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ENGINEERING TRIPOS, Part IB 2005
Paper 6 - INFORMATION ENGINEERING
Solutions

1 (a) A system is asymptotically stable if its impulse response $g(t)$ satisfies the condition:

$$\int_0^\infty |g(t)|dt < \infty$$

The transfer function of the system can be calculated by taking the Laplace transform of the ODE. If the real parts of all poles of the transfer function are negative then the system is asymptotically stable. If there exist single poles on the imaginary axis and all the other poles have the real part negative then the system is marginally stable. In all the other cases the system is unstable.

(b) (i)

$$g(t) = u(t-1) - u(t-2) \rightarrow G(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

(ii) The integral of $g(t)$ over all possible values of t is 1, therefore the system is asymptotically stable, according to the definition of asymptotic stability.

(c) Final value theorem states that:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\bar{y}(s)$$

where $\bar{y}(s) = \bar{x}(s)G(s)$ is the Laplace transform of the output signal.

Use the final value theorem for (i) and (iii) and the frequency response with $\omega = \pi$ for (ii).

(i) $\bar{x}(s) = 1/s$

By L'Hopital's rule:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\frac{1}{s}\frac{e^{-s} - e^{-2s}}{s} = \lim_{s\to 0}\frac{e^{-s} - e^{-2s}}{s} = \lim_{s\to 0}\frac{-e^{-s} + 2e^{-2s}}{1} = 1$$

(ii)

$$G(j\omega) = \frac{e^{-j\omega} - e^{-2j\omega}}{j\omega}$$

1(cont'd) For $\omega = \pi$:

$$G(j\omega) = \frac{e^{-j\pi} - e^{-2j\pi}}{j\pi} = \frac{-2}{j\pi}$$

Modulus of the transfer function at desired frequency is $2/\pi$ and argument is $\pi/2$, therefore the response of the system settles at $2/\pi sin(\pi t + \pi/2)$

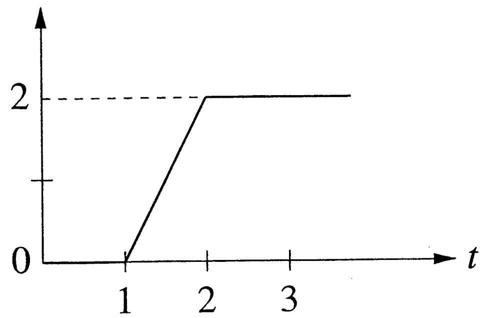
(iii)

$$x(t) = sin\pi t + sin\pi(t-1) \to \bar{x}(s) = \frac{\pi(1-e^{-s})}{s^2+\pi^2}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\frac{\pi(1-e^{-s})}{s^2+\pi^2}\frac{e^{-s}-e^{-2s}}{s} = 0$$

(d) The impulse response is the inverse Laplace transform of the return ratio $2G(s)/s$, which is simply $2\int g(t)dt$:

impulse response of the return ratio



Close-loop transfer function:

$$H(s) = \frac{G(s)}{1+K(s)G(s)} = \frac{e^{-s}-e^{-2s}}{s^2+2(e^{-s}-e^{-2s})}$$

Using again the final value theorem and L'Hopital's rule:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\frac{e^{-s}-e^{-2s}}{s^2+2(e^{-s}-e^{-2s})} = \lim_{s\to 0}\frac{e^{-s}-e^{-2s}-se^{-s}+2se^{-2s}}{2s-2e^{-s}+4e^{-2s}} = 0$$

Note that this is not the response to an input but to a disturbance, since $K(s)$ appears on the feedback loop, therefore the steady-state value is 0 and not 1, as might have been expected.

