid in V $= \int_V \rho \, dV$

(ii) Rate at which mass leaves surface element
$= \rho (\underline{u} . \underline{n}) \, \delta A = \rho \underline{u} . d\underline{A}$

$\Rightarrow$ net loss of mass through S $= \int_S \rho \underline{u} . d\underline{A}$

(iii) Conservation of mass:

$$\frac{\partial}{\partial t} \int_V \rho \, dV + \int_S \rho \underline{u} . d\underline{A} = 0$$

$\partial/\partial t$ can be taken inside integral since V is fixed

Applying Gauss's theorem to second integral gives

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla . (\rho \underline{u}) \right] dV = 0$$

This is true for <u>any</u> control volume. Hence

$$\nabla . (\rho \underline{u}) + \frac{\partial \rho}{\partial t} = 0$$

2 (b) (i) $\quad \nabla \cdot \underline{u} = \nabla \cdot \nabla \times (\varphi \underline{k}) = 0$

since "div curl" is always zero (see data book)

Hence $\underline{u}$ is solenoidal.

$$\nabla \times (\varphi \underline{k}) = \cancel{\varphi \nabla \times \underline{k}} + \nabla \varphi \times \underline{k}$$

(since $\underline{k}$ is constant)

Hence $\underline{u}$ is $\perp$ to $\underline{k}$, so it lies in (x, y)-plane.

(ii) The rate of change of φ along streamlines of $\underline{u}$ equals $\underline{u} \cdot \nabla \varphi / |\underline{u}|$. But

$$\underline{u} \cdot \nabla \varphi = \nabla \varphi \times \underline{k} \cdot \nabla \varphi = 0$$

Hence φ is constant along streamlines of $\underline{u}$.

(iii)

$$0 = \nabla \cdot (\rho \underline{u})$$

$$= \rho \cancel{\nabla \cdot \underline{u}} + \nabla \rho \cdot \underline{u} \qquad \text{(from data book)}$$

(= 0 since $\underline{u}$ is solenoidal)

Hence $\nabla \rho \cdot \underline{u} = 0$, so ρ is constant along streamlines of $\underline{u}$.

Question 2. Although part 2(a) was familiar bookwork, many candidates were unable to reproduce the logical sequence of the argument to derive the mass conservation equation. Part (b) was well-answered by many candidates, though many did not clearly explain why $\mathbf{u}$ orthogonal to $\nabla \phi$ implies that ϕ is constant along streamlines of $\mathbf{u}$.

3 (a) In linear PDEs the dependent variables, and their derivatives, appear linearly. If terms like $\phi \frac{\partial \phi}{\partial x}$ or $\left(\frac{\partial^2 \phi}{\partial x^2}\right)^2$ etc. appear then the PDEs are nonlinear.
The principle of superposition applies to linear but not nonlinear PDEs.

Question 3. The separation of variables argument was familiar to most candidates though a few incorrectly mimicked the proper approach by failing to produce two terms which were dependent only on θ and r but nevertheless set them equal to a constant. Part (d) caused problems for the majority of candidates who did not realise that T would need to be expressed as a Fourier series in order to match the boundary conditions.

Laplace's equation: $\nabla^2 \phi = 0$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Diffusion equation: $\frac{\partial \phi}{\partial t} = \alpha \nabla^2 \phi$

$$\Rightarrow \frac{\partial \phi}{\partial t} = \alpha \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$$

Wave equation: $\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi$

$$\Rightarrow \frac{\partial^2 \phi}{\partial t^2} = c^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$$

(b) $f = R(r)\,X(\theta) \Rightarrow$

$$\ddot{R} X + \frac{1}{r} \dot{R} X + \frac{1}{r^2} R \ddot{X} = 0$$

$$\Rightarrow \left(\ddot{R} + \frac{1}{r} \dot{R} \right) X + \frac{1}{r^2} R \ddot{X} = 0$$

$$\Rightarrow \frac{\ddot{R} + \frac{1}{r}\dot{R}}{\frac{1}{r^2} R} = -\frac{\ddot{X}}{X} = \text{const.} = \lambda \text{ say}$$

indep. of θ indep. of R

$$\Rightarrow \begin{cases} r^2 \ddot{R} + r \dot{R} - \lambda R = 0 \\ \ddot{X} + \lambda X = 0 \end{cases}$$

