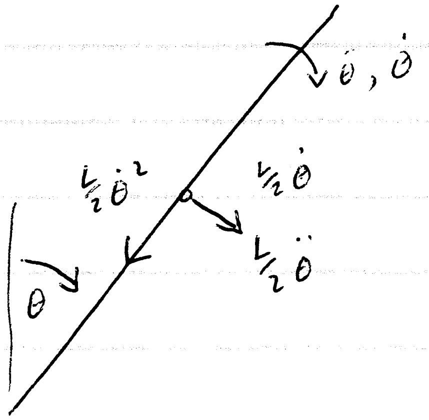
ENGINEERING TRIPOS PART IB 2005 DR A SESHIA

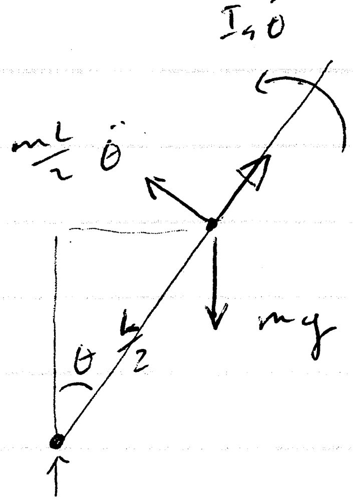
PAPER 1 SECTION A

MECHANICS

CRIBS

1)





a) $$\frac{m}{2}\left(\frac{L}{2}\dot\theta\right)^2 + \frac{m}{2}\frac{L^2}{12}\dot\theta^2 = mg\frac{L}{2}(1-\cos\theta)$$

$$\frac{L^2}{6}\dot\theta^2 = \frac{gL}{2}(1-\cos\theta)$$

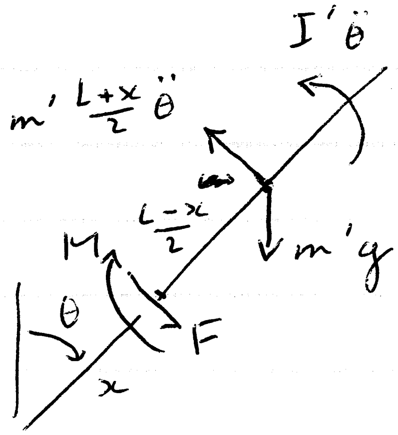
$$\underline{\dot\theta = \sqrt{\frac{3g}{L}(1-\cos\theta)}}$$

b) $$\frac{mL^2}{4}\ddot\theta + \frac{mL^2}{12}\ddot\theta = mg\frac{L}{2}\sin\theta$$

$$\frac{L}{3}\ddot\theta = \frac{g}{2}\sin\theta$$

$$\underline{\ddot\theta = \frac{3g}{2L}\sin\theta}$$

c)



$$\frac{dM}{dx} = S \text{ (shear force)}$$

$\therefore$ max $M \rightarrow S = 0$

$$F = 0 \quad \text{then} \quad m' \frac{L+x}{2} \ddot{\theta} = m' g \sin\theta$$

$$\frac{L+x}{2} \frac{3g}{2L} \sin\theta = g \sin\theta$$

$$\frac{3(L+x)}{4L} = 1 \quad \Rightarrow \quad x = L/3$$

$$\text{Then,} \quad M = m' \frac{L+x}{2} \frac{L-x}{2} \ddot{\theta} + I_G' \ddot{\theta} - m'g \frac{L-x}{2} \times \sin\theta$$

$$= \frac{2m}{3} \left(\frac{2}{9} L^2 \cancel{\theta} + \frac{4L^2}{9 \times 12} \right) \ddot{\theta} - \frac{2m}{3} g \frac{L}{3} \sin\theta$$

$$= \frac{2m}{3} \left(\frac{7L^2}{27} \frac{3g}{2L} \sin\theta - \frac{gL}{3} \sin\theta \right)$$

$$= \frac{2mgL\sin\theta}{3} \left(\frac{7}{18} - \frac{1}{3} \right)$$

$$= \underline{\frac{mgL\sin\theta}{27}}$$

Q2 Let mass of whole cylinder = m

(a) mass removed by hole = $m/4$

∴ To find location of G (say distance x away from O radially.

$$\frac{m}{4}\left(\frac{R}{2}+x\right) = mx$$

$$\frac{R}{2}+x = 4x$$

$$x = R/6$$

$$I_G = \frac{1}{2}mR^2 - \frac{1}{2}\frac{m}{4}\left(\frac{R}{2}\right)^2 + m\left(\frac{R}{6}\right)^2 - \frac{m}{4}\left(\frac{R}{6}+\frac{R}{2}\right)^2$$