Examiner's comment: Almost all candidates have answered the theoretical part of this question in a satisfactory manner and most were able to calculate the transfer function correctly. Some common errors regarding the Laplace transform were to identify the delay but consider its effect wrongly, or to consider the impulse response as a step function, or to confuse the step and delta functions. Some solutions even used the Fourier transform instead of Laplace! The final value theorem was familiar to practically all candidates, although some stated it incorrectly. Only few students remembered L'Hopital's rule, but others calculated the limit correctly by using Taylor expansion of the numerator and denominator. A mistake often encountered was to try to calculate the steady-state value for Part b(i) by using the final value theorem instead of the frequency response. Some candidates solved this part by calculating the inverse Laplace transform or by convolution, which received the same number of marks as the method involving the final value theorem. However, a few students tried to used the close-loop system, assuming arbitrary values for $K(s)$, instead of calculating the open-loop steady-state response. Only one or two candidates noticed that the graph of Part (d) was simply the integral of the impulse response, but a large number of students calculated the inverse Laplace transform of the return ratio instead. Many candidates understood the requirements of the last part of the question and at least attempted to solve it.

2 (a) If we input into the system harmonic functions of different frequencies, then the response will oscillate on the same frequency. The ratio of the output to input amplitudes will give us the magnitude of the transfer function, while the phase difference between input and output is the argument of the transfer function for the corresponding frequency.

(b) (i) Yes, but with a very small phase margin (about 3 degrees).

(ii) By inspection of the Bode diagram, $G(s)$ has one pole in 0 (the modulus has negative slope of 20dB/dec and the argument starts at $-\pi/2$). There is a resonance at approximately $\omega = 1$ rad/s, therefore $G(s)$ contains a second-order term at the denominator. It can also be noticed, from the interception of the asymptotes, that $G(s)$ must have a zero around $\omega = 0.1$rad/s, therefore:

$$G(s) = \frac{\alpha(s + 0.1)}{s(s^2 + \beta s + 1)}$$

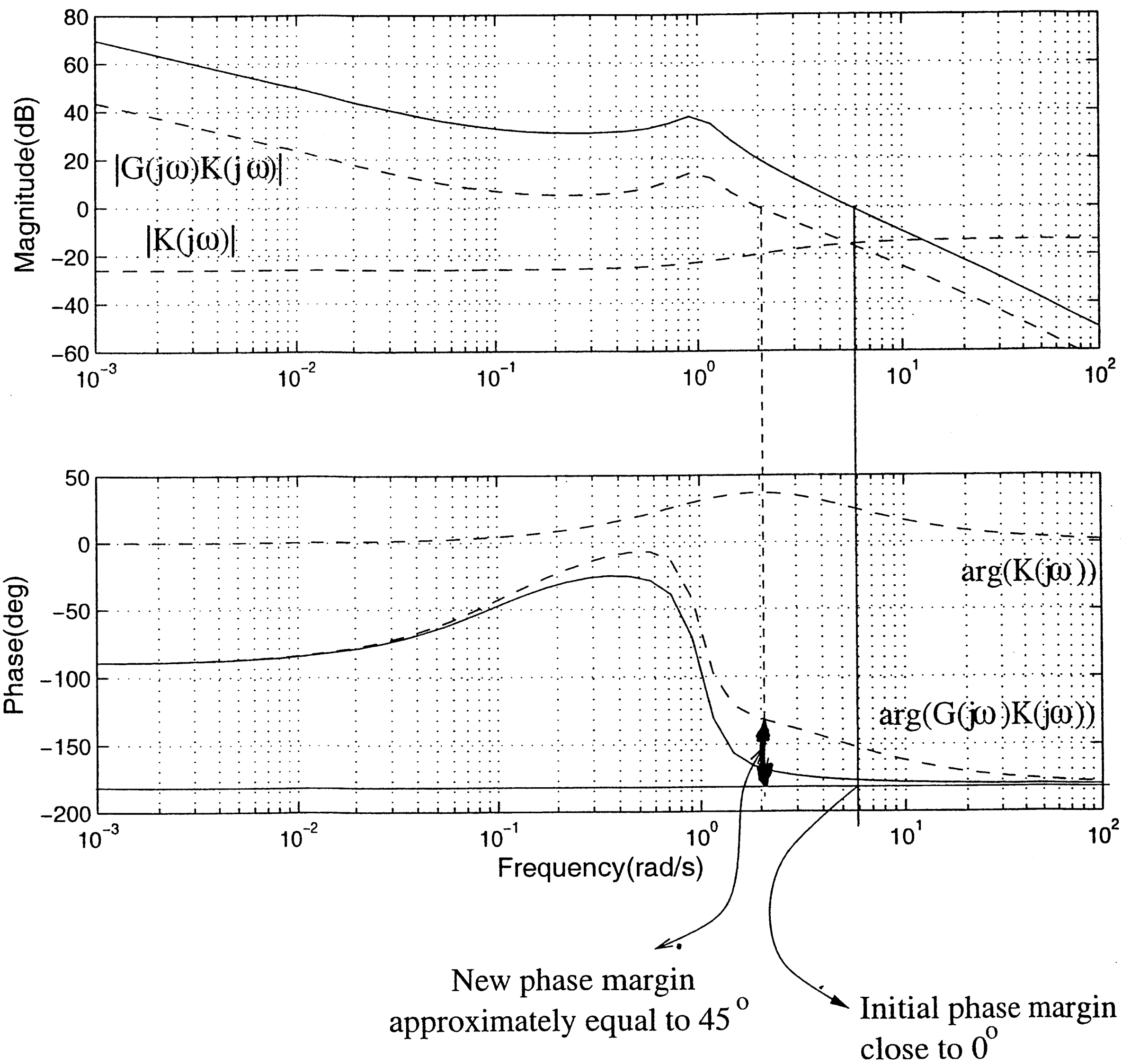
To estimate α, use for instance $G(0.003j) = 60$dB (from the graph): $0.1\alpha/0.003 = 1000 \rightarrow \alpha = 30$ The damping ratio can be approximated by reading the magnitude of the peak in the modulus of $G(s)$ and equating it with $30/\beta$. It follows that:

