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Solutions: Paper 7 – Mathematical Methods, IB 2005

Section A

1. (a) Express the integral I in polar coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$.

The substitution is: $x^2 + y^2 = r^2$. The Jacobian is r and therefore $dxdy$ is substituted by $rdrd\theta$ to give:

$$\begin{aligned} I &= \int_{r=0}^{R}\int_{\theta=0}^{\pi} r^2 \; rdrd\theta \\ &= \pi\left[\frac{r^4}{4}\right]_0^R \\ &= \frac{\pi R^4}{4} \end{aligned}$$

(b) The volume V is a stack of the semi-circular slices evaluated in part (a).

$$\begin{aligned} J &= \int\int\int_V (x^2+y^2) \; dxdydz \\ &= \int_{z=0}^{1} I(z) \; dz \\ &= \int_{z=0}^{1} \frac{\pi R^4}{4} \; dz \qquad \text{with } R = (1-z) \\ &= -\frac{\pi}{4}\left[\frac{(1-z)^5}{5}\right]_0^1 \\ &= \frac{\pi}{20} \end{aligned}$$

(c)(i) $\nabla \cdot \mathbf{F} = y^2 + x^2$

(ii) Flat triangular surface: $\mathbf{n} = -\mathbf{j}$. Therefore $\mathbf{F}\cdot\mathbf{n} = -x^2y$. But $y = 0$ everywhere on this surface so

$$\int\int_{S(triangular)} \mathbf{F}\cdot\mathbf{n} \; dS = 0$$

Flat semicircular surface: $\mathbf{n} = -\mathbf{k}$. Therefore $\mathbf{F}\cdot\mathbf{n} = -1$. The area of this surface is $\pi/2$ so

$$\int\int_{S(semicircular)} \mathbf{F}\cdot\mathbf{n} \; dS = -\pi/2$$

(iii) The divergence (Gauss) theorem in 3D is

$$\int\int\int_V \nabla\cdot\mathbf{F} \; dV = \int\int_S \mathbf{F}\cdot\mathbf{n} \; dS$$

From part (b), $\int\int\int_V \nabla\cdot\mathbf{F}\;\; dV = \pi/20$. Hence:

$$\begin{aligned}\int\int_{S(conical)} \mathbf{F}\cdot\mathbf{n}\;\; dS &= \frac{\pi}{20} - \int\int_{S(triangular)} \mathbf{F}\cdot\mathbf{n}\;\; dS - \int\int_{S(semicircular)} \mathbf{F}\cdot\mathbf{n}\;\; dS \\ &= \frac{\pi}{20} - 0 + \frac{\pi}{2} \\ &= \frac{11\pi}{20}\end{aligned}$$

[A very popular question. Most candidates showed good understanding of the principles of the question. A common mistake was to evaluate the Jacobian as $1/r$, rather than r in part (a). Another common error was to assume that $\nabla\cdot\mathbf{F}$ was equal to zero in part (c)iii, even though nearly all candidates had evaluated it as x^2+y^2 in part (c)i.]

2. (a) (i) $\nabla\cdot\mathbf{F} = 6$ and $\nabla\times\mathbf{F} = 0$.

(ii) Because $\nabla\times\mathbf{F} = 0$, there is a scalar potential ϕ which satisfies $\mathbf{F} = \nabla\phi$. This is given by:

$$\phi = x^2 + y^2 + z^2 + C$$

where C is an arbitrary constant.

(iii) Because $\nabla\times\mathbf{F} = 0$, the line integral is independent of the path taken and $\int_L \mathbf{F}\cdot d\mathbf{L} = \phi_{end} - \phi_{start} = 1 - 0 = 1$.

(b) (i) By Stokes' theorem, $\oint_C \mathbf{G}.\mathbf{dl} = \int\int_A \nabla\times\mathbf{G}\;\; dA$, where A is the bottom face of the cube, which has area 1 and unit normal $\mathbf{k}$ (defined by the direction of the arrows). From the definition of $\mathbf{G}$, the vector product $\nabla\times\mathbf{G} = -2\mathbf{k}$. Hence $\oint_C \mathbf{G}.\mathbf{dl} = -2$.