$$I_G = mR^2\left(\frac{1}{2}-\frac{1}{32}+\frac{1}{36}-\frac{1}{9}\right)$$

$$I_G = \frac{37\,mR^2}{96}$$

$$I_G = 0.385\,mR^2$$

(b) $PE = \frac{3}{4}mg\frac{R}{6}(1-\cos\theta)$; potential energy

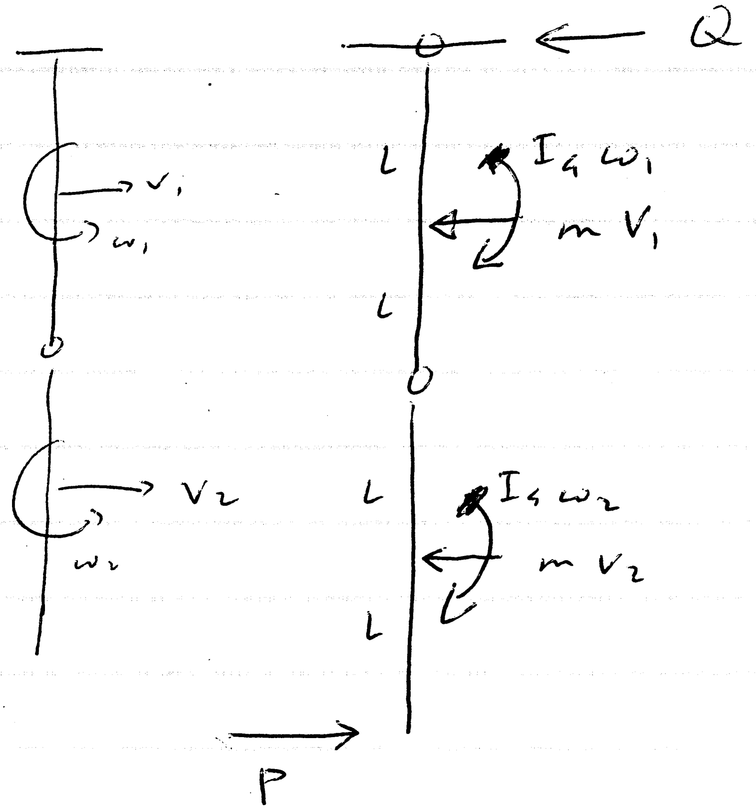
$$PE = \frac{mgR(1-\cos\theta)}{8}$$

$KE = \frac{1}{2}I_G\dot{\theta}^2 + \frac{1}{2}m\left(\frac{3}{4}\right)\left(\frac{5}{6}R\dot{\theta}\right)^2$; kinetic energy

$$KE = \frac{1}{2}(0.385mR^2)\dot{\theta}^2 + \frac{3}{8}mR^2\dot{\theta}^2\left(\frac{25}{36}\right)$$

$$KE = 0.453\,mR^2\dot{\theta}^2$$

3



a) $V_1 = L\omega_1 \rightarrow \omega_1 = V_1/L$

$V_2 = 2V_1 + L\omega_2 \rightarrow \omega_2 = \dfrac{V_2 - 2V_1}{L}$

b) $Q + m(v_1 + v_2) = P$

$Q = P - m(v_1 + v_2)$

c) $2PL = mV_2 L + \dfrac{mL^2}{3}\,\dfrac{V_2 - 2V_1}{\cancel{3}L}$ (1

$4PL = 3mV_2L + mV_1L + \dfrac{mL^2}{3L}\left(V_2 - V_1\right)$ (2

$\rightarrow \dfrac{2P}{m} = V_2 + \dfrac{V_2}{3} - \dfrac{2V_1}{3}$

$\dfrac{4P}{m} = 3V_2 + V_1 + \dfrac{V_2}{3} - \dfrac{V_1}{3}$

So $$\frac{2P}{m} = \frac{4V_2}{3} - \frac{2V_1}{3}$$

$$\frac{4P}{m} = \frac{10V_2}{3} + \frac{2V_1}{3}$$

$$\frac{6P}{m} = \frac{14V_2}{3} \rightarrow V_2 = \cancel{\frac{18P}{14}} \; \frac{9P}{7m}$$

