ENGINEERING TRIPOS PART IB
PAPER 1
2006
AA SESHIA

PART IB

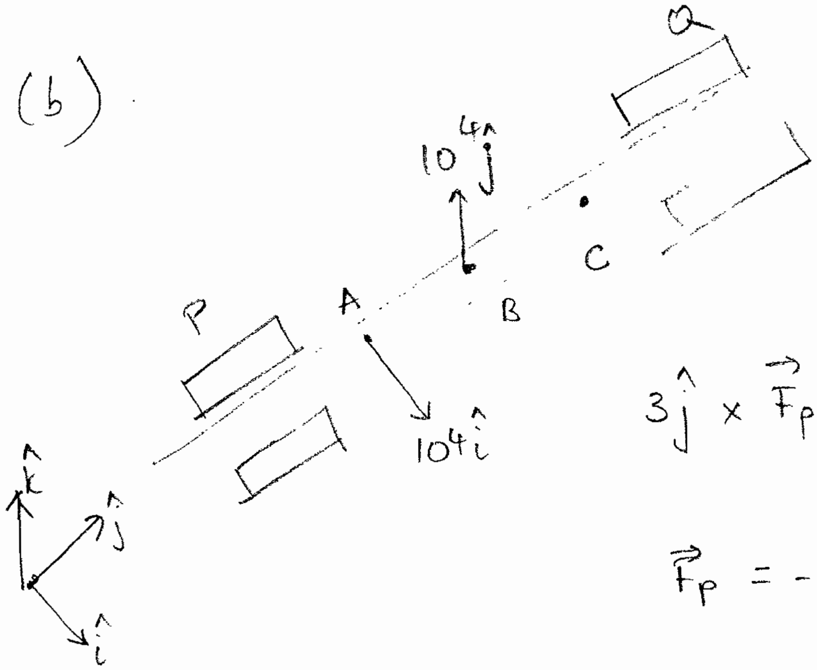
CRIB FOR PAPER 1

MECHANICS

JUNE 2006.

Q1 (a) A shaft is statically balanced if the overall center of gravity of the rotating masses lies on the shaft axis. In static balance the shaft bearing reactions are not neccessarily zero for constant rotation.
For dynamic balance for constant rotation the bearing forces are equal to zero eliminating undesired bearing reactions

(b)



set up co-ordinate system

$$m r \omega^2 = 0.01 \times 10^6 = 10^4 \text{ N}$$

Let bearing force at P and Q be $\vec{F}_P$ and $\vec{F}_Q$

$$3\hat{j} \times \vec{F}_P = -2.5\hat{j} \times 10^4 \hat{i} - 1.5\hat{j} \times 10^4 \hat{k}$$

$$\vec{F}_P = \frac{-2.5 \times 10^4 \hat{i}}{3} - 0.5 \times 10^4 \hat{k}$$

$$\vec{F}_P + \vec{F}_Q + 10^4 \hat{i} + 10^4 \hat{k} = 0 \qquad (\Sigma \vec{F} = 0)$$

$$\therefore \vec{F}_Q = \frac{-0.5 \times 10^4}{3} \hat{i} - 0.5 \times 10^4 \hat{k}$$

(c) Let m_A and m_C be masses added to A and C for static and dynamic balance

Static Balance $\sum_i m_i r_i = 0$

$$\therefore m_S \vec{r}_S = \frac{-10^4 \hat{i} - 10^4 \hat{k}}{10^6}$$

$\therefore\ m_A \vec{r}_A + m_C \vec{r}_C = m_S \vec{r}_S$ (1)

Take moments about A (DYNAMIC BALANCE)

$-\hat{j} \times 10^4 \hat{k} - 2\hat{j} \times m_C \vec{r}_C = 0$

$$-2\hat{j} \times m_C \vec{r}_C = 10^4 \hat{i} / 10^6$$

$$m_C \vec{r}_C = \frac{5 \times 10^3}{10^6} (-\hat{k})$$

$$\therefore\ m_C = \frac{5 \times 10^3}{0.5 \times 10^6} = 10\,g \text{ at } -\hat{k} \text{ on disc C}$$