(c)

$R = r^\alpha$ and $X = \sin(\beta\theta)$

$$\Rightarrow \begin{cases} r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - \lambda r^\alpha = 0 \\ -\beta^2 \sin(\beta\theta) + \lambda \sin(\beta\theta) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha^2 - \lambda = 0 \\ -\beta^2 + \lambda = 0 \end{cases} \Rightarrow \alpha^2 = \beta^2 \Rightarrow \alpha = \pm\beta$$

(d) From Part(c) we have

$$T = r^{\pm n} \sin n\theta \qquad (n \text{ an integer})$$

is a solution. (we will only need integer solutions to satisfy the boundary condition.) Superposition tells us that

$$T = \sum_{n=1}^{\infty} (a_n r^n \sin n\theta + b_n r^{-n} \sin n\theta)$$

is also a solution. To avoid a singularity at $r=0$ we need $b_n = 0$. Thus, a physically acceptable solution is:

$$T = \sum_{n=1}^{\infty} a_n r^n \sin n\theta$$

To satisfy the boundary condition we need

$$T(r=A) = \theta(\pi^2 - \theta^2) = 12 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin n\theta$$

Thus,

$$a_n A^n = 12 \frac{(-1)^{n+1}}{n^3}$$

Hence

$$T = 12 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \left(\frac{r}{A}\right)^n \sin n\theta$$

Section B.

4. (a)

$$A = \begin{bmatrix} 4 & 2 & 0 & 3 \\ 8 & 6 & 1 & 6 \\ -4 & 0 & 3 & -3 \end{bmatrix} \begin{array}{l} \\ \text{Multiplier } 8/4 = 2 \\ \text{Multiplier } -4/4 = -1 \end{array}$$

Pivot

$$\rightarrow \begin{bmatrix} 4 & 2 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \text{Multiplier } 2/2 = 1 \end{array}$$

Pivot

$$U = \begin{bmatrix} 4 & 2 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Pivot

Rank = 3 (= no. of pivots)

Examiner's comments on sections B and C of 1B paper 7.

The sections were answered with mixed success. A few questions were answered very well, while others were avoided or answered badly. Where questions were answered badly it was usually because candidates were not familiar with bookwork material from the lecture notes or examples from the examples sheets, so a good piece of advice to candidates in future would be to re-focus their revision on the lecture notes.

(b) Null Space:

$$A x = 0$$

$$\Rightarrow LU x = 0$$

$$U x = 0$$

$$\begin{bmatrix} 4 & 2 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \underline{x} = 0$$

Setting $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

& solving gives:

$$x_2 = 0, \quad x_3 = 0, \quad 4x_1 = -3x_4$$

So nullspace is:

$$\lambda \begin{bmatrix} -3/4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Column space is any 3 independent columns of A (or U, or indeed any three independent 3-vectors): $\begin{bmatrix} 4 \\ 8 \\ -4 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

(c) Can always find a solution for any $\underline{A}\underline{x} = \underline{b}$, since A has rank 3 and therefore any $\underline{b}$ is in the range space of $\underline{A}$.

$$\underline{L}\,\underline{U}\,\underline{x} = \underline{b}$$

$$\Rightarrow \underline{U}\,\underline{x} = \underline{L}^{-1}\underline{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(by Gaussian elimination)

x_4 is the free variable

Solve first with e.g. $x_4 = 0$ for a particular solution:

$$\begin{bmatrix} 4 & 2 & 0 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Solving $\Rightarrow$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$

(c) contd.