$$\frac{2V_1}{3} = \frac{24P}{14m} - \frac{2P}{m} = \frac{-4P}{14m}$$

$$V_1 = \frac{-3P}{7m}$$

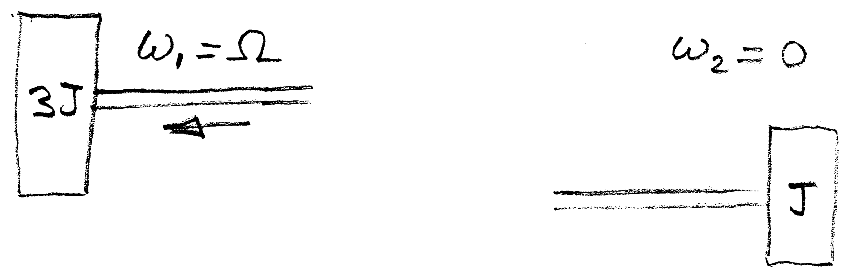
DR HEM HUNT

ENGINEERING TRIPOS PART IB 2005
PAPER 1 MECHANICS SOLUTIONS SECTION B

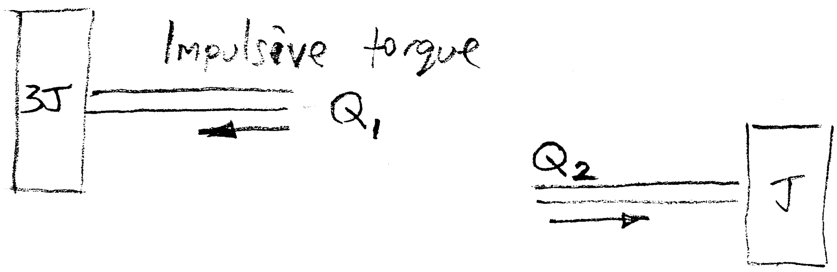
4.(a) There is no axis about which moment of momentum is conserved so consider each shaft separately.

Shaft 1 Shaft 2

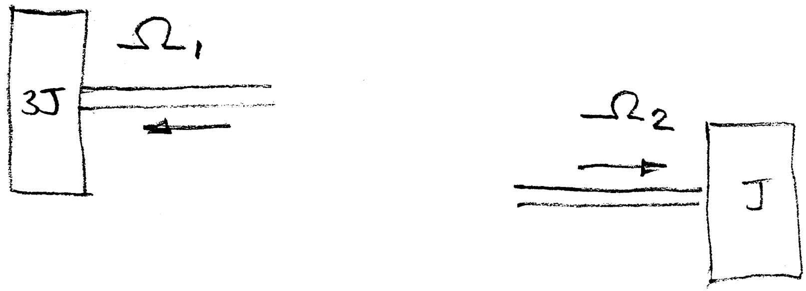
Before :



During :

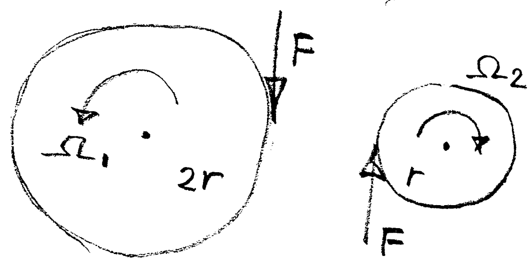


After :



Gearbox : $\Omega_2 = 2\Omega_1$

$Q_2 = \frac{1}{2} Q_1 = Fr$



Moment of Momentum

Shaft 1 : $Q_1 = 3J(\omega_1 - \Omega_1)$ ①

Shaft 2 : $Q_2 = J\,\Omega_2$ ②

$$\frac{①}{②} \rightarrow \frac{Q_1}{Q_2} = 2 = \frac{3(\Omega - \Omega_1)}{2\Omega_1} \quad \therefore 7\Omega_1 = 3\Omega$$

$$\therefore \Omega_1 = \frac{3}{7}\Omega \qquad \Omega_2 = \frac{6}{7}\Omega$$

4(b) Consider shaft 1, subject to constant torque T for time τ. The impulsive torque Q_1

$$Q_1 = \int_0^{\tau} T\,dt = T\tau = 3J(\omega_1 - \Omega_1)$$

$$\therefore \tau = \frac{3J}{T}\left(\Omega - \frac{3}{7}\Omega\right)$$

$$\underline{\underline{\tau = \frac{12J\Omega}{7T}}}$$

(c) Final KE − Initial KE

$$= \tfrac{1}{2}3J\Omega_1^2 + \tfrac{1}{2}J\Omega_2^2 - \tfrac{1}{2}3J\omega_1^2$$

$$= \tfrac{1}{2}J\Omega^2\left(3\left(\tfrac{3}{7}\right)^2 + \left(\tfrac{6}{7}\right)^2 - 3\right)$$

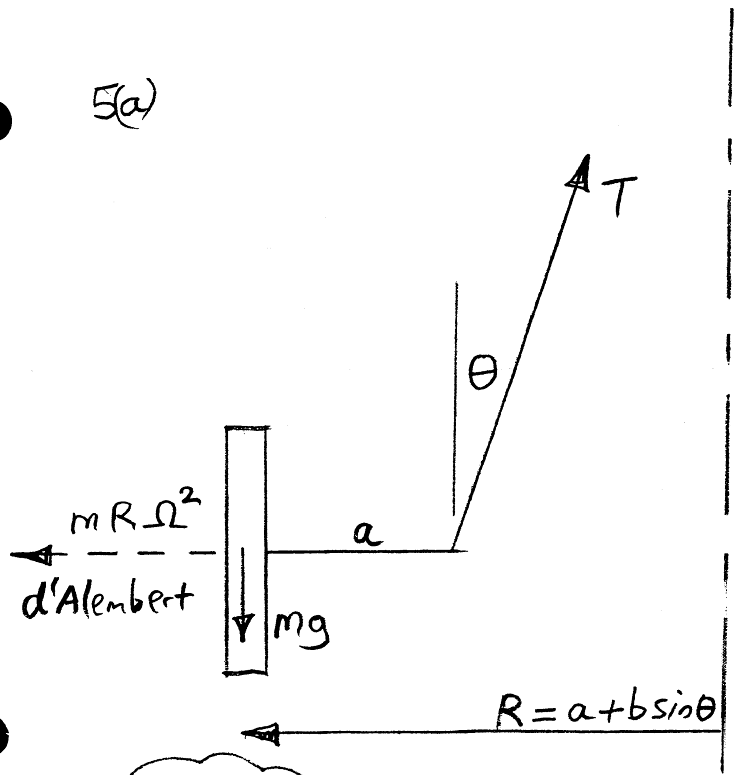
$$= \tfrac{1}{2}J\Omega^2\left(27 + 36 - 147\right)/49$$