From (1) and dynamic balance —

$$m_A \vec{r}_A = \frac{-10^4 \hat{i} - 10^4 \hat{k}}{10^6} \quad \frac{+0.5 \times 10^4 \hat{k}}{10^6}$$

$$m_A \vec{r}_A = \frac{-10^4 \hat{i} - 0.5 \times 10^4 \hat{k}}{10^6}$$

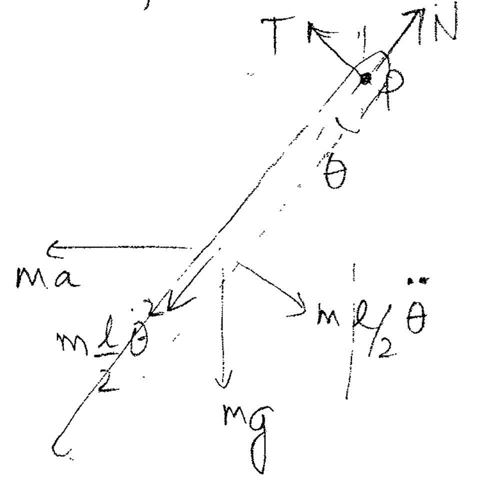
$$\therefore\ m_A \vec{r}_A = \left(\frac{10^4}{10^6}\right) \times 0.5 \times \sqrt{5} \left(\frac{-2\hat{i} - \hat{k}}{\sqrt{5}}\right)$$

$$\vec{r}_A = 0.5 \left(\frac{-2\hat{i} - \hat{k}}{\sqrt{5}}\right)$$

$$m_A = \sqrt{5} \times 10^{-2}\,kg = 22.4\,g \text{ at } \vec{r}_A$$

Q2

(a) Shown forces on od



Taking moments about P —

$$I_0 \ddot{\theta} = ma\cos\theta - mg\sin\theta$$

$$\left(\frac{ml^2}{12} + \frac{ml^2}{4}\right)\ddot{\theta} = ma\cos\theta - mg\sin\theta$$

$$\therefore \ddot{\theta} = \frac{3}{2l}(a\cos\theta - g\sin\theta)$$

$$\int_0^{\dot{\theta}} \dot{\theta}\, d\dot{\theta} = \int_0^{\theta} \ddot{\theta}\, d\theta$$

$$\frac{\dot{\theta}^2}{2} = \frac{3}{2l}\int_0^{\theta}(a\cos\theta - g\sin\theta)\, d\theta$$

$$= \frac{3}{2l}\Big[g\cos\theta + a\sin\theta\Big]_0^{\theta}$$

$$\therefore \dot{\theta}^2 = -\frac{3}{l}\left[g(1-\cos\theta) - a\sin\theta\right]$$

(b) Normal reactions N and T as shown above

$\nearrow \Sigma F = 0 \qquad N = mg\cos\theta + \frac{ml}{2}\dot{\theta}^2 + ma\sin\theta$

$$N = mg\cos\theta + \frac{ml}{2}\cdot\frac{3}{l}\left[g(\cos\theta - 1) + a\sin\theta\right] + ma\sin\theta$$

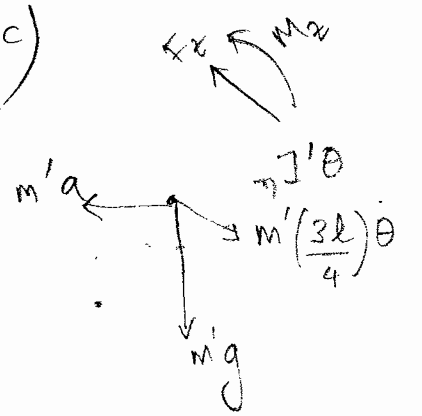
$$N = \frac{5}{2}m(g\cos\theta + a\sin\theta) - \frac{3}{2}mg$$

$+\theta$; $\Sigma F = 0$

$$T = -ma\cos\theta + mg\sin\theta + \frac{ml}{2}\,\frac{l/2}{l^2/3}\left(a\cos\theta - g\sin\theta\right)$$

$$T = \frac{m}{4}\left(g\sin\theta - a\cos\theta\right)$$

(c)



