ENGINEERING TRIPOS, Part IB 2006
Paper 6 - INFORMATION ENGINEERING
Solutions

1 (a) The phase margin is given by the difference between the angle corresponding to the frequency where the modulus of $K(j\omega)G(j\omega)$ crosses the 0dB axis, and -180 degrees (if this difference is positive). The gain margin (in dB) is the difference (when positive) between 0dB and the modulus of the loop gain at the frequency where the argument becomes -180 degrees.

(b) The modulus of G_1 changes slope from 20dB/dec to 40 dB/dec around $\omega = 1$, then it displays a resonance around $\omega = 10$, therefore G_1 is the function represented with a continuous line.

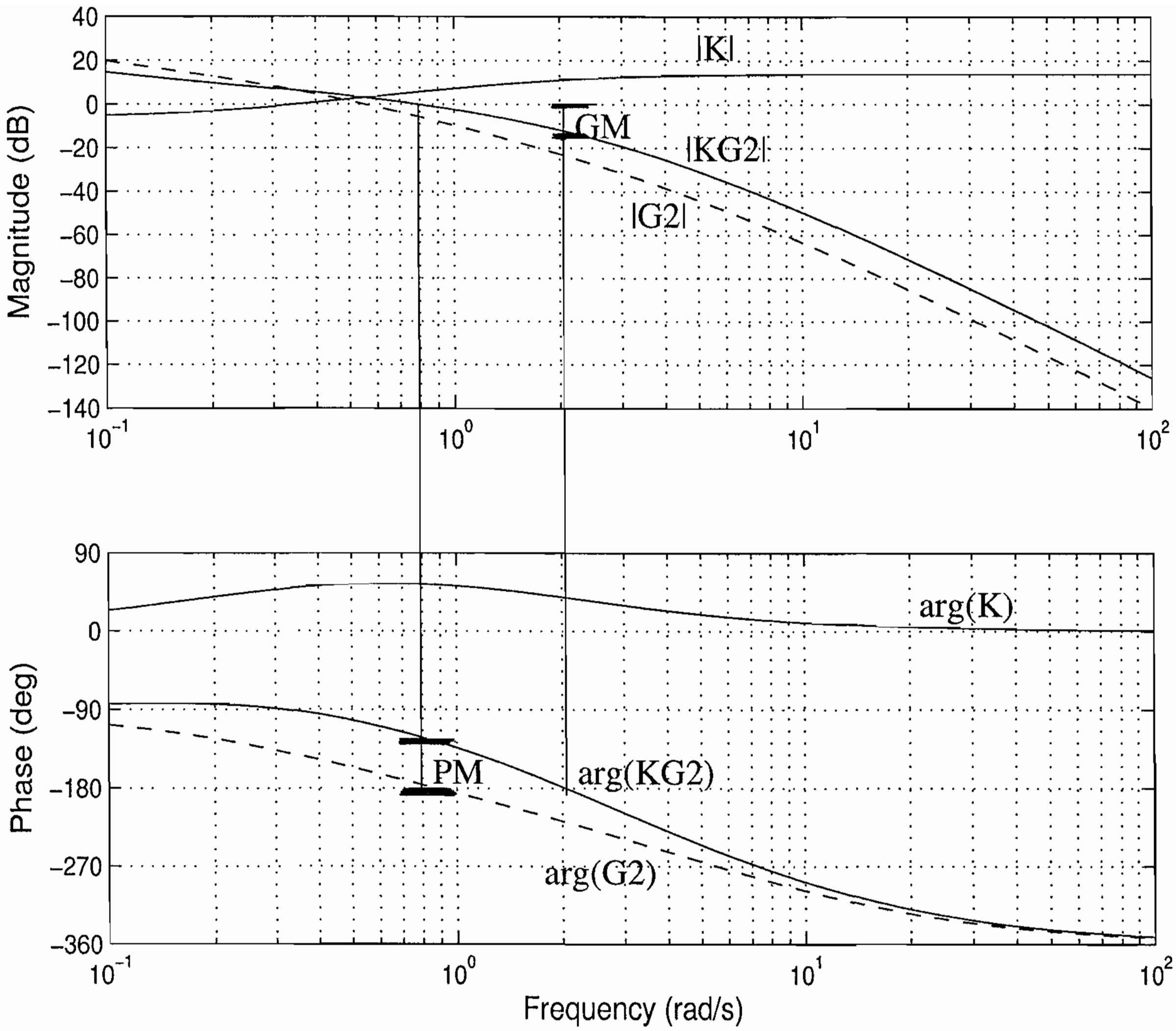
The second order function in the denominator of G_2 has real roots, therefore no resonance peak is present in its diagram. It follows that G_2 is the function represented with a dashed line.

For G_1, gain margin is 19.5 and phase margin is 50 degrees.

For G_2, gain margin is 2.3 and phase margin is 19.4 degrees.

(c) The new Bode diagram is shown below. The new GM and PM are calculated as shown in the figure. Adding controller $K(s)$ has the effect of increasing the gain margin to 4 and the phase margin to 59 degrees (however at the expense of a reduced dc gain).

Examiner's comment: This question was generally well answered, although, surprisingly, there were quite a few confusions about the correct method to calculate the phase and gain margins of a closed-loop system using Bode diagrams. Most candidates distinguished aptly between the two systems and there were many qualitatively correct constructions in part (c). A common mistake was to not realise that the given controller was likely to improve the performances of the closed-loop system.



|K|
GM
|KG2|
|G2|
Magnitude (dB)
arg(K)
PM
arg(KG2)
arg(G2)
Phase (deg)
Frequency (rad/s)

2 (a) If we input into the open-loop system harmonic