UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IB

Friday 4 June 2004 9 to 11

Paper 7

MATHEMATICAL METHODS

*Answer not more than **four** questions.*

*Answer at least **one** question from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

There are no attachments to this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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SECTION A

*Answer at least **one** question from this section.*

1 (a) Sketch the region R in the first quadrant of the (x,y)-plane which lies between the curves: $y = x^2$, $y = 2x^2$, $y = 1 - x^2$ and $y = 2 - x^2$. [3]

(b) Using the substitutions $u = y/x^2$ and $v = x^2 + y$, or otherwise, evaluate the integral

$$I_1 = \int\int_R x \, dx \, dy. \qquad [10]$$

(c) Consider the vector field

$$\mathbf{V} = xy\,\mathbf{i} - x^2\mathbf{j}$$

where $\mathbf{i}$, $\mathbf{j}$ are the unit vectors in the directions of the x- and y-axes. Using Stokes' theorem and the result of Part (b), evaluate the integral

$$I_2 = \int_C \mathbf{V} \cdot d\mathbf{r}$$

where C is the curve which encloses R in an anticlockwise direction. [7]

2 (a) Suppose a fluid has density $\rho(x, y, z, t)$ and velocity $\mathbf{u}(x, y, z, t)$.

(i) Write down an expression for the mass of fluid in a given control volume V . [2]

(ii) Derive an integral expression for the net loss of mass through the surface S of V . [2]

(iii) Deduce the mass conservation equation for fluid flow

$$\nabla \cdot (\rho \mathbf{u}) + \frac{\partial \rho}{\partial t} = 0 .$$ [4]

(b) Suppose $\mathbf{u} = \nabla \times (\phi \mathbf{k})$ where ϕ is a scalar field and $\mathbf{k}$ is the unit vector along the z-axis.

(i) Show that $\mathbf{u}$ is solenoidal and lies in the (x, y) -plane. [4]

(ii) Show that $\mathbf{u}$ is orthogonal to $\nabla\phi$ and deduce that ϕ is constant along streamlines of $\mathbf{u}$. [4]

(iii) If $\dfrac{\partial \rho}{\partial t} = 0$, show that ρ is constant along streamlines of $\mathbf{u}$. [4]

3 (a) Distinguish between linear and non-linear PDEs and write down examples of *two* common linear PDEs in two-dimensional cartesian coordinates. [4]

(b) Laplace's equation transformed to two-dimensional polar coordinates is

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0 .$$

Assuming a separable solution $f = R(r)X(\theta)$ find differential equations which R and X must satisfy. [5]

(c) Show that solutions of the form $R(r) = r^\alpha$ and $X(\theta) = \sin\beta\theta$ can satisfy the equations in Part (b), and find the relation between α and β. [5]

(d) The steady-state temperature distribution in a thin metal disc is governed by Laplace's equation. The outer rim of the disc, $r = A$, is held at a temperature $T = \theta(\pi^2 - \theta^2)$ for $-\pi \le \theta \le \pi$. Find the temperature distribution in the disc. [6]

[You may find it useful to use the Fourier series expansion

$$\theta(\pi^2 - \theta^2) = 12\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin n\theta .]$$

SECTION B

*Answer at least **one** question from this section.*

4 A matrix is defined as

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 0 & 3 \\ 8 & 6 & 1 & 6 \\ -4 & 0 & 3 & -3 \end{bmatrix} .$$

(a) Determine the **LU** decomposition of $\mathbf{A}$. What is the rank of $\mathbf{A}$? [8]

(b) Determine a basis for the nullspace and column space of $\mathbf{A}$. [6]

(c) Determine whether there is a solution to the equation

$$\mathbf{Ax} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} .$$

If so, find all solutions. If not, explain your reasoning. [6]

5 A scientist has made measurements of a physical phenomenon at regular time intervals spaced $T = 1\,\mu s$ apart. He believes the phenomenon comprises two components: a low amplitude oscillation with frequency $f_0 = 2/\pi\,\text{MHz}$ and an exponential decay term with decay rate $a = 1 \times 10^6\,\text{s}^{-1}$. The overall signal is expected to be

$$x(kT) = b\sin(2\pi f_0 kT) + c\exp(-akT)$$

where b and c are unknown constants. Noisy measurements of the process over a $5\,\mu s$ time interval are shown in Fig. 1. Measurements are shown as crosses (X) and joined with dashed lines.

(a) It is wished to determine the unknown amplitudes b and c from the data. This estimation problem can be expressed as a least squares problem of the form $\mathbf{y} \approx \mathbf{A}\boldsymbol{\theta}$. Write down the elements of $\boldsymbol{\theta}$ and $\mathbf{y}$ and show that $\mathbf{A}$ is given to 2 decimal places by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -0.76 & 0.37 \\ 0.99 & 0.14 \\ -0.54 & 0.05 \\ -0.29 & 0.02 \end{bmatrix}.$$

What is the rank of $\mathbf{A}$? [6]

(b) Determine the least squares solution for the parameter vector $\boldsymbol{\theta}$, stating clearly the steps involved in your calculations. What is the least squares error? [10]

(c) A second scientist contends that there should be no oscillation at frequency f_0 in the signal. Briefly discuss how you would evaluate the merits of this suggestion. No detailed statistical analysis is required. [4]