M_z (Bending Moment) at $l/2$ mid-point of rod

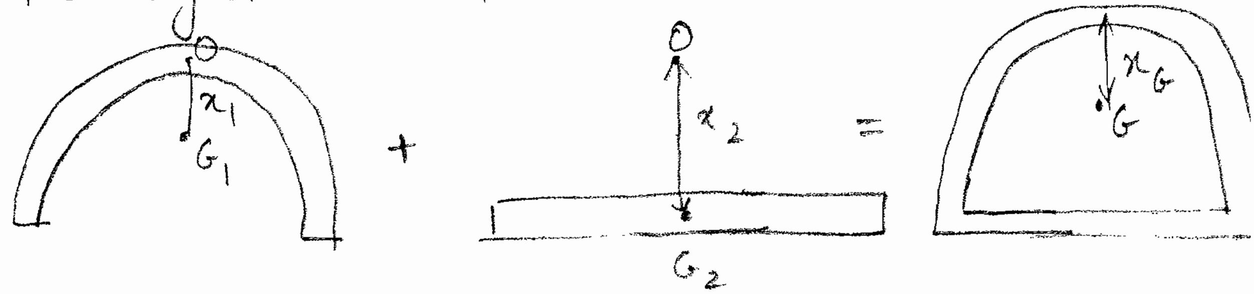
$$= -\frac{l}{4}\left(m'g\sin\theta - m'a\cos\theta + m'\left(\frac{3l}{4}\right)\ddot{\theta}\right) - I'\ddot{\theta}$$

$$= l/4\left(-m/2\, g\sin\theta + m/2\, a\cos\theta - \frac{m}{2}\left(\frac{3l}{4}\right)\left(\frac{3}{2l}\right)\left(a\cos\theta - g\sin\theta\right)\right) - \frac{1}{2}\,\frac{m\,(l/2)^2}{12}\left(3/2l\right)\left(a\cos\theta - g\sin\theta\right)$$

$$= \frac{ml}{4}\left(-\frac{g}{2}\sin\theta + \frac{a}{2}\cos\theta + \frac{5}{8}\left(a\cos\theta - g\sin\theta\right)\right)$$

$$= +\frac{ml}{32}\left(g\sin\theta - a\cos\theta\right)$$

Q3 (a) Shape formed of a curved rod + straight rod



mass of curved rod - m_c

" " straight rod - m_s

$$m_c + m_s = m$$

$$m_c = \left(\frac{\pi}{\pi+2}\right) m \quad \text{and} \quad m_s = \frac{2\,m}{\pi+2}$$

x_1 (from mechanics databook) $= r - \dfrac{r \sin \pi/2}{\pi/2} = \left(\dfrac{\pi-2}{\pi}\right) r$

$$x_2 = r$$

$$x_G = \left(\frac{\pi}{\pi+2}\right)\left(\frac{\pi-2}{\pi}\right) r + \left(\frac{2}{\pi+2}\right) r$$

$$x_G = \frac{\pi r}{\pi+2}$$ location of center of mass below peg

$$I_O = I_{G_1} + I_{G_2} + m_c r^2_{O\text{-}G_1} + m_s r^2_{O\text{-}G_2}$$

$$= m_c\left[r^2 - \left(\frac{1}{\pi/2}\right)^2 r^2\right] + \frac{m_s}{12}(2r)^2 + m_c x_1^2 + m_s x_2^2$$

$$= m_c\left[r^2 - \frac{4r^2}{\pi^2} + \left(\frac{\pi-2}{\pi}\right)^2 r^2\right] + \frac{m_s}{3} r^2 + m_s r^2$$

$$= 2\left(\frac{\pi-2}{\pi+2}\right) m r^2 + \frac{4}{3}\left(\frac{2}{2+\pi}\right) m r^2 = \frac{(6\pi-4)}{3(\pi+2)} m r^2$$