$$= -\frac{6}{7}J\Omega^2$$

$$\therefore \underline{\underline{\text{Energy loss} = \frac{6}{7}J\Omega^2}} \qquad \left(< \tfrac{1}{2}3J\Omega^2 \quad \underline{\underline{\text{ok}}}\right)$$

This was a popular question, generally well done but many candidates applied momentum conservation to the whole system to get absurd answers – earning zero marks for part (a). Energy conservation also earned zero. Many in part (b) applied torque T to shaft 2 (a neat shortcut, it seems) but the gearbox means that the torque on shaft 2 is $T/2$.
In (c), if the answer turns out to be an energy gain as happens if (a) is wrong as described above then candidates who noticed this got a bonus mark. In fact no-one did notice it! Be alert.

5(a)



(i) $T\cos\theta = mg$
(vertical equilibrium)

$$\therefore \quad T = \frac{mg}{\cos\theta}$$

(ii) $T\sin\theta = mR\Omega^2$
(horizontal equilibrium)

$$\therefore \quad \tan\theta = \frac{mR\Omega^2}{mg} = \frac{(a+b\sin\theta)\Omega^2}{g}$$

Small θ $\therefore$ $\tan\theta \approx \theta$, $\sin\theta \approx \theta$

$$\therefore \quad g\theta \simeq a\Omega^2 + b\Omega^2\theta$$

$$\therefore \quad \theta \simeq \frac{a\Omega^2}{g - b\Omega^2}$$

Don't use moment equilibrium because a precessing rotor is not in equilibrium. The precession is the result of a net couple.

(b) $Q = T\cos\theta\, a = mga = J\Omega\omega$

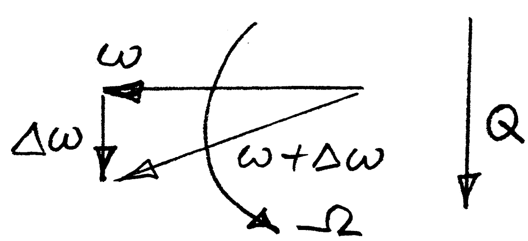
$J = \frac{1}{2}Ma^2$ (from data book $I_{xx} = I_{yy} = \frac{1}{4}Ma^2$ and perpendicular axis theorem $I_{zz} = I_{xx} + I_{yy}$)

