ENGINEERING TRIPOS PART IB

Friday 10 June 2005 9 to 11

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer at least **one** *question from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

There are no attachments to this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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SECTION A

*Answer at least **one** question from this section*

1 (a) Figure 1 shows a semi-circular surface S_1 of radius R. Express the integral $I = \int\int_{S_1} (x^2 + y^2)\, dxdy$ in polar coordinates, and evaluate I in terms of R. [5]

(b) A volume V is bounded by the two planes $y = 0$ and $z = 0$, and by the cone $x^2 + y^2 = (1 - z)^2$, as shown in Fig. 2. By considering V as a stack of semi-circular slices of thickness dz, or otherwise, evaluate the volume integral

$$J = \int\int\int_V (x^2 + y^2)\ dxdydz$$

[7]

(c) The vector $\boldsymbol{F}$ is given by $\boldsymbol{F} = y^2 x\mathbf{i} + x^2 y\mathbf{j} + \mathbf{k}$, where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are the unit Cartesian vectors.

(i) What is the divergence of $\boldsymbol{F}$? [1]

(ii) The flux integral of $\boldsymbol{F}$ through a surface S, with unit outward normal $\mathbf{n}$, is defined by $\int\int_S \boldsymbol{F} \cdot \mathbf{n}\ dS$. Evaluate the fluxes of $\boldsymbol{F}$ through the flat triangular surface of V, and the flat semi-circular surface of V , where V is defined in Part (b). [4]

(iii) Hence find the flux of $\boldsymbol{F}$ through the conical face of volume V. [3]

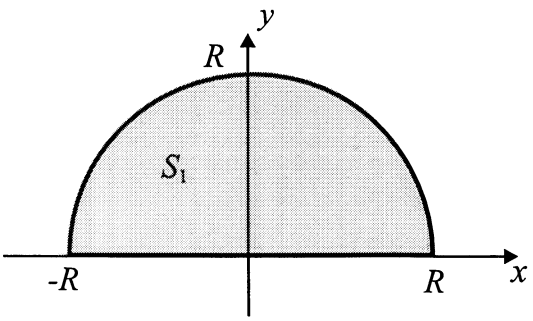

Fig. 1

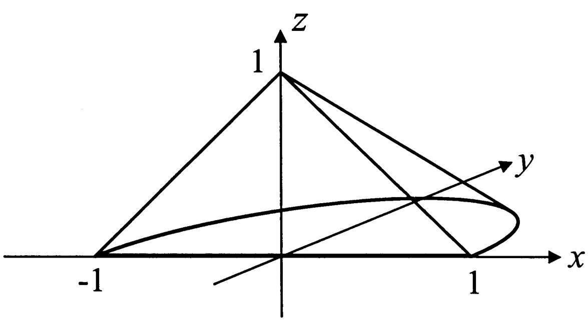

Fig. 2

2 (a) The 3D vector field $\boldsymbol{F}$ is given by $\boldsymbol{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are the unit Cartesian vectors.

(i) Evaluate $\nabla \cdot \boldsymbol{F}$ and $\nabla \times \boldsymbol{F}$. [2]

(ii) Find a scalar function ϕ which satisfies $\boldsymbol{F} = \nabla \phi$. [3]

(iii) What is the value of $\int_L \boldsymbol{F} \cdot \mathbf{dl}$ along the line L shown in Fig. 3? [4]

(b) The vector field $\boldsymbol{G}$ is given by $\boldsymbol{G} = y\mathbf{i} - x\mathbf{j}$.

(i) Calculate $\oint_C \boldsymbol{G} \cdot \mathbf{dl}$, where C is the line which encloses the bottom face of the cube shown in Fig. 4. [5]

(ii) Evaluate $\nabla \cdot (\nabla \times \boldsymbol{G})$. What is the flux of $\nabla \times \boldsymbol{G}$ through the surface S, which comprises the other five faces of the cube? [6]