(b) $T = \frac{1}{2} I_0 \dot{\theta}^2$

$$= \frac{1}{2} \frac{(6\pi-4)}{3(\pi+2)} m r^2 \dot{\theta}^2$$

$$V = mg x_G \left[1 - (\cos\theta) \right]$$

expand as $\left(1-\frac{\theta^2}{2}\right)$ for small θ

$$V \approx mg \left(\frac{\pi r}{\pi+2}\right) \frac{\theta^2}{2}$$

(c) $T + V = E$ (conservative system)

$$\frac{1}{2} \frac{(6\pi-4)}{3(\pi+2)} m r^2 \dot{\theta}^2 + \frac{\pi m g r}{\pi+2} \frac{\theta^2}{2} = E$$

$$\frac{(6\pi-4)}{3(\pi+2)} m r^2 \ddot{\theta} + \frac{\pi m g r}{\pi+2} \theta = 0$$ (Equation of Motion)

$$\omega^2 = \frac{\frac{\pi m g r}{\pi+2}}{\frac{6\pi-4}{3(\pi+2)} m r^2} = \frac{3\pi g}{(6\pi-4) r}$$

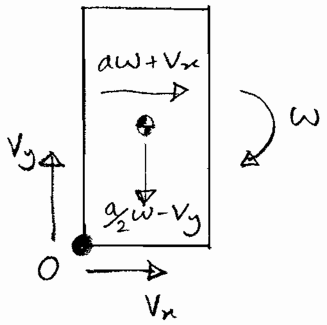
natural frequency of small oscillation

4)

(a) Using the perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy}$$
$$= \frac{1}{12} m b^2 + \frac{1}{12} m a^2 = \frac{1}{12} m (a^2 + b^2)$$

(b)



After impact O has velocity V_x, V_y

$$I_Q = \frac{1}{12} m \left((2a)^2 + a^2 \right) = \frac{5}{12} m a^2$$

Conservation of linear momentum:

$\rightarrow$ $5m(a\omega + V_x) + mV_x = 0$ $\therefore$ $5a\omega + 6V_x = 0$ ①

$\downarrow$ $5m\left(\frac{a}{2}\omega - V_y\right) - mV_y = 5m.u - m.5u = 0$

$\therefore$ $\frac{5}{2}a\omega - 6V_y = 0$ ②

Conservation of moment of momentum:

About O: ↻

$$5m.(a\omega + V_x).a + 5m\left(\frac{a}{2}\omega - V_y\right)\frac{a}{2} + \frac{5}{12} m a^2 \omega = 5m.u.\frac{a}{2}$$

$$\therefore \left(5 + \frac{5}{4} + \frac{5}{12}\right) a\omega + 5V_x - \frac{5}{2}V_y = \frac{5}{2}u$$

$$\frac{4}{3}a\omega + V_x - \frac{1}{2}V_y = \frac{1}{2}u \quad ③$$

① & ② into ③ $\frac{4}{3}a\omega - \frac{5}{6}a\omega - \frac{1}{2}\frac{5}{12}a\omega = \frac{1}{2}u$

$\left(\frac{8}{3} - \frac{5}{3} - \frac{5}{12}\right)a\omega = u$ $\therefore$ $\omega = \frac{12u}{7a}$ clockwise

and $V_x = -\frac{5a}{6} \times \frac{12u}{7a} = -\frac{10}{7}u$

$V_y = \frac{5}{7}u$

4) (cont.) Velocity of C.O.G. of Q : $a\omega + V_x = \frac{12}{7}u - \frac{10}{7}u = \frac{2}{7}u$

$\frac{a}{2}\omega - V_y = \frac{6}{7}u - \frac{5}{7}u = \frac{1}{7}u$

(c) Energy dissipated in impact = Initial K.E. − Final K.E.

$$\left\{\frac{1}{2}\cdot 5m\cdot u^2 + \frac{1}{2}\cdot m\cdot(5u)^2\right\} - \left\{\frac{1}{2}\cdot 5m\left[\left(\frac{-10}{7}u\right)^2 + \left(\frac{5}{7}u\right)^2\right]\right.$$