$$\therefore \quad mga = \frac{1}{2}Ma^2\Omega\omega$$

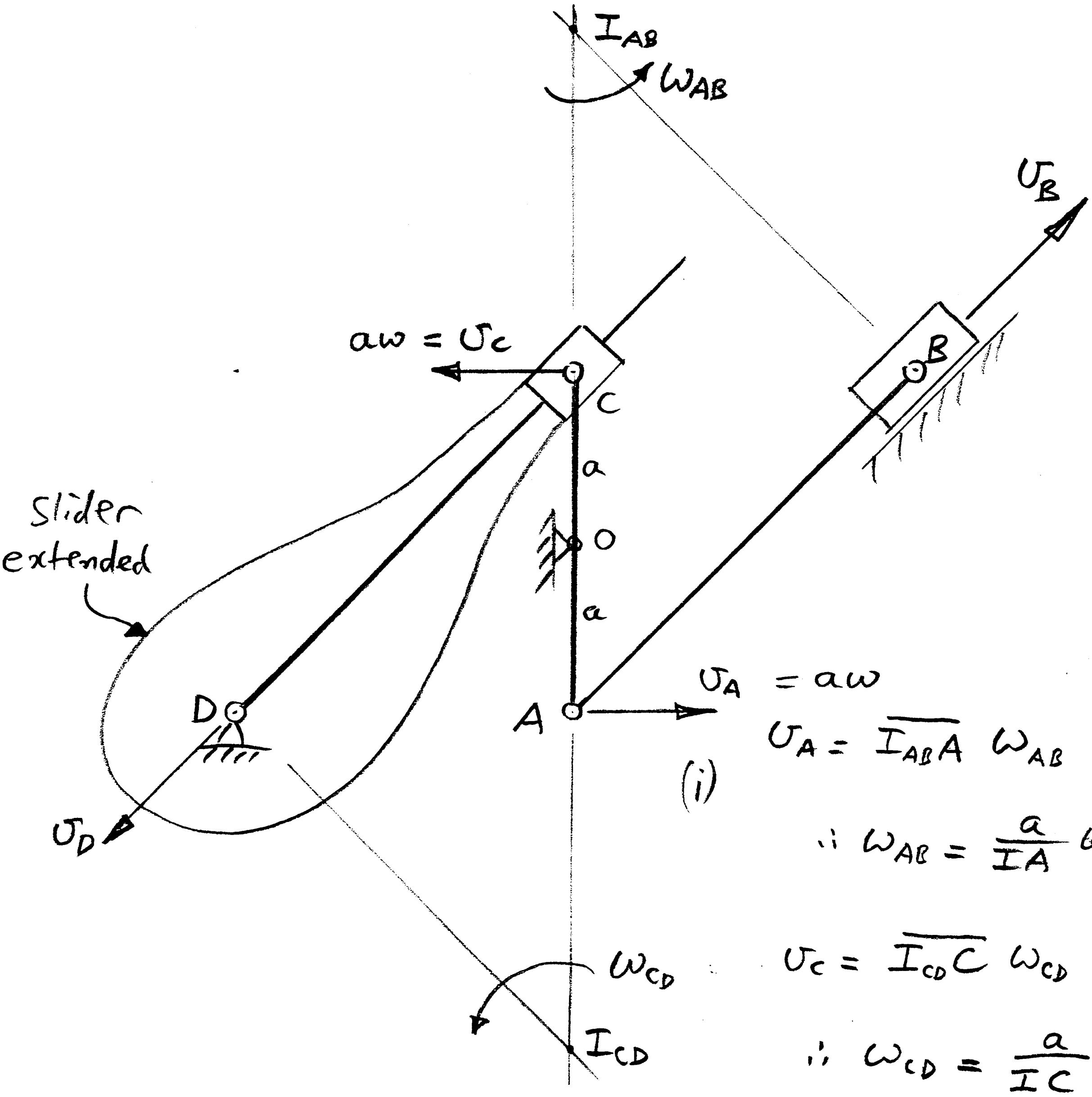
$$\therefore \quad \Omega = \frac{2g}{a\omega}$$

(c) Anticlockwise. Angular momentum changes in the direction of the applied couple.

Any explanation based on "the right hand rule" was not enough.



6(a) Instantaneous Centres



(i)

$$v_A = \overline{I_{AB}A}\,\omega_{AB} = a\omega$$

$$\therefore \omega_{AB} = \frac{a}{IA}\omega = \frac{\omega}{4}$$

$$v_C = \overline{I_{CD}C}\,\omega_{CD} = a\omega$$

$$\therefore \omega_{CD} = \frac{a}{IC}\omega = \frac{\omega}{4}$$

$$= \omega_{DE}$$

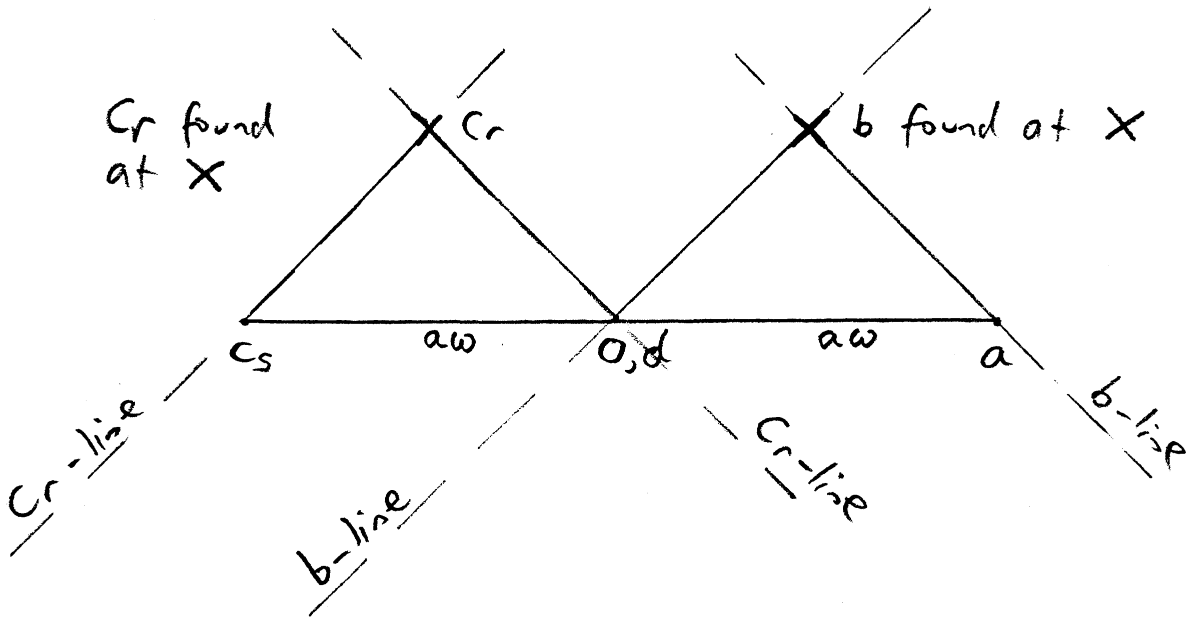
(ii) Sliding velocity at B and at C found by resolving v_A and v_C in the sliding direction