Now simply add any multiple of the null space vector, since this contributes zero to RHS,

Full solution is:

$$\underline{x} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -3/4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Particular soln.

Null-space vector.

EXAMINER'S COMMENTS

Q4 – LU decomposition

This highly popular question was answered most satisfactorily. The majority of candidates were well able to compute the LU decompositon, and most understood how to calculate column and null spaces. The last part on finding the full set of solutions was answered more variably, but many perfect answers were presented

5. (a) Measurements are (from graph):

t (μs)	0	1	2	3	4
$y(t)$	9	0	5	-1	-1

$$\underset{\sim}{y} = \begin{bmatrix} 9 \\ 0 \\ 5 \\ -1 \\ -1 \end{bmatrix}$$

Functions to fit are:

t (μs)	0	1	2	3	4
$\sin(\Omega_0 kT)$	0	-0.76	0.99	-0.54	-0.29
$\exp(-akT)$	1	0.37	0.14	0.05	0.02

So: $$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 \\ -0.76 & 0.37 \\ 0.99 & 0.14 \\ -0.54 & 0.05 \\ -0.29 & 0.02 \end{bmatrix} \qquad \underline{\theta} = \begin{bmatrix} b \\ c \end{bmatrix}$$

$\underline{A}$ has clearly rank 2 since column 1 independent of column 2.

5. (b) Least squares solution:

$$\underline{\hat{\theta}} = (\underline{A}^T\underline{A})^{-1}\underline{A}^T \underline{y}$$

(matrix projection operator)

Multiplying out:

$$(\underline{A}^T\underline{A}) = \begin{bmatrix} 1.92 & -0.18 \\ -0.18 & 1.16 \end{bmatrix}$$

$$\underline{A}^T \underline{y} = \begin{bmatrix} 5.77 \\ 9.60 \end{bmatrix}$$

$$(\underline{A}^T\underline{A})^{-1} = \begin{bmatrix} 0.53 & 0.08 \\ 0.08 & 0.88 \end{bmatrix}$$

$$\underline{\hat{\theta}} = \begin{bmatrix} 3.82 \\ 8.89 \end{bmatrix}$$

5 (b) Alternatively: (this method takes more time and was more prone to algebraic errors);

Using QR decomposition:

$$\underline{A} = \underline{Q} \begin{bmatrix} \underline{R} \\ 0 \end{bmatrix}$$

$$\underline{Q} = \begin{bmatrix} 0 & -0.94 \\ 0.55 & -0.28 \\ -0.71 & -0.21 \\ 0.39 & 0 \\ 0.21 & 0 \end{bmatrix} ; \quad \underline{R} = \begin{bmatrix} -1.39 & 0.13 \\ 0 & -1.07 \end{bmatrix}$$

$$\hat{\underline{\theta}} = \underline{R}^{-1} \underline{Q}^T y = \begin{bmatrix} 3.82 \\ 8.89 \end{bmatrix}$$

Finally least □'s error is:

$$E = y^T y - y^T \underline{A} \, \hat{\underline{\theta}}$$

$$= 0.53 .$$

5.c) The model would be reanalysed in the form:

$$x(kT) = c \exp(-akT)$$

The least squares error could then be compared with part b/ and the smallest error gives the best fit model.

[Note that a full statistical analysis would allow for the fact that model a) has more degrees of freedom and hence could 'overfit' the data. However, candidates were not expected to know this degree of subtlety.]