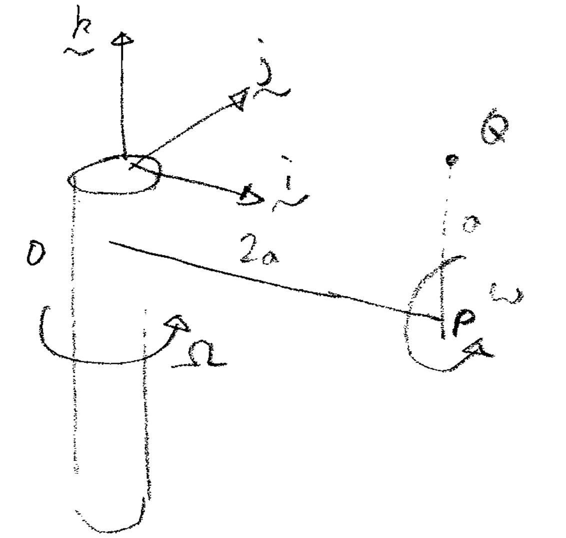
$$\left. + \frac{1}{2}\cdot m\left[\left(\frac{2}{7}u\right)^2 + \left(\frac{1}{7}u\right)^2\right] + \frac{1}{2}\cdot\frac{5}{12}ma^2\cdot\left(\frac{12u}{7a}\right)^2\right\}$$

$$= \frac{mu^2}{2}\left(\{5+25\} - \left\{\frac{500}{49} + \frac{125}{49} + \frac{4}{49} + \frac{1}{49} + \frac{5.12}{49}\right\}\right)$$

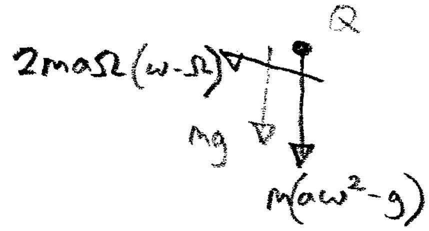
$$= \frac{mu^2}{2}\left(30 - \frac{690}{49}\right)$$

$$= \frac{390}{49}mu^2$$

5



Forces on Q:



$$\underset{\sim}{r} = 2a\underset{\sim}{i} + a\underset{\sim}{k}$$

$$\left[\frac{d\underset{\sim}{r}}{dt}\right]_R = -a\omega\underset{\sim}{j} \qquad \left[\frac{d^2\underset{\sim}{r}}{dt^2}\right]_R = -a\omega^2\underset{\sim}{k}$$

(a) $$\underset{\sim}{v}_Q = \left[\frac{d\underset{\sim}{r}}{dt}\right]_R + \underset{\sim}{\Omega}\times\underset{\sim}{r} = -a\omega\underset{\sim}{j} + \Omega\underset{\sim}{k}\times(2a\underset{\sim}{i} + a\underset{\sim}{k})$$

$$= -a\omega\underset{\sim}{j} + 2a\Omega\underset{\sim}{j} = (2\Omega-\omega)a\underset{\sim}{j}$$

(b) $$\underset{\sim}{a}_Q = \left[\frac{d^2\underset{\sim}{r}}{dt^2}\right]_R + \overbrace{\frac{d\underset{\sim}{\Omega}}{dt}}^{=0}\times\underset{\sim}{r} + 2\underset{\sim}{\Omega}\times\left[\frac{d\underset{\sim}{r}}{dt}\right]_R + \underset{\sim}{\Omega}\times(\underset{\sim}{\Omega}\times\underset{\sim}{r})$$

$$= -a\omega^2\underset{\sim}{k} + 2\Omega\underset{\sim}{k}\times -a\omega\underset{\sim}{j} + \Omega\underset{\sim}{k}\times 2a\Omega\underset{\sim}{j}$$