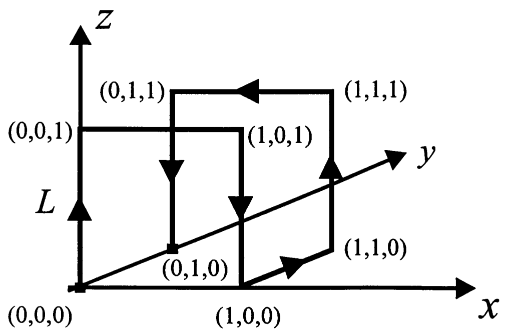

Fig. 3

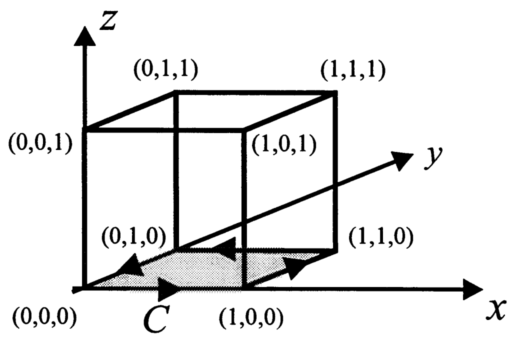

Fig. 4

3 (a) The function $\psi(x,y)$ satisfies Laplace's equation $\nabla^2\psi = 0$. If $\psi(x,y)$ is expressed as the product of two functions $X(x)$ and $Y(y)$, show that X and Y must satisfy :

$$\left\{\frac{X''}{X} = -k^2 \text{ and } \frac{Y''}{Y} = k^2\right\} \quad \text{or else} \quad \left\{\frac{X''}{X} = k^2 \text{ and } \frac{Y''}{Y} = -k^2\right\}$$

where X'' and Y'' are understood to mean $\dfrac{d^2X}{dx^2}$ and $\dfrac{d^2Y}{dy^2}$ respectively. [2]

(i) Find the general solution for $\psi(x,y)$ if k is zero. [2]

(ii) Find the general solution for $\psi(x,y)$ if X''/X is non-zero and negative. [2]

(iii) Find the general solution for $\psi(x,y)$ if X''/X is non-zero and positive. [2]

(b) As part of an annealing process, a sheet of glass of thickness H is placed on a hot sheet of metal and heated unevenly from above, as shown in Fig. 5. At steady state, the bottom face of the glass is at a constant temperature T_0. The top face has the temperature distribution $T = T_0 + T_1 \sin(2\pi x/D)$, where D is the distance between the heaters and T_1 is a constant. The temperature distribution does not vary normal to the $x-y$ plane and obeys Laplace's equation. Show that the rescaled temperature $\bar{T} = T - T_0$ also satisfies Laplace's equation and hence evaluate T throughout the sheet. [12]

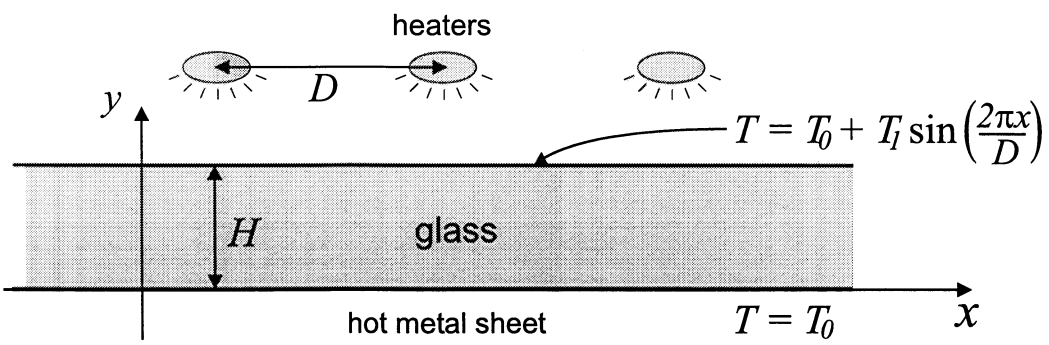

Fig. 5

SECTION B

*Answer at least **one** question from this section*

4 (a) For an $n \times m$ matrix A, define the concept of *rank* and explain the meaning of each of the four fundamental subspaces of A.

If the dimensions of the row, column, null and left null spaces are R_{RS}, R_{CS}, R_{NS} and R_{LNS} respectively, write down two equations relating these dimensions to n and m and explain briefly how these equations are obtained. [5]

(b) The matrix A is given as

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 7 & 8 & 7 \\ 1 & 5 & 7 & 8 \end{bmatrix}$$

By performing an LU decomposition on A, or otherwise, find a basis for each of the four fundamental subspaces.