$$v_B = \frac{a\omega}{\sqrt{2}} \nearrow$$

$$v_{C/C_{rod}} = \frac{a\omega}{\sqrt{2}} \swarrow$$

6(a) or Velocity diagram

scale $5\,\text{cm} \equiv a\omega$



(i) $$\underline{v}_{B/A} = \underset{\sim}{\omega}_{AB} \times \underline{r}_{B/A} = \frac{a\omega}{\sqrt{2}} \nwarrow$$

$$\therefore \quad \omega_{AB} = \frac{\frac{a\omega}{\sqrt{2}}}{2\sqrt{2}a} = \frac{\omega}{4}$$

$$v_{Cr/D} = \underline{\omega}_{CD} \times \underline{r}_{C/D} = \frac{a\omega}{\sqrt{2}} \nwarrow$$

$$\therefore \quad \omega_{CD} = \frac{\frac{a\omega}{\sqrt{2}}}{2\sqrt{2}a} = \frac{\omega}{4}$$

(ii) Sliding velocities from diagram

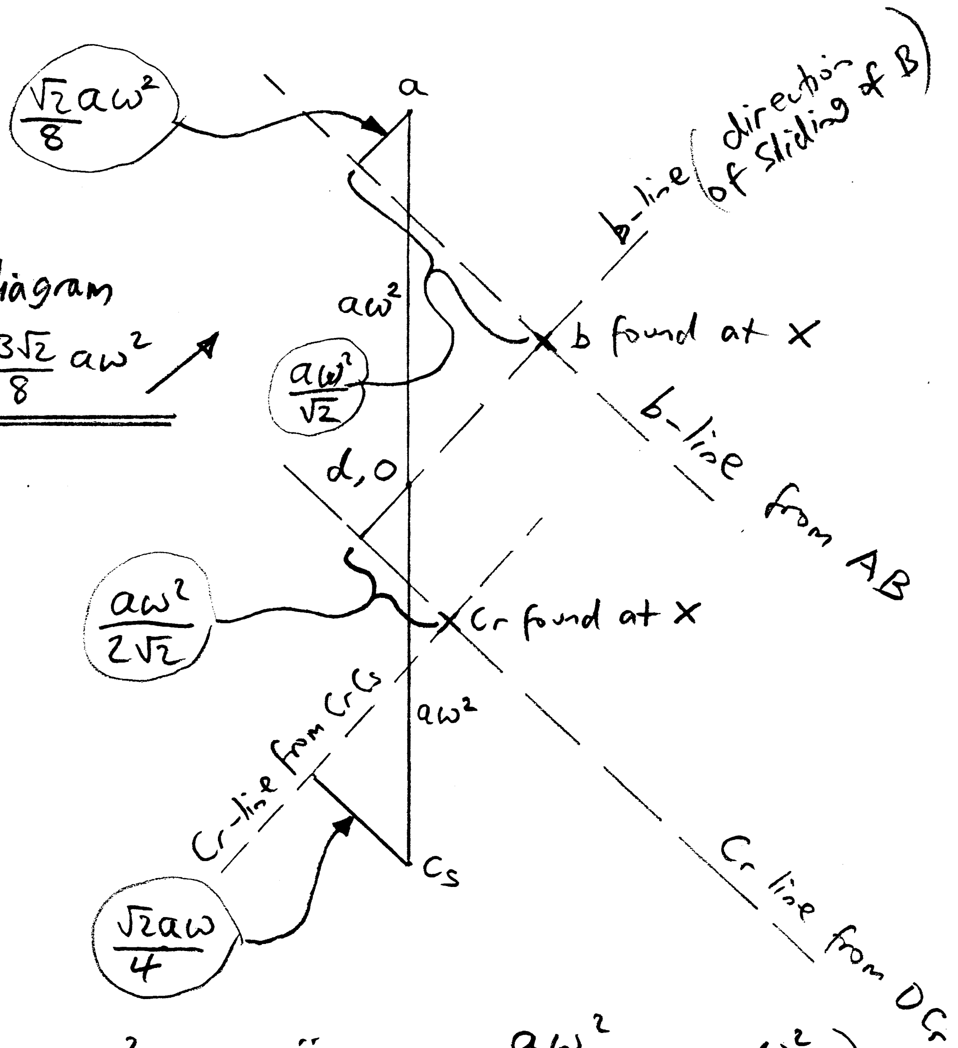
$$v_b = \frac{a\omega}{\sqrt{2}} \nearrow$$

$$v_{Cs/Cr} = \frac{a\omega}{\sqrt{2}} \swarrow$$

6(b) <u>acceleration diagram</u> scale $5\,\text{cm} = a\omega^2$

$$\underline{\ddot{r}} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e_\theta}$$