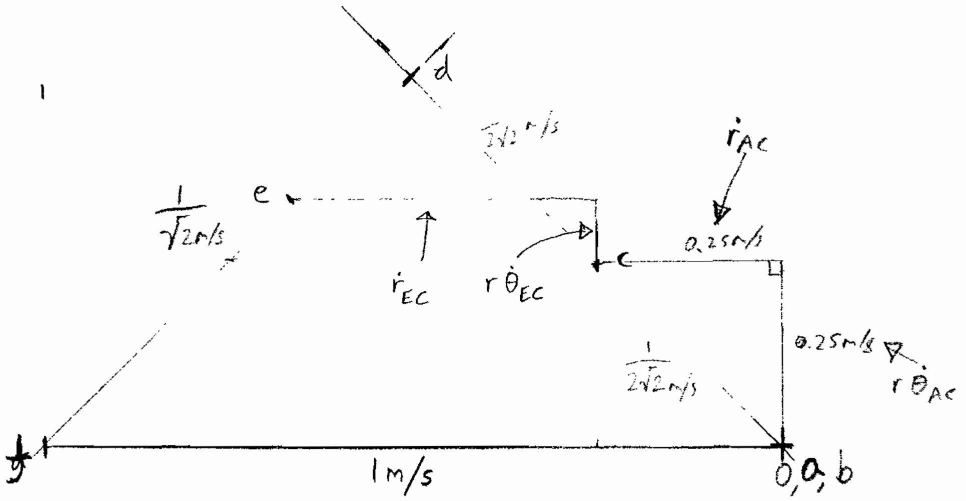
$$= -a\omega^2\underset{\sim}{k} + 2\Omega\omega a\underset{\sim}{i} + -2a\Omega^2\underset{\sim}{i}$$

$$= 2a\Omega(\omega-\Omega)\underset{\sim}{i} - a\omega^2\underset{\sim}{k}$$

(c) Forces in bar PQ: Axial tension $= m(a\omega^2 - g)$

Shear force (in the $\underset{\sim}{i}$ direction) $= 2ma\Omega(\omega-\Omega)$

6) Velocity diagram (Scale 100mm = 1m/s)



$\omega_{BD} = \frac{1}{\sqrt{2}}/2 = \frac{1}{2\sqrt{2}}$ rad/s anti-clockwise $\therefore r\omega^2 = 2\left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{4}$ m/s²

$\omega_{DF} = \frac{1}{\sqrt{2}}/3 = \frac{1}{3\sqrt{2}}$ rad/s clockwise $\therefore r\omega^2 = 3\left(\frac{1}{3\sqrt{2}}\right)^2 = \frac{1}{6}$ m/s²

$\dot{r}_{AC} = -0.25$ m/s $\quad \dot{\theta}_{AC} = \frac{0.25}{AC} = \frac{0.25}{\sqrt{2}} = \frac{1}{4\sqrt{2}}$ rad/s anti-clockwise

$\dot{r}_{EC} = -0.25 - \frac{1}{3}0.5 = -\frac{1}{4} - \frac{1}{6} = -\frac{5}{12}$ m/s

$\dot{\theta}_{EC} = \frac{\frac{1}{3}0.5}{EC} = \frac{\frac{1}{6}}{\sqrt{2}} = \frac{1}{6\sqrt{2}}$ rad/s anti-clockwise