$$G(s) = \frac{30(s + 0.1)}{s(s^2 + 0.4s + 1)}$$

(c) For $\omega = 2$rad/s the modulus of $G(j\omega)$ can be read of the graph as approximately 20dB, so the modulus of $K(j\omega)$ at $\omega = 2$rad/s has to be around 0.1, therefore $A \simeq 4$

(d) See figure on the next page. The phase margin is increased to 49 degrees, therefore the stability of the closed-loop system is increased.

Examiner's comment: Many candidates answered this question well, although there have been quite a few silly algebra mistakes. The corner frequencies of the second part were noticed by most students who attempted this problem, although calculating k and β proved a more challenging task. It was encouraging to note almost all candidates realised what was the desired role of adding the controller, and were able to explain it in a satisfactory manner.



80
60
40
20
0
-20
-40
-60
Magnitude(dB)
|G(jω)K(jω)|
|K(jω)|
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
50
0
-50
-100
-150
-200
Phase(deg)
arg(K(jω))
arg(G(jω)K(jω))
Frequency(rad/s)
New phase margin approximately equal to 45°
Initial phase margin close to 0°

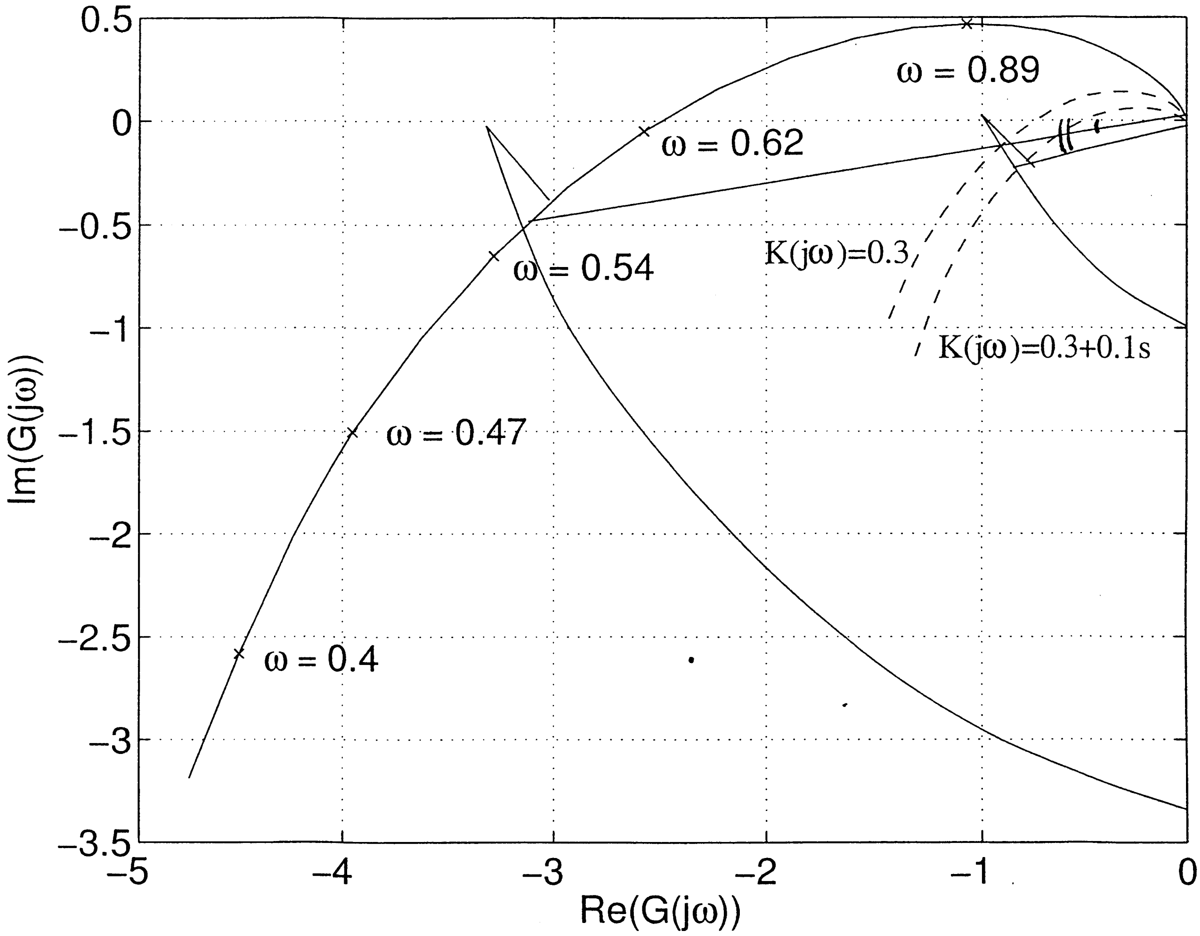
3 (a) Plot the Nyquist diagram of $K(s)G(s)$. The system is asymptotically stable if the curve leaves out point -1

(b) (i) unstable

(ii) $2.48k \leq 1 \rightarrow k \leq 0.4$

(iii) gain margin approximately 1.33, phase margin 11 degrees.

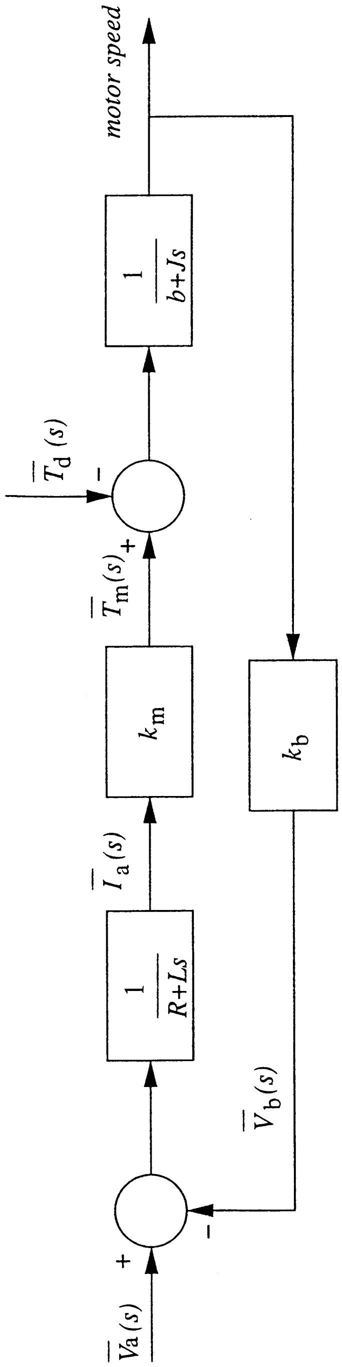
(c) See figure below. The new gain margin is 2, and the new phase margin approximately 21 degrees.