$$
\begin{array}{lllll}
OA & (0 - a\omega^2)\downarrow & + & (0 + 0) & \rightarrow \\
AB & \left(0 - 2\sqrt{2}a\left(\frac{\omega}{4}\right)^2\right)\nearrow & + & (0 + ?) & \nwarrow \\
 & & & & \\
OC_s & (0 - a\omega^2)\uparrow & + & (0 + 0) & \leftarrow \\
C_rC_s & (? - 0)\swarrow & + & \left(2\cdot\frac{a\omega}{\sqrt{2}}\cdot\frac{\omega}{4} + 0\right) & \searrow \\
DC_r & \left(0 - 2\sqrt{2}a\left(\frac{\omega}{4}\right)^2\right)\nearrow & + & (0 + ?) & \nwarrow
\end{array}
$$



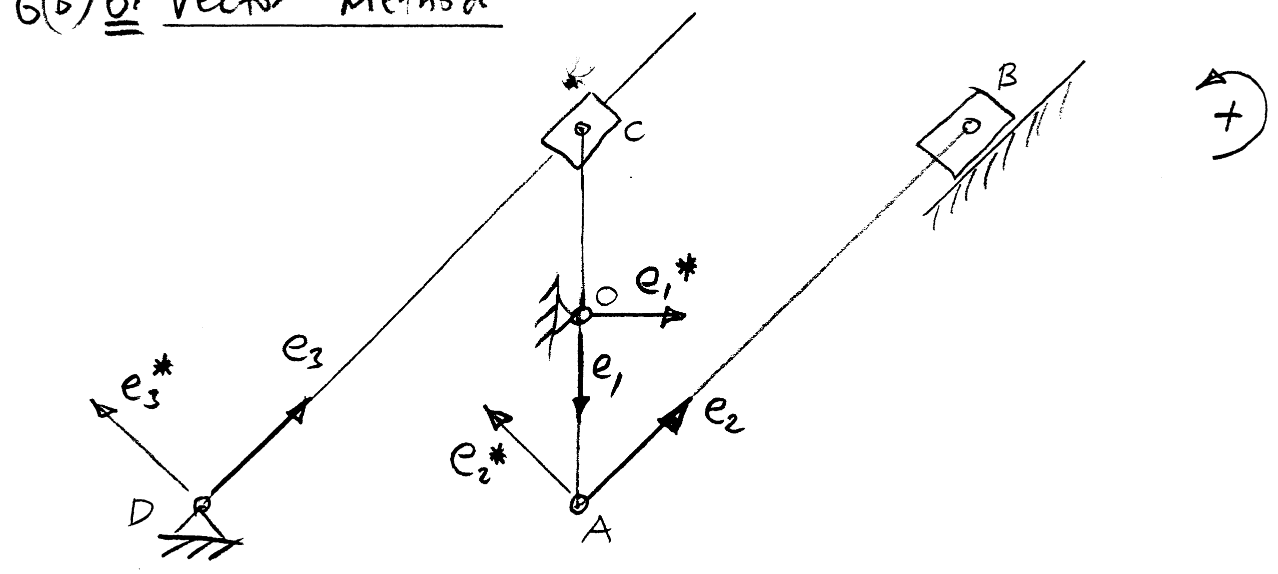
(i) from the diagram

$$\underline{\ddot{r}_B = \frac{3\sqrt{2}}{8}a\omega^2 \nearrow}$$

(ii) $\overline{AB}\,\ddot{\theta}_{AB} = \dfrac{a\omega^2}{\sqrt{2}}$ $\therefore$ $\underline{\ddot{\theta}_{AB} = \dfrac{a\omega^2}{2\sqrt{2}a\sqrt{2}} = \dfrac{\omega^2}{4}}$ ↻

$\overline{DC_r}\,\ddot{\theta}_{DC_r} = \dfrac{a\omega^2}{2\sqrt{2}}$ $\therefore$ $\underline{\ddot{\theta}_{DC_r} = \ddot{\theta}_{DE} = \dfrac{a\omega^2}{2\sqrt{2}a\,2\sqrt{2}} = \dfrac{\omega^2}{8}}$ ↻

6(b) or Vector method



$$\underline{r}_{OB} = a\,\underline{e}_1 + 2\sqrt{2}a\,\underline{e}_2$$
$$\dot{\underline{r}}_{OB} = a\omega\,\underline{e}_1^* + 2\sqrt{2}a\,\dot{\theta}_{AB}\,\underline{e}_2^*$$

but B moves in $\nearrow$ direction

$\therefore\ \dot{\underline{r}}_{OB} \cdot \nwarrow = 0 \qquad \therefore\ \dot{\underline{r}}_{OB} \cdot \underline{e}_2^* = 0$

$\therefore\ a\omega\,\underline{e}_1^* \cdot \underline{e}_2^* + 2\sqrt{2}a\,\dot{\theta}_{AB}\,\underline{e}_2^* \cdot \underline{e}_2^* = 0$