(ii) $\nabla\cdot(\nabla\times\mathbf{G}) = 0$. Hence the flux of $\nabla\times\mathbf{G}$ through surface S, the other five faces of the cube, is also -2. Because a direction is not specified, $+2$ is also an acceptable answer.

[This was the most popular question in section A and was very straightforward. Most candidates spotted the short way to do parts (a)iii and (b)i, showing good understanding of the principles.]

3. (a) Start from Laplace's equation and use separation of variables:

$$\nabla^2\psi(x,y) = \nabla^2(X(x)Y(y)) = YX'' + XY'' = 0$$

$$\Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

Therefore, either

$$\frac{X''}{X} = -k^2 \qquad \text{and} \qquad \frac{Y''}{Y} = k^2$$

or else

$$\frac{X''}{X} = k^2 \qquad \text{and} \qquad \frac{Y''}{Y} = -k^2$$

(i) The general solution if $k = 0$ is $\psi(x, y) = (Ax + B)(Cy + D)$.

(ii) The general solution if $\frac{X''}{X}$ is non-zero and negative is any of the following:

$$\begin{aligned}
\psi(x,y) &= (Ae^{ikx} + Be^{-ikx})(Ce^{ky} + De^{-ky}) \\
\psi(x,y) &= (A'\cos(kx) + B'\sin(kx))(C'\cosh(ky) + D'\sinh(ky)) \\
\psi(x,y) &= (A'\cos(kx) + B'\sin(kx))(Ce^{ky} + De^{-ky}) \\
\psi(x,y) &= (Ae^{ikx} + Be^{-ikx})(C'\cosh(ky) + D'\sinh(ky))
\end{aligned}$$

(iii) The general solution if $\frac{X''}{X}$ is non-zero and positive is any of the following:

$$\begin{aligned}
\psi(x,y) &= (Ae^{kx} + Be^{-kx})(Ce^{iky} + De^{-iky}) \\
\psi(x,y) &= (A'\cosh(kx) + B'\sinh(kx))(C'\cos(ky) + D'\sin(ky)) \\
\psi(x,y) &= (A'\cosh(kx) + B'\sinh(kx))(Ce^{iky} + De^{-iky}) \\
\psi(x,y) &= (Ae^{kx} + Be^{-kx})(C'\cos(ky) + D'\sin(ky))
\end{aligned}$$

(b) $\nabla^2\overline{T} = \nabla^2 T - \nabla^2 T_0 = 0$ because T_0 is a constant and we are told that the temperature distribution obeys Laplace's equation: $\nabla^2 T = 0$.

Find the general solution by separating the variables: $\overline{T}(x, y) = \overline{T}_X(x)\overline{T}_Y(y)$. We have already worked out these general solutions in part (a). The form of the solution will depend on the boundary conditions and, since the top boundary condition is of the form $\overline{T} = \sin(kx)$, one can see by inspection that the general solution will be of the form $\overline{T}(x, y) = (A'\cos(kx)+B'\sin(kx))(C'\cosh(ky)+D'\sinh(ky))$ or equivalently $\overline{T}(x, y) = (A'\cos(kx) + B'\sin(kx))(Ce^{ky} + De^{-ky})$. The first of these shall be used here since we have finite boundary conditions.

The bottom boundary condition is $\overline{T}(x, y = 0) = 0$. For non-zero A' and B', this requires that $C' = 0$.