EXAMINER'S COMMENTS

Q5 – Least squares

A less popular question. The format will have looked less familiar to candidates, but the material was well within the course remit. Most could set up the LS equations in part a). In part b) most used QR decomposition to solve, and a pleasing number managed to get the correct solution without algebraic error. Most failed to spot that the question was much easier to handle using the matrix projection operator directly – (A^TA)^-1 A^Ty. Part c) was quite well answered

6. (a)

Bookwork as follows:

$$\int_{-\infty}^{+\infty} f(t)^2 \, dt$$

$$= \frac{1}{2\pi} \int f(t) \underbrace{\int_{-\infty}^{+\infty} F(\omega) \, e^{+j\omega t} \, d\omega}_{2\pi f(t)} \, dt$$

$$= \frac{1}{2\pi} \int F(\omega) \left(\int f(t) \, e^{+j\omega t} \, dt \right) d\omega$$

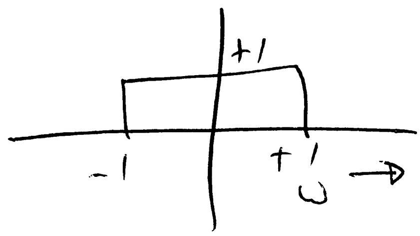
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \, F^*(\omega) \, d\omega$$

QED.

6. (b)

Using databook duality result for 'sinc':

$F(\omega) =$



Therefore, by Parseval's theorem:

$$\text{Energy} = \frac{1}{2\pi}\int_{-1}^{+1} |F(\omega)|^2 = \underline{\underline{\frac{1}{\pi}}}$$

(c) Output of filter:

$$Y(\omega) = H(\omega)\,F(\omega)$$

Energy @ output is:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |Y(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-1}^{+1} \overbrace{1}^{|F(\omega)|^2} \cdot \overbrace{\frac{A_0^2}{1+\omega^2}}^{|H(\omega)|^2} d\omega$$

(Using integral from tables)

$$= \frac{2}{2\pi} A_0^2 \tan^{-1} 1$$

$$= \frac{1}{\pi} A_0^2 \frac{\pi}{4} = \frac{A_0^2}{4}$$

6 (c) contd.

Setting input energy = output energy,

$$A_0^2/4 = 1/\pi, \quad \Rightarrow A_0 = 2/\sqrt{\pi}$$

EXAMINER'S COMMENTS

Q6 – Fourier transforms

Not a popular question. It was disappointing that very few could show part a) (Parseval's theorem) even though it had been derived fully in lectures. I think this highlights that candidates are not paying enough attention to learning material from the lecture notes. Common errors included saying that the integral of a function squared equals the square of the integral. Part b) on applying Parseval was handled better, although many made algebraic slips in calculating the FT of the sinc function (using the duality theorem from the tables). Nevertheless, the principle was well understood. Finally, part c) on filtered signal energy caused a few problems. Again, this was a bit of a surprise since there was a very similar example on the examples sheet.

7. (a) DTFT is

$$F_\delta(\omega) = \sum_{k=-\infty}^{+\infty} f_k \, e^{-j\omega kT}$$

Continuous-time signal:

$$f_s(t) = \sum_{k=-\infty}^{+\infty} f_k \, \delta(t-kT)$$

Take FT $\Rightarrow$ $F_s(\omega) = \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} f_k \, \delta(t-kT) e^{-j\omega t} \, dt$

$$= \sum_{k=-\infty}^{+\infty} f_k \, e^{-j\omega kT}$$

(b) $F_s(\omega) = \sum_{k=0}^{99} 0.5^k \, e^{i\pi k} \, e^{-i\omega kT}$

$\boxed{G.P.}$

$r = 0.5 e^{(i\pi - i\omega T)}$

$$= \frac{1 - r^{100}}{1 - 0.5 e^{i(\pi - \omega T)}} \qquad r^{100} \approx 0$$

$$= \frac{1}{1 + 0.5 e^{-i\omega T}}$$

(c) MGF:

$$\Phi(z) = \sum_{n=0}^{\infty} p_n z^n$$

where $p_n = \Pr\{X = n\}$

Mean $\mu = E[X] = \sum_{0}^{\infty} n\, p_n$

Variance $E[(X-\mu)^2] = \sum n^2 p_n - \left(\sum n\, p_n\right)^2$

$$\mu = \left.\frac{d\,\Phi(z)}{dz}\right|_{z=1}$$

$$\sigma^2 = \left.\frac{d^2\,\Phi(z)}{dz^2}\right|_{z=1} + E[X] - E[X]^2$$

7. (d)