(d) Maximum value for the modulus of sensitivity is the inverse of the length of the perpendicular taken from point $(-1, 0)$ to the curve. For proportional control the value is approximately 5, for proportional and derivative control, 3, therefore in the second case the errors are amplified less than in the first case.

Examiner's comment: The large majority of the candidates attempting this question answered Part (a) and (b) perfectly. Some of the sketches in the next part were wildly different from the correct shape, and unfortunately very few candidates noticed that added derivative control is not likely to worsen the performances of the closed-loop system. In Part (d), most solutions demonstrated a good understanding of the meaning of the sensitivity function and of how its value can be estimated using the Nyquist diagram.

4 (a)



4 (a) (cont'd)

$$J\dot{\omega} + b\omega = T_m - T_d \rightarrow Js\bar{\omega}(s) + b\bar{\omega}(s) = \bar{T_m}(s) - \bar{T_d}(s)$$

$$T_m = k_m \frac{V_a - V_b}{R + Ls} = k_m \frac{V_a - k_b\omega}{R + Ls}$$

Now we separate ω from the first equation and replace the expression of T_m from the second equation:

$$\bar{\omega}(s) = \frac{\bar{T_m}(s) - \bar{T_d}(s)}{Js + b} = k_m \frac{\bar{V_a}(s) - k_b\bar{\omega}(s)}{R + Ls} \frac{1}{Js + b} - \frac{\bar{T_d}(s)}{Js + b}$$

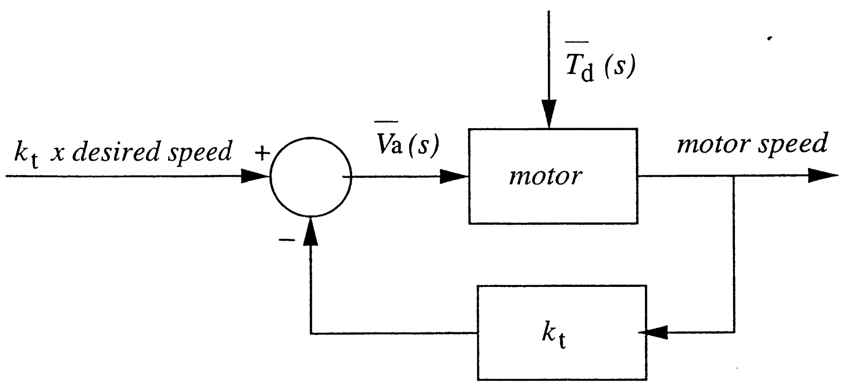
$$\bar{\omega}(s)(1 + \frac{k_m k_b}{(R + Ls)(Js + b)}) = \bar{V_a}(s) \frac{k_m}{(R + Ls)(Js + b)} - \frac{\bar{T_d}(s)}{Js + b}$$

It follows that:

$$\bar{\omega}(s) = \bar{V_a}(s) \frac{k_m}{(R + Ls)(Js + b) + k_m k_b} - \bar{T_d}(s) \frac{R + Ls}{(R + Ls)(Js + b) + k_m k_b}$$

$$\bar{\omega}(s) = G_v \bar{V_a}(s) - G_T \bar{T_d}(s)$$

(b)



$$V_a = G_1 k_t(\omega_d - \omega) = G_1 k_t(\omega_d - G_v \bar{V_a}(s) + G_T \bar{T_d}(s))$$

$$\frac{\bar{\omega}(s)}{\bar{\omega_d}(s)} = \frac{G_1 G_v k_t}{1 + G_1 G_v k_t}$$

$$\frac{\bar{\omega}(s)}{\bar{\omega_d}(s)} = \frac{G_1 k_m k_t}{G_1 k_m k_t + k_m k_b + (R + Ls)(Js + b)}$$