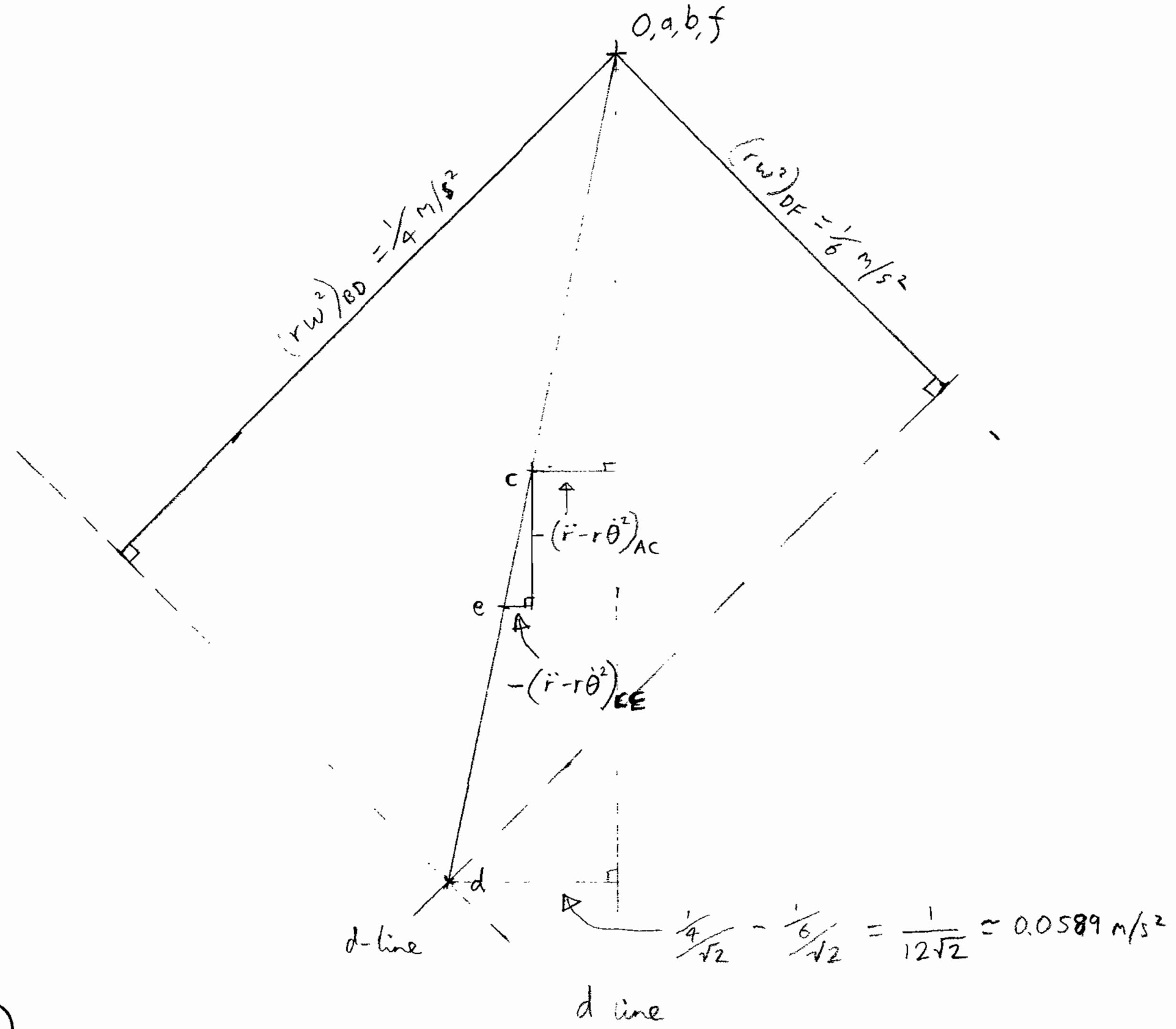
(a) The extension speed of each ram:

AC is shortening at 0.25 m/s

EC " " " $\frac{5}{12} \simeq 0.417$ m/s

6) (cont.)

Acceleration diagram (100mm = 0.25 m/s²)



(b)

For AC $\quad -(\ddot{r} - r\dot{\theta}^2)_{AC} = \frac{1}{2}\frac{1}{12\sqrt{2}}$

$$\therefore \quad \ddot{r}_{AC} = -\frac{1}{2}\frac{1}{12\sqrt{2}} + \sqrt{2} \times \left(\frac{1}{4\sqrt{2}}\right)^2 = \frac{-1}{24\sqrt{2}} + \frac{\sqrt{2}}{32}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{16} - \frac{1}{24}\right) = \frac{1}{8\sqrt{2}}\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{48\sqrt{2}}\ m/s^2$$

For CE $\quad -(\ddot{r} - r\dot{\theta}^2)_{CE} = \frac{1}{3} \times \frac{1}{24\sqrt{2}}$

$$\therefore \quad \ddot{r}_{CE} = \frac{-1}{72\sqrt{2}} + \sqrt{2} \times \left(\frac{1}{6\sqrt{2}}\right)^2 = \frac{-1}{72\sqrt{2}} + \frac{1}{36\sqrt{2}}$$

$$= \frac{1}{72\sqrt{2}}\ m/s^2$$