$$\Phi(z) = \sum_{n=0}^{\infty} \exp(-\lambda) \frac{\lambda^n}{n!} z^n$$

$$= \exp(-\lambda) \exp(\lambda z)$$

From Taylor expansion of $e^{\lambda z}$

(e)

$$\mu = \Phi'(z) \Big|_{z=1}$$

$$= e^{-\lambda} \lambda e^{\lambda z} \Big|_{z=1}$$

$$= \underline{\underline{\lambda}}$$

$$\sigma^2 = \Phi''(z)\Big|_{z=1} + E[X] - E[X]^2$$

$$= \lambda^2 e^{-\lambda} e^{\lambda s} \Big|_{s=1} + \lambda - \lambda^2$$

$$= \lambda$$

$$\Rightarrow \sigma = \underline{\underline{\sqrt{\lambda}}}$$

Q7 – Discrete time Fourier Transform and moment generating functions

The least popular questions of all. Most candidates did not answer the initial bookwork part at all well, confusing the DTFT with the DFT in most cases and being unable to derive the DTFT. Part b) was also not very well handled – again lots of confusion between DFT and DTFT. However, moment generating functions and their application were well answered by many in c) and d).

8. (a) $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$ (conditional probability)

$$= \frac{P(B|A)\, P(A)}{P(B)}$$ (Bayes' Theorem)

(b) Let N denote choosing biased coin

" O " " " 2-headed coin

" P " " " 2-tailed coin

Let H_i^u denote flipping heads-up on the i^{th} flip.

T_i^d " " tails-down on the i^{th} flip.

Then:

$$p(H_1^u) = p(H_1^u|N)\,p(N) + p(H_1^u|O)\,p(O) + p(H_1^u|P)\,p(P)$$

$$= \tfrac{1}{3} \times \tfrac{1}{3} + 1 \times \tfrac{1}{3} + 0 \times \tfrac{1}{3}$$

$$= \tfrac{4}{9}.$$

(c) (i) $$p(T_1^a | H_1^u) = \frac{p(T_1^a \cap H_1^u)}{p(H_1^u)}$$

$$= \frac{p(N) \times p(H_1^u|N)}{p(H_1^u)}$$

$$= \frac{\tfrac{1}{3} \times \tfrac{1}{3}}{\cancel{\tfrac{4}{7}}\ \tfrac{4}{9}}$$

$$= \underline{\underline{\tfrac{1}{4}}}$$

(c) (ii) $p(H_2^d \mid H_1^u)$

$$= p(H_2^d \mid N, H_1^u)\, p(N \mid H_1^u) + p(H_2^d \mid O, H_1^u)\, p(O \mid H_1^u) + p(H_2^d \mid P, H_1^u)\, p(P \mid H_1^u)$$

$$= \tfrac{2}{3}\, p(N \mid H_1^u) + 1 \times p(O \mid H_1^u) + 0$$

But, $$p(N \mid H_1^u) = \frac{p(H_1^u \mid N)\, p(N)}{p(H_1^u)} = \frac{\frac{1}{3} \times \frac{1}{3}}{4/9} = \tfrac{1}{4}$$

$$\Rightarrow \quad p(O \mid H_1^u) = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

(since $p(P \mid H_1^u) = 0$)

$$\Rightarrow \quad p(H_2^d \mid H_1^u) = \tfrac{2}{3} \times \tfrac{1}{4} + 1 \times \tfrac{3}{4} = \tfrac{11}{12}$$

EXAMINER'S COMMENTS

Q8 – Probability

This was the most popular question, and most candidates were able to manipulate conditional probabilities correctly. In some cases, candidates stated the conditional probabilities and Bayes' theorem incorrectly in part a) but were still able to do some of the subsequent calculations correctly. This may mean that our students have a good intuitive understanding of probability, which is a positive outcome.