The top boundary condition is $\overline{T}(x, y = H) = T_1 \sin(2\pi x/D)$. Hence, by inspection, $A' = 0$, $k = 2\pi/D$ and:

$$T_1 \sin\left(\frac{2\pi x}{D}\right) = B'D' \sin\left(\frac{2\pi x}{D}\right) \sinh\left(\frac{2\pi H}{D}\right)$$

Hence

$$B'D' = \frac{T_1}{\sinh\left(\frac{2\pi H}{D}\right)}$$

Therefore the general solution is:

$$T = T_0 + \overline{T}$$
$$T = T_0 + T_1 \frac{\sinh\left(\frac{2\pi y}{D}\right)}{\sinh\left(\frac{2\pi H}{D}\right)} \sin\left(\frac{2\pi x}{D}\right)$$

[This was the least popular question in section A and indeed in the whole paper. Part (a) was well answered by most candidates, although a common mistake was to state that the general solution if $k = 0$ is $\psi(x, y) = 0$, rather than $\psi(x, y) = (Ax + B)(Cy + D)$. In Part (b) a common mistake was to assume that the coefficient in front of the e^{ky} term must be zero, although this would only be true in an infinitely-wide sheet.]

Section B

4. (a) Definitions are in the data book, so this should have been an easy start to the question. Note the question says an $n \times m$ matrix (ie n rows and m columns) whereas the databook quotes results for an $m \times n$ matrix.

Row Space: space of vectors $\{\mathbf{a}_i\}$, with each $\mathbf{a} \in R^m$, spanned by the rows of A. OR, the space formed from the independent rows of A.

Column Space: space of vectors $\{\mathbf{a}_j\}$, with each $\mathbf{a} \in R^n$, spanned by the columns of A. OR, the space formed from the independent columns of A.

[Right] Null Space: space spanned by the set of independent vectors $\{\mathbf{x}_i\} \in R^m$ which are mapped onto $\mathbf{0} \in R^n$ by A, i.e. $A\mathbf{x}_i = 0$. It is the orthogonal complement of the row space.

Left Null Space: space spanned by the set of independent vectors $\{\mathbf{y}_i\} \in R^n$ which satisfy $\mathbf{y}_i^T A = \mathbf{0}$ $(\mathbf{0} \in R^m)$. Alternatively we can think of the LNS as the null space of A^T or the orthogonal complement of the column space.

The **rank** of a matrix A is the dimension of the column space, which is the number of independent rows or columns.

As the row space is the orthogonal complement of the null space, and the total dimension of the space must be m, we have

$$R_{RS} + R_{NS} = m$$

$$R_{CS} + R_{LNS} = n$$

(b) Perform the LU decomposition

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \end{bmatrix}$$

$l_{21} = 2,\quad 3l_{21}{+}u_{22} = 7,\quad \text{therefore } u_{22} = 1 \ :\quad l_{31} = 1,\quad 3l_{21}{+}u_{23} = 8,\quad \text{therefore } u_{23} = 2$

$2l_{21} + u_{24} = 7,\quad \text{therefore } u_{24} = 3 \ :\ 3l_{31} + l_{32}u_{22} = 5,\quad \text{therefore } l_{32} = 2$

$3l_{31}{+}l_{32}u_{23}{+}u_{33} = 7,\quad \text{therefore } u_{33} = 0 \ :\ 2l_{31}{+}l_{32}u_{24}{+}u_{34} = 8,\quad \text{therefore } u_{34} = 0$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The 4 fundamental subspaces are:

(i) Row space: has rank 2 and is given by the indpendent rows of U, i.e.

$$[1,\ 3,\ 3,\ 2],\quad [0,\ 1,\ 2.\ 3]$$

Note, that other vectors are possible as long as they lie in the 'plane' formed by the above two vectors.

(ii) Column space: has rank 2 and is given by the columns of A corresponding to the columns of U without pivots.

$$[1,\ 2,\ 1]^T,\quad [3,\ 7,\ 5]$$

Note that lots of candidates gave the column space as the rows of U without pivots, which is incorrect.