(c) To find the natural frequency of oscillation ω_n and the damping ratio ξ for a second-order system, the denominator of the transfer function has to be arranged in the form: $s^2 + 2\xi s\omega_n + \omega_n^2$.

By identification, the parameters of the motor in Part (a) are:

$$\omega_n^a = \sqrt{\frac{k_m k_b + Rb}{LJ}} = \sqrt{\frac{1.5}{2}} = 0.866$$

$$\xi^a = \frac{Lb + RJ}{2LJ\omega_n^a} = \frac{2.5}{2 * 2 * 0.0866} = 0.723$$

For Part (b), the natural frequency of oscillation and damping ratio become:

$$\omega_n^b = \sqrt{\frac{k_m k_b + Rb + G_1 k_m k_t}{LJ}} = \sqrt{\frac{11.5}{2}} = 2.398$$

$$\xi^b = \frac{Lb + RJ}{2LJ\omega_n^b} = \frac{2.5}{2 * 2 * 2.398} = 0.26$$

The controller added in Part (b) has increased the natural frequency of oscillation, which means that the bandwidth has increased, but the damping ratio is reduced, therefore the overall feedback system in Part (b) will be have a more oscillatory behaviour than the motor in Part (a).

Examiner's comment: This question was answered excellently and it was pleasing to notice that candidates were not confused by the closing of a second feedback loop and by the fact that the controller actually deteriorates the stability of the system. Most solutions have shown a perfect grasp of how the performances of a second-order system are affected by the natural frequency of oscillation and the damping ratio.

5 (a) If signal $x(t)$ is sampled at time intervals of length T, then the sampled signal is:

$$x_s(t) = x(t)p(t)$$

where

$$p(t) = T\sum_{m=-\infty}^{\infty}\delta(t-mT) = \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

and $\omega_s = \frac{2\pi}{T}$

Then x_s becomes:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t)e^{jn\omega_s t}$$

with the Fourier transform:

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

where $X(\omega)$ is the Fourier transform of $x(t)$

If follows that the spectrum $X_s(\omega)$ is the spectrum of x repeated at multiples of ω_s. If ω_n is the maximum frequency of components in $X(\omega)$ then ω_s has to satisfy: $\omega_s \geq 2\omega_n$. If the sampling frequency is lower than this limit, components from two separate spectra overlap, leading to **aliasing errors**.

The **quantising errors** appear due to the finite precision of the ADC.

(b) The RMS value of the signal is: $\frac{0.8}{\sqrt{2}}$.

If the noise is modelled with uniform pdf of magnitude $\frac{1}{\delta v}$ between $-\frac{\delta v}{2}$ and $\frac{\delta v}{2}$ then the error noise power can be calculated as: $\int_{-\frac{\delta v}{2}}^{\frac{\delta v}{2}} x^2 \frac{1}{\delta v} dx = \frac{\delta v^2}{12}$

It follows that the RMS value of the noise is $\frac{\delta v}{\sqrt{12}}$

The quantising step size is: $\delta v = \frac{2}{2^n\sqrt{12}}$, where n is the number of bits of the ADC, therefore for $n = 4$ $S/N = 0.8\sqrt{2} \times 2^4\sqrt{12}/2$ which gives 23.9dB.

(c)

The ADC step is: $(V_{max} - V_{min})/2^n = 0.125$V

The ADC operation is described in the following table.