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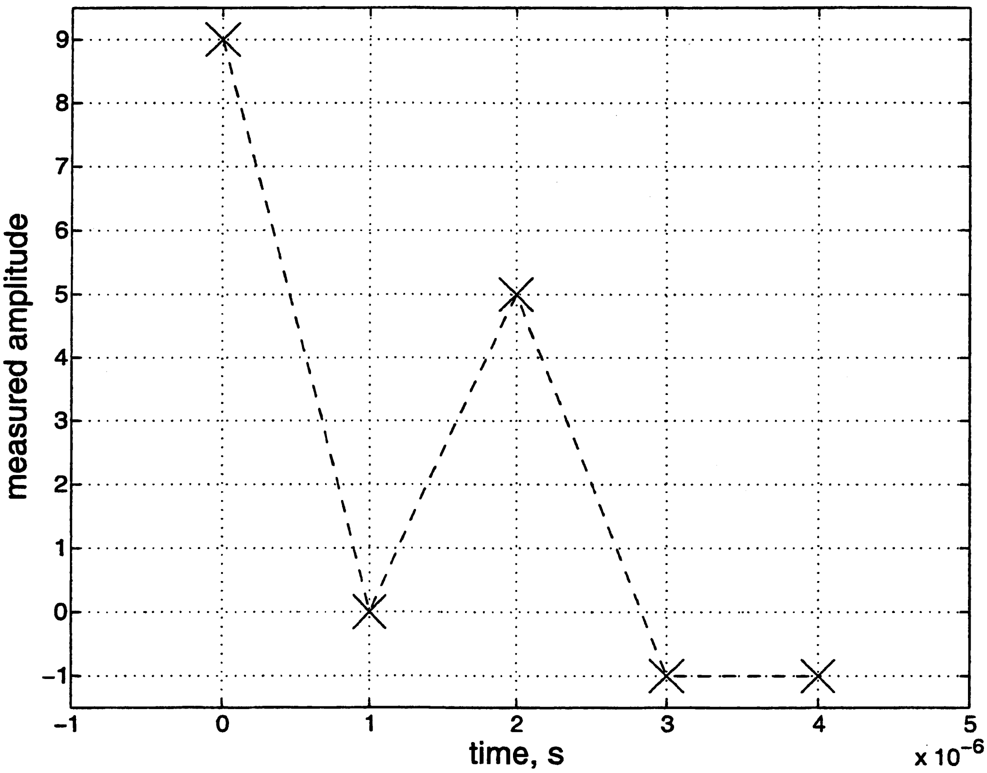

Fig. 1

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SECTION C

*Answer at least **one** question from this section.*

6 (a) For a real-valued function $f(t)$ with Fourier transform $F(\omega)$, show from first principles that

$$\int_{-\infty}^{+\infty} f(t)^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega \, .$$ [7]

(b) Determine the total energy of the function

$$f(t) = \frac{\sin t}{\pi t} \, .$$ [6]

(c) The same function as in Part (b) is passed through a filter having frequency response

$$H(\omega) = \frac{A_0}{1 + j\omega} \, .$$

Determine the value of the constant A_0 such that the total energy at the output of the filter equals the energy at the input of the filter. [7]

7 (a) Define the discrete time Fourier transform (DTFT) of a signal f_k, $k = -\infty, \ldots, +\infty$, having sampling period T. Explain briefly how the DTFT is derived from the standard Fourier transform of a sampled signal f_k represented as an appropriate continuous time function $f(t)$. [5]

(b) A discrete time signal is defined as

$$f_k = \begin{cases} 0.5^k \exp(i\pi k) & k = 0, 1, \ldots, 99 \\ 0 & \text{otherwise.} \end{cases}$$

Write down and simplify an expression for the DTFT of this function. [5]

(c) Define the terms moment generating function, mean and variance for a discrete random variable. Explain briefly how the moment generating function can be used to determine the mean and variance of a random variable. [4]

(d) A random variable X is Poisson distributed, so that

$$P(X = n) = \exp(-\lambda)\frac{\lambda^n}{n!}, \qquad n = 0, 1, \ldots$$

Show that the moment generating function for such a random variable is

$$\exp(\lambda(z-1)) .$$ [3]

(e) Hence show that the mean and standard deviation for the Poisson distribution are λ and $\sqrt{\lambda}$, respectively. [3]

8 (a) Two random events A and B have probabilities $P(A)$ and $P(B)$, respectively. The probability of both events occurring is denoted $P(A \cap B)$. What is the probability $P(A|B)$, i.e. the probability of A given that B has occurred? Rewrite your expression for $P(A|B)$ in terms of $P(B|A)$ and other quantities. [4]

(b) Three coins are used in a magic trick. One is a biased coin with the probability of a head equal to 1/3 and the probability of a tail equal to 2/3, a second has two heads, and the third has two tails. If a coin picked at random is flipped, what is the probability that a head is the outcome? [4]

(c) One of the three coins described in (b) is picked at random and flipped. The outcome is a head.

(i) What is the probability that the lower side of the coin is a tail? [6]

(ii) The same coin is flipped a further time. What is the probability that the lower side is a head? [6]

END OF PAPER