(iii) Null space: If $A\mathrm{x} = 0$, $U\mathrm{x} = 0$, so we find the nullspace of U (which is much easier).

$$U\mathrm{x} = 0 \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Have 4 equations in 2 unknowns:

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 2x_4 &= 0 \\ x_2 + 2x_3 + 3x_4 &= 0 \end{aligned}$$

Can use any of the xs as *free variables*, x_3, x_4 is a good choice to reduce the amount of algebra:

$$x_2 = -2x_3 + -3x_4$$
$$x_1 = -3(-2x_3 - 3x_4) - 3x_3 - 2x_4 = -2x_3 - 3x_4$$

$$\Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Thus a basis for the null space is:

$$[3,\ -2,\ 1,\ 0]^T,\quad [7,\ -3,\ 0,\ 1]^T$$

[of course any two independent vectors which are in the 'plane' spanned by the above vectors, will do as a basis of the null space. Candidates gave a variety of correct answers, of which the above were the most common. There were also a variety of incorrect answers!]

(iv) Left Null Space: either find $A^T\mathbf{y} = 0$ or find L^{-1} and take bottom row, or find the orthogonal complement to the column space (this is the easiest way as we are dealing with 3d vectors and can therefore take the cross product).

Using $A^T\mathbf{y} = 0$ we have

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 5 \\ 3 & 8 & 7 \\ 2 & 7 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Gaussian eliminate: row 2 - (3xrow 1), row 3 - (3xrow 1), row 4 - (2xrow1):

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

row 3 - (2xrow 2), row 4 - (3xrow 2)

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which tells us $y_2 + 2y_3 = 0$ and $y_1 + 2y_2 + y_3 = 0$, or $y_2 = -2y_3$ and $y_1 = -2(-2y_3) - y_3 = 3y_3$; therefore a basis for the left nullspace is

$$[3, \ -2, \ 1]^T$$

Much easier, however, to cross the two basis vectors that form the column space

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Check if the equations are satisfied:

$R_{RS} + R_{NS} = m \quad 2 + 2 = 4 \quad$ correct

$R_{CS} + R_{LNS} = n \quad 2 + 1 = 3 \quad$ correct

(c) $A\mathbf{x} = \mathbf{b}$ has a solution if $\mathbf{b}$ lies in the column space of A. Thus, if there is to be no solution, $\mathbf{b}$ has a component outside the column space, i.e. $\mathbf{b} \cdot \mathbf{n} \neq 0$, or

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \neq 0 \quad \Longrightarrow \quad 3b_1 - 2b_2 + b_3 \neq 0$$

[*This was the most popular question in the paper with almost all candidates attempting it.*]

5. (a) If b does not lie in the column space of A, there will be no exact solution but we can find a least-squares solution via the solution to the equation:

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

Thus, if we can decompose A into an orthogonal matrix and an upper triangular matrix, $A = QR$, we have

$$A^T = R^T Q^T \implies A^T A = R^T Q^T Q R = R^T R$$

since $Q^T Q = I$. We then have

$$R^T R\mathbf{x} = R^T Q^T\mathbf{b} \implies R\mathbf{x} = Q^T\mathbf{b}$$

which can be easily solved by back substitution as R is upper triangular, therefore giving us the LS solution.

(b) To do the QR decomposition of M we first perform a Gram-Schmidt orthogonalisation. Let $\mathbf{u}_i$ be the vector corresponding to the ith column of M – want to find the $\mathbf{q}_i$ from the $\mathbf{u}_i$ such that the $\mathbf{q}_i$ are mutually orthonormal.

Set $\mathbf{q}_1 = \frac{\mathbf{u}_1}{|\mathbf{u}_1|} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

Now form $\mathbf{q}_2 = \frac{\mathbf{u}'_2}{|\mathbf{u}'_2|}$ where

$$\mathbf{u}'_2 = \mathbf{u}_2 - (\mathbf{q}_1 \cdot \mathbf{u}_2)\mathbf{q}_1$$

$$= \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

Therefore $\mathbf{q}_2 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

Similarly

$$\mathbf{u}_3' = \mathbf{u}_3 - (\mathbf{q}_1 \cdot \mathbf{u}_3)\mathbf{q}_1 - (\mathbf{q}_2 \cdot \mathbf{u}_3)\mathbf{q}_2$$

$$= \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} - \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} - \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Thus, $\mathbf{q}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

Q is therefore given by $\frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$

R is now obtained using $R = \begin{bmatrix} \mathbf{q}_1 \cdot \mathbf{u}_1 & \mathbf{q}_1 \cdot \mathbf{u}_2 & \mathbf{q}_1 \cdot \mathbf{u}_3 \\ 0 & \mathbf{q}_2 \cdot \mathbf{u}_2 & \mathbf{q}_2 \cdot \mathbf{u}_3 \\ 0 & 0 & \mathbf{q}_3 \cdot \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Can check that QR does indeed give M.