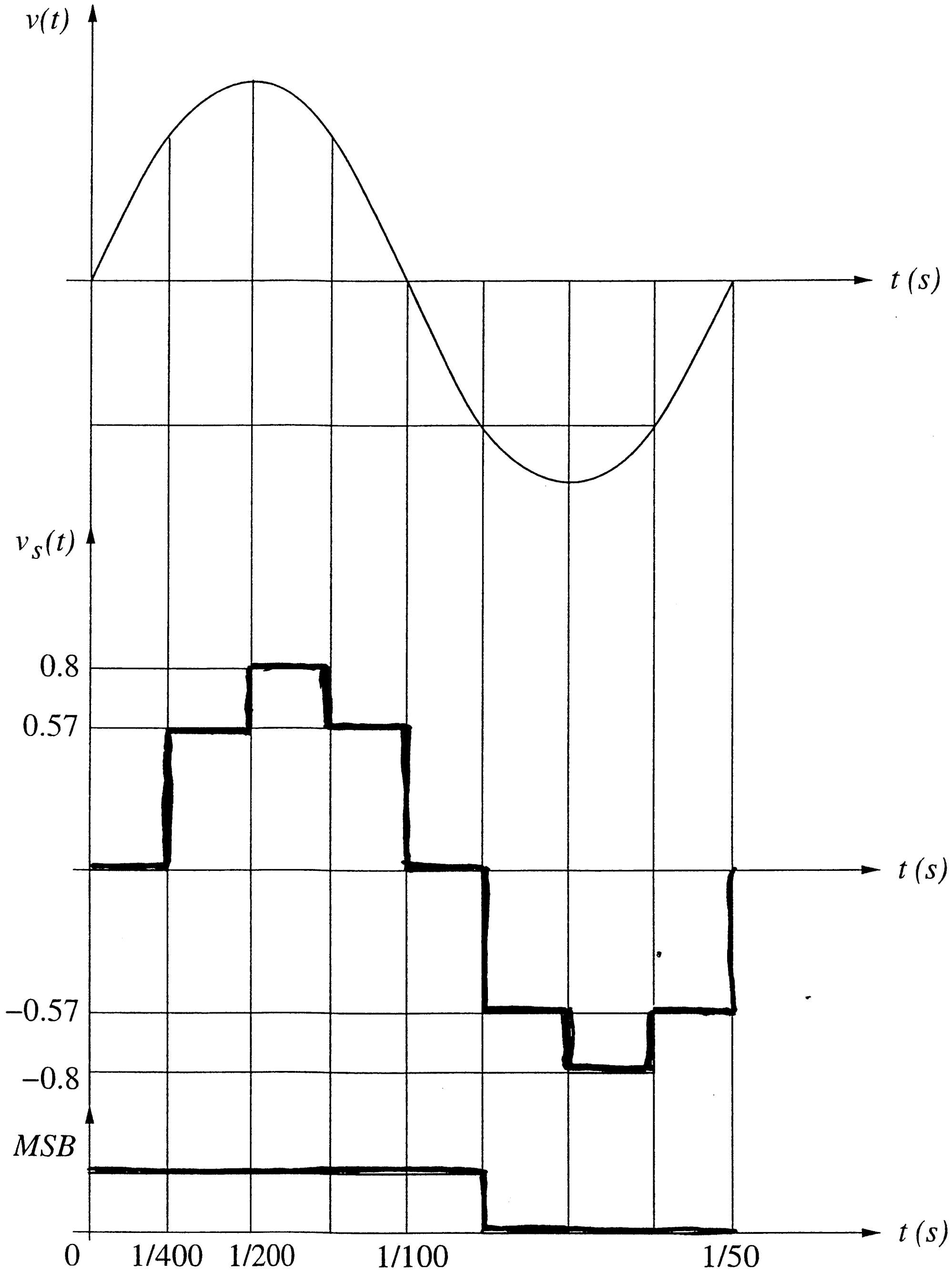
level	output code
-1V	0000
-0.875V	0001
-0.75V	0010
-0.625V	0011
-0.5V	0100
-0.375V	0101
-0.25V	0110
-0.125V	0111
0V	1000
0.125V	1001
0.25V	1010
0.375V	1011
0.5V	1100
0.625V	1101
0.75V	1110
0.875V	1111