$\therefore\ \omega\left(\frac{-1}{\sqrt{2}}\right) + 2\sqrt{2}\,\dot{\theta}_{AB} = 0$

$$\therefore\ \dot{\theta}_{AB} = \frac{\omega}{4}$$

$$\therefore\ \ddot{\underline{r}}_{OB} = -a\omega^2\underline{e}_1 + 2\sqrt{2}a\,\ddot{\theta}_{AB}\,\underline{e}_2^* - 2\sqrt{2}a\,\dot{\theta}_{AB}^2\,\underline{e}_2$$
$$= -a\omega^2\underline{e}_1 + 2\sqrt{2}a\,\ddot{\theta}_{AB}\,\underline{e}_2^* - \frac{\sqrt{2}}{8}a\omega^2\underline{e}_2$$

but B moves in $\nearrow$ direction

$\therefore\ \ddot{\underline{r}}_{OB} \cdot \nwarrow = 0 \qquad \therefore\ \ddot{\underline{r}}_{OB} \cdot \underline{e}_2^* = 0$

$\therefore\ -a\omega^2\,\underline{e}_1 \cdot \underline{e}_2^* + 2\sqrt{2}a\,\ddot{\theta}_{AB}\,\underline{e}_2^* \cdot \underline{e}_2^* = 0$

$\therefore\ \ddot{\theta}_{AB} = \frac{a\omega^2}{2\sqrt{2}a}\left(-\frac{1}{\sqrt{2}}\right) = \frac{-\omega^2}{4}$

$$\therefore\ \ddot{\theta}_{AB} = \frac{\omega^2}{4}$$

Acceleration of B $= \ddot{\underline{r}}_{OB} \cdot \underline{e}_2 = -a\omega^2\,\underline{e}_1 \cdot \underline{e}_2 - \frac{\sqrt{2}}{2}a\omega^2\,\underline{e}_2 \cdot \underline{e}_2$

$$= -a\omega^2\left(-\frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{2}a\omega^2$$
$$= \frac{3\sqrt{2}}{8}a\omega^2 \nearrow$$

6(b) vector method cont'd

$\underline{r}_{DO} = (2\sqrt{2}a + x)\,\underline{e}_3 + a\,\underline{e}_1$ where x is the distance between C_s & C_r such that $x = 0$ but $\dot{x} \neq 0$ at the instant shown

$\dot{\underline{r}}_{DO} = (2\sqrt{2}a + x)\,\dot{\theta}_{DE}\,\underline{e}_3^* + \dot{x}\,\underline{e}_3 + a\omega\,\underline{e}_1^*$

$= 0$ since $\overline{DO}$ is a fixed vector and put $x = 0$

$\therefore\ 2\sqrt{2}a\,\dot{\theta}_{DE}\,\underline{e}_3^* + \dot{x}\,\underline{e}_3 + a\omega\,\underline{e}_1^* = 0$

dot with e_3^* $\therefore$ $2\sqrt{2}a\,\dot{\theta}_{DE} + a\omega\,\underline{e}_1^* \cdot \underline{e}_3^* = 0$

$$\therefore\ \dot{\theta}_{DE} = -\frac{a\omega\left(-\frac{1}{\sqrt{2}}\right)}{2\sqrt{2}a} = \frac{\omega}{4}$$

dot with e_3 $\therefore$ $\dot{x} + a\omega\,\underline{e}_1^* \cdot \underline{e}_3 = 0$

$$\therefore\ \dot{x} = -\frac{a\omega}{\sqrt{2}} \nearrow$$

$\therefore$ sliding velocity of C_s relative to $C_r = \frac{a\omega}{\sqrt{2}} \swarrow$

$$\ddot{\underline{r}}_{DO} = (2\sqrt{2}a + x)\ddot{\theta}_{DE}\,\underline{e}_3^* - 2\sqrt{2}a\,\dot{\theta}_{DE}^2\,\underline{e}_3 + \ddot{x}\,\underline{e}_3 + 2\dot{x}\,\dot{\theta}_{DE}\,\underline{e}_3^* - a\omega^2\underline{e}_1$$

$= 0$

dot with e_3^* : $2\sqrt{2}a\ddot{\theta}_{DE} + 2\dot{x}\,\dot{\theta}_{DE} - a\omega^2\,\underline{e}_1 \cdot \underline{e}_3^* = 0$

$$\therefore\ \ddot{\theta}_{DE} = \frac{1}{2\sqrt{2}a}\left(a\omega^2\left(\frac{-1}{\sqrt{2}}\right) - 2\left(\frac{-a\omega}{\sqrt{2}}\right)\frac{\omega}{4}\right)$$

$$= -\frac{a\omega^2}{8}$$

$$\therefore\ \ddot{\theta}_{DE} = \frac{a\omega^2}{8}$$

The most popular method was velocity diagram followed by acceleration diagram.
A few used the e, e^* method but none got the right answers with it.