(c) To find the eigenvalues of Q we solve $Q\mathrm{e} = \lambda\mathrm{e}$ or $\det(Q - \lambda I)\mathrm{e} = 0$

$$\Longrightarrow \begin{vmatrix} 2/3 - \lambda & -1/3 & 2/3 \\ 2/3 & 2/3 - \lambda & -1/3 \\ -1/3 & 2/3 & 2/3 - \lambda \end{vmatrix} = 0$$

$$\therefore (2/3-\lambda)[(2/3-\lambda)^2+2/9]+(1/3)[2/3(2/3-\lambda)-1/9]+(2/3)[4/9+1/3(2/3-\lambda)] = 0$$

$$\Longrightarrow (2-3\lambda)(4-12\lambda+9\lambda^2+2)+(4-6\lambda-1)+2(4+2-3\lambda) = 0$$

$$\Longrightarrow 8-24\lambda+18\lambda^2+4-12\lambda+36\lambda^2-27\lambda^3-6\lambda+3-6\lambda+12-6\lambda = -27\lambda^3+54\lambda^2-54\lambda+27 = 0$$

$$\Longrightarrow \lambda^3 - 2\lambda^2 + 2\lambda - 1 = (\lambda - 1)(\lambda^2 - \lambda + 1) = 0$$

Therefore, $\lambda = 1$ or $\lambda = (1 \pm \sqrt{3}i)/2$. Thus the real eigenvalue of matrix Q is 1. To find the eigenvector corresponding to this eigenvalue we solve $(Q - I)\mathbf{x} = 0$

$$\begin{bmatrix} -1/3 & -1/3 & 2/3 \\ 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now solve this – Gaussian elimination is one way:

2x(row1)+row2, -row1+row3 $\implies$ $$\begin{bmatrix} -1/3 & -1/3 & 2/3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

row2 + row3: $\implies$ $$\begin{bmatrix} -1/3 & -1/3 & 2/3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies \quad -x_1 - x_2 + 2x_3 = 0, \quad -x_2 + x_3 = 0$

Therefore, $x_1 = x_2 = x_3$, so that $\mathrm{e} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Consider the plane $x + y + z = 0$, this passes through the origin and has normal vector $[1, 1, 1]^T$, which we note is one of the eigenvalues, e, of Q . If $\mathbf{v}$ lies in the plane, then $\mathbf{v} \cdot \mathrm{e} = 0$. Consider $Q\mathbf{v}$,

$$Q\mathbf{v} \cdot \mathrm{e} = \mathbf{v}^T Q^T \mathrm{e} = \mathbf{v}^T \mathrm{e} = \mathbf{v} \cdot \mathrm{e} = 0$$

as $Q^T\mathrm{e}$ is also equal to e. Since $Q\mathbf{v} \cdot \mathrm{e} = 0$ we know that $Q\mathbf{v}$ also lies in the plane.

[Note that since $Q\mathrm{e} = \mathrm{e}$, multiply both sides by Q^T to give $\mathrm{e} = Q^T\mathrm{e}$].