Output sequence: 8, 13, 14, 13, 8, 3, 2, 3

(d) Aliasing distortions are avoided if the filter lets the spectrum of the initial signal pass undistorted, and rejects completely the repeated spectrum centred on ω_s, hence the pass band of the filter has to be at least equal to ω_m, the maximum frequency present in the spectrum of the original signal, and the pass band ends at $\omega_s/2$:

$$f_m + 0.4 f_m \leq \frac{f_s}{2} \rightarrow f_m \leq 143\text{Hz}$$

Since the sampling frequency remains constant, f_m can be increased only if the transition band of the filter is reduced. This can be achieved by introducing extra poles in the transfer function of the filter, for instance by adding more capacitors to the circuit, thus making it more complex and prone to noise.


v(t)
t (s)
$v_s(t)$
0.8
0.57
t (s)
−0.57
−0.8
MSB
t (s)
0
1/400
1/200
1/100
1/50

Examiner's comment: This question was generally well answered. A number of candidates miscalculated the signal power when evaluating the SNR in part (b), i.e., they assumed the signal amplitude was 1V, not the 0.8 V specified in the question. Surprisingly, many of the answers to part (c) were quite poor, given the relatively straightforward nature of the task. Part (d) caused a few problems, which betrayed a lack of comprehension concerning the function of the low pass filter prior to analogue to digital conversion.

6 (a) As in the notes.

(b) Assuming that the horizontal and vertical resolutions are the same and neglecting extra requirements for interlaced scanning, we have:

625 lines/frame $\times$ 25 frames/s $\times$ (625$\times$4/3) pixel/line=13Mpixel/s

(c) The signal transmitted is a rectangular pulse of width one pixel and period given by the line frequency. From the Databook:

$$x(t) = \frac{a}{T}(1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin\frac{n\pi a}{T}}{\frac{n\pi a}{T}} e^{jn\omega_0 t})$$

where $\omega_0 = 2\pi f_{line} = 2\pi \times 625 \times 25$, $T = 2\pi/\omega$ and $a = 1/13\text{M}$ (the duration of one pixel).

To calculate how many harmonics have to be transmitted before the amplitude decreases to 3dB compared to the fundamental component:

$$\text{sinc}(\frac{\text{n}\pi\text{a}}{\text{T}}) = \frac{1}{\sqrt{2}} \rightarrow \frac{\text{n}\pi\text{a}}{\text{T}} = 1.392 \rightarrow \text{n} \simeq 371$$

The 3dB frequency is the frequency of the n-th harmonic:

$$f_{3dB} = n \times 625 \times 25 = 5.79\text{MHz}$$

(d) The signal transmitted becomes: $x(t) + x(t - \tau)$, where $\tau = 100 \times a$. Writing each signal as an exponential Fourier series, the amplitudes of the harmonics can be identified as:

$$c_n = c_{n1} + c_{n2} = \frac{a}{T} \frac{\sin\frac{n\pi a}{T}}{\frac{n\pi a}{T}} (1 + e^{-jn\omega_0\tau})$$

and the DC amplitude:

$$c_0 = 2\frac{a}{T}$$

The attenuation of the n-th harmonic is:

$$|\frac{c_n}{c_0}| = |\text{sinc}\frac{n\pi}{833}| \times |\cos 0.38n|$$

The coefficients are similar to those in Part (c), but modulated with $|\cos 0.38n|$, hence the bandwidth will be comparable with the one calculated before (very slightly increased, since the magnitude of a cos function is less than 1).

Examiner's comment: For the most part this question was reasonably well answered. Parts (a) and (b) caused few problems, but a number of candidates had difficulty finding the number of Fourier Series coefficients required in part (c). Part (d) caused most difficulties and no candidate produced a complete correct solution. Some candidates realised that superposition and the time-shift theorem could be applied, but did not reach the final solution. In view of the problems experienced in part (d) it was decided to slightly modify the marks allocated to part (c) and to part (d) from 4 to 6 and from 8 to 6 respectively.