From these results, verify that the equations obtained in Part (a) are satisfied. [12]

(c) In the equation $A\mathbf{x} = \mathbf{b}$, find a condition on $\mathbf{b}$ for there to be no exact solution for $\mathbf{x}$. [3]

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5 (a) Explain how QR decomposition is used for solving least-squares problems, where Q is an orthogonal matrix and R is an upper triangular matrix. [3]

(b) Find the QR decomposition of the matrix M, given by [7]

$$M = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 4 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

(c) Find the eigenvalues of the orthogonal matrix Q obtained in Part (b), verifying that its real eigenvalue is 1. Find the unit eigenvector $\mathbf{e}_1$ corresponding to this real eigenvalue.

For any vector $\mathbf{v}$ lying in the plane $x + y + z = 0$, show that $Q\mathbf{v}$ also lies in this plane. [10]

SECTION C

*Answer at least **one** question from this section*

6 (a) For two functions $f(t)$ and $g(t)$, the convolution $h(t)$ is given by

$$h(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau$$

If $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$ respectively, derive an expression for $H(\omega)$, the Fourier transform of $h(t)$, in terms of $F(\omega)$ and $G(\omega)$. [5]

(b) Consider the function $f(t)$ where

$$f(t) = \begin{cases} 1, & |t| < a \\ 0, & \text{otherwise} \end{cases}$$

Show that the Fourier transform of $f(t)$ is given by $F(\omega) = 2a \text{ sinc } a\omega$. Sketch $\text{sinc } a\omega$ for $a = 1$ and $a = 2$, drawing both curves on the same axes. [5]

(c) Using the concept of Fourier transform pairs, or otherwise, show that the convolution of $F_1(\omega) = 2 \text{ sinc } \omega$ and $F_2(\omega) = 2 \text{ sinc } 2\omega$ is proportional to $F_1(\omega)$. [6]

(d) If a signal is sampled at the Nyquist frequency, describe briefly how the original signal can be perfectly reconstructed from its samples and indicate how this is equivalent to convolving the sampled signal with a sinc function. [4]

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7 (a) Define the *Moment Generating Function* $g(s)$ for a continuous random variable X that has a probability density function $f(x)$. [3]

(b) If X has a normal distribution with mean μ and standard deviation σ $(X \sim N(\mu, \sigma))$, the moment generating function of X is given by

$$g(s) = \exp\left(-s\mu + \frac{1}{2}\sigma^2 s^2\right)$$

Let X_1 and X_2 be two independent random variables distributed normally as $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$ respectively. Using moment generating functions, or otherwise, show that the sum $S = X_1 + X_2$ is distributed as

$$N\left(\mu_1 + \mu_2, \sqrt{(\sigma_1^2 + \sigma_2^2)}\right)$$

[6]

(c) Components manufactured by an engineering company have a length which is a random variable and is normally distributed with mean 2 cm and standard deviation 0.05 cm. These are to be packaged, 4 at a time, into tubes which are exactly 8.1 cm long. Assume that each component is independent and identically distributed.

(i) Find the probability that any 4 components will not fit into a given tube. [5]

(ii) Suppose that we have some reason to believe that the mean value of each component may be greater than 2 cm, with the standard deviation remaining at 0.05 cm. To test this hypothesis we take a random sample of 100 components and measure their lengths, X_i. From these we form a mean length, $\bar{X}$, given by

$$\bar{X} = \frac{1}{100}\sum_{i=1}^{100} X_i$$

We observe that our sample mean is 2.01 cm. By forming the probability density function of $\bar{X}$, determine if this result is significant at the 0.05 level and at the 0.01 level. Explain carefully any assumptions you make. [6]

8 (a) Define the discrete Fourier transform (DFT) of a set of N samples $\{f_0, f_1, \ldots, f_{N-1}\}$ in terms of N discrete frequency components $\{F_0, F_1, \ldots, F_{N-1}\}$. [2]

Show that, for real f_n, where n ranges from 0 to $N-1$, and k ranges from 1 to $N-1$,

$$F_k = F^*_{N-k}$$

where * denotes complex conjugate. [3]

By forming the discrete Fourier transform of the sequence $\{1, 1, 1, -1\}$, verify that $F_1 = F^*_3$. [5]

(b) Define the *expected value*, $E[X]$, of a continuous random variable X with probability density function $f(x)$. [2]

For $f(x)$ given by

$$f(x) = \begin{cases} \frac{1}{2}\mathrm{e}^{x}, & x \leq 0 \\ \frac{1}{2}\mathrm{e}^{-x}, & x \geq 0 \end{cases}$$

sketch $f(x)$ and explain why $E[X]$ and all odd moments of X are zero. Find both the expected value and the variance of $|X|$. [8]

END OF PAPER