[*A less popular linear algebra question. Most candidates managed to do a correct QR decomposition or at least knew how to attempt it. In (c) a good number of people left out a factor of 1/3 in Q and therefore got an eigenvalue of 3 instead of 1 and the wrong eigenvector. Very few people were able to do the very last part.*]

6. (a)

$$H(\omega) = \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} f(\tau) g(t-\tau) \mathrm{e}^{-i\omega t} d\tau dt = \int_{\tau=-\infty}^{\infty} f(\tau) \left[\int_{t=-\infty}^{\infty} g(t-\tau) \mathrm{e}^{-i\omega t} dt \right] d\tau$$

Substitute $u = t - \tau$ so that $du = dt$ to give

$$\int_{\tau=-\infty}^{\infty} f(\tau) \mathrm{e}^{-i\omega\tau} d\tau \int_{t=-\infty}^{\infty} g(u) \mathrm{e}^{-i\omega u} du = F(\omega) G(\omega)$$

Thus $H(\omega) = F(\omega)G(\omega)$, ie convolution in the time domain is equal to multiplication in the Fourier domain.

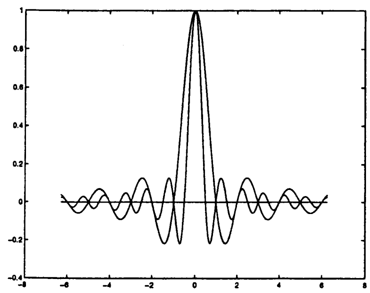
(b)

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}dt = \int_{-a}^{a} \mathrm{e}^{-i\omega t}dt = -\frac{1}{i\omega}\left[\mathrm{e}^{-i\omega t}\right]_{-a}^{a}$$

$$\frac{1}{i\omega}\left[\mathrm{e}^{i\omega a} - \mathrm{e}^{-i\omega a}\right] = 2a \operatorname{sinc} a\omega$$

[could also use result in databook]

Zeros of $\operatorname{sinc} a\omega$ occur when $\sin a\omega = 0$ ie when $a\omega = \pi$. Thus for $a = 1$, first zero is at π and for $a = 2$ first zero is at $\pi/2$ – the two curves therefore look like:



(c)

$$F_1(\omega) = 2 \operatorname{sinc}\omega \text{ is the FT of } f_1(t) = \begin{cases} 1, & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$F_2(\omega) = 2 \operatorname{sinc}2\omega \text{ is the FT of } f_2(t) = \begin{cases} 1/2, & |t| < 2 \\ 0, & \text{otherwise} \end{cases}$$

If we convolve in the frequency domain we multiply in the time domain:

$$F_1 * F_2 \quad \text{is proportional to the FT of} \quad f_1(t)f_2(t) \propto f_1(t)$$

since multiplying the two top hat functions together gives a multiple of the narrower top hat.

(d) If we sample at the Nyquist frequency we know that the FT of the sampled signal is the FT of the original signal repeated every interval of the sampling frequency, ω_s.

Since there is no 'overlap' when we sample at the Nyquist frequency, we can now low pass filter this to obtain the original spectrum.

If

$$G(\omega) = \begin{cases} 1, & |\omega| < \frac{1}{2}\omega_s \\ 0, & \text{otherwise} \end{cases}$$

is the lowpass filter, then clearly $F(\omega) = F_s(\omega)G(\omega)$. Multiplying in one domain implies convolving in the other domain, therefore we know that $f(t) \propto f_s(t) * \operatorname{sinc}(\omega_s\omega/2)$, since we know that the FT of a pulse is a sinc function. Hence the original signal $f(t)$ can be formed by convolving the sampled signal, $f_s(t)$, with a sinc function.

[*The most popular question of Section C, well done by most. Part (d), a qualitative explanation of reconstruction of a sampled signal was done well by most who attempted it.*]

7. (a) The moment generating function $g(s)$ for a continuous random variable, X, with pdf $f(x)$, is defined by

$$g(s) = \int_{\text{all } x} \mathrm{e}^{-sx} f(x) dx$$

(b) Know that if $W = X_1 + X_2$, ie the sum of two random variables, then the mgf of W, $g_W(s)$, is given by the product of the mgfs for X_1 and X_2, ie

$$g_W(s) = g_{X_1}(s) g_{X_2}(s)$$

If X_1 and X_2 are normal, as given, then we see that

$$g_W(s) = \mathrm{e}^{-s\mu_1 + \frac{1}{2}\sigma_1^2 s^2} \mathrm{e}^{-s\mu_2 + \frac{1}{2}\sigma_2^2 s^2}$$

giving

$$g_W(s) = \mathrm{e}^{-s(\mu_1+\mu_2)+\frac{1}{2}(\sigma_1^2+\sigma_2^2)s^2} = N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$$

i.e another normal distribution with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

(c) $X \sim N(2\text{cm},\ 5 \times 10^{-2}\text{cm})$. If we have 4 components end to end, the total length is $X_s = \sum_{i=1}^{4} X_i$ and from (b) we know that

$$X_s \sim N(8\text{cm}, \sqrt{4 \times 25 \times 10^{-4}}\text{cm}) = N(8\text{cm}, 0.1\text{cm})$$

(i) Tubes are 8.1cm long, so we want $P(X_s > 8.1) = 1 - P(X_s < 8.1)$

$$= 1 - \Phi\left(\frac{8.1 - 8}{0.1}\right) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

from tables. Thus the probability of *not* fitting into a given tube is about 15.9%.

(ii) If each X_i is distributed as $N(\mu, \sigma)$, we know that

$$\bar{X} \sim N\left(\sum_{i=1}^{100} \mu_i, \sqrt{\sum_{i=1}^{100} \sigma_i{}^2}\right) = N(100\mu, \sqrt{100\sigma^2}) = N(100\mu, 10\sigma) = N(200\text{cm}, 0.5\text{cm})$$

Since $X \sim N(\mu, \sigma) \implies \alpha X \sim N(\alpha\mu, \alpha\sigma)$

$$\bar{X} \sim N(2\text{cm}, 5 \times 10^{-3}\text{cm})$$

Thus, assuming that all our components are independent and identically distributed with $X_i \sim N(\mu, \sigma)$, $\bar{X}$ is also distributed as a normal distribution. Therefore, to undertake a significance test, we see if our observation lies in the 5% or 1% regions.

We therefore need $P(\bar{X} \geq 2.01)$

$$P(\bar{X} \geq 2.01) = 1 - \Phi\left(\frac{2.01 - 2.0}{5 \times 10^{-3}}\right) = 1 - \Phi(2) = 1 - 0.9772 = 0.0228$$

Thus, the chances of observing a sample mean of 2.01 are *not significant* at the 1% level, but *are significant* at the 5% level.

Note that here we are performing a one-sided hypothesis test.

[*A very easy question if you knew what you were doing. However, many people went wrong on part (b), thinking that finding the mean and variance of the $S = X_1 + X_2$, where X_1 and X_2 are normally distributed random variables, was enough to show that S was also distributed normally. Many also tried to do this via MGFs, which made it very painful.*]

8. (a) DFT $F_k = \sum_{n=0}^{N-1} f_n e^{-2\pi i \frac{kn}{N}} \quad 0 \leq k \leq N-1$

IDFT $f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{2\pi i \frac{kn}{N}} \quad 0 \leq n \leq N-1$

$$F_{N-k} = \sum_{n=0}^{N-1} f_n e^{-2\pi i (N-k)\frac{n}{N}} = \sum_{n=0}^{N-1} f_n e^{2\pi i k \frac{n}{N}} e^{-2\pi i n} = \sum_{n=0}^{N-1} f_n e^{2\pi i k \frac{n}{N}} = F_k^*$$

since $e^{-2\pi i n} = 1$ and if $f_n^* = f_n$, ie the f_n are real.

Now form the DFT of $\{1, 1, 1, -1\}$ (N=4);

$$F_0 = \sum_{n=0}^{3} f_n e^{-2\pi i 0 n/4} = \sum_{n=0}^{3} f_n = 1 + 1 + 1 - 1 = 2$$

$$F_1 = \sum_{n=0}^{3} f_n e^{-2\pi i n/4} = 1 + 1e^{-2\pi i/4} + 1e^{-2\pi 2i/4} - 1e^{-2\pi 3i/4} = 1 - i - 1 - i = -2i$$

$$F_2 = \sum_{n=0}^{3} f_n e^{-2\pi i 2n/4} = 1 + 1e^{-2\pi 2i/4} + 1e^{-2\pi 4i/4} - 1e^{-2\pi 6i/4} = 1 - 1 + 1 + 1 = 2$$

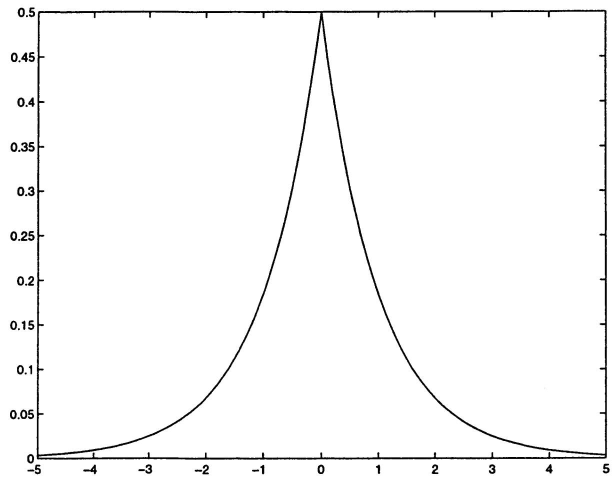
$$F_3 = \sum_{n=0}^{3} f_n e^{-2\pi i 3n/4} = 1 + 1e^{-2\pi 3i/4} + 1e^{-2\pi 6i/4} - 1e^{-2\pi 9i/4} = 1 + i - 1 + i = 2i$$

Therefore, $F_k = \{2,\ -2i,\ 2,\ 2i\}$ and we can see that $F_1 = -2i$ is the same as $F_3^* = (2i)^* = -2i$.

(b) The expected value is given by $E[X] = \int_{allx} x f(x) dx$.

[*Note that many people stated that the expected value was the negative of the derivative of the moment generating function evaluated at zero – while this is correct (and was marked as such) it was not the answer the examiner expected!*]

$f(x)$ as given looks like



Since $f(x)$ is symmetric, or even, $x^n f(x)$ is an odd function if n is odd and an even function if n is even. Moments of X are given by

$$\int_{-\infty}^{\infty} x^n f(x) dx$$

for n odd this must be zero as $\int_{-\infty}^{0} = -\int_{0}^{\infty}$.

$$E[|X|] = \int_{-\infty}^{0} -x\frac{1}{2}\mathrm{e}^{x}dx + \int_{0}^{\infty} x\frac{1}{2}\mathrm{e}^{-x}dx$$

$$= 2\int_{0}^{\infty} x\frac{1}{2}\mathrm{e}^{-x}dx \quad \text{putting } x \longrightarrow -x \text{ in first integral}$$

$$= \left[-x\mathrm{e}^{-x}\right]_0^{\infty} + \int_{0}^{\infty} \mathrm{e}^{-x}dx = -\left[\mathrm{e}^{-x}\right]_0^{\infty} = 1$$

Therefore $E[|X|] = 1$.

$$E[|X|^2] = \int_{-\infty}^{0} x^2\frac{1}{2}\mathrm{e}^{x}dx + \int_{0}^{\infty} x^2\frac{1}{2}\mathrm{e}^{-x}dx$$

$$= 2\int_{0}^{\infty} x^2\frac{1}{2}\mathrm{e}^{-x}dx \quad \text{putting } x \longrightarrow -x \text{ in first integral}$$

$$= \left[-x^2 e^{-x}\right]_0^\infty + 2\int_0^\infty x e^{-x} dx = 2 \quad \text{using result for mean}$$

Therefore $E[|X|^2] = 2$.

Thus $V[|X|] = E[|X|^2] - (E[|X|])^2 = 2 - 1 = 1$

[*Not a popular question, which was strange as it was probably the easiest on the paper. Part (a), on DFTs, was generally well done.Part (b) was less well done. Candidates had surprising difficulty in evaluating quantities for* $|X|$*, indeed, many just ignored the modulus signs and did it for* X*. Poorly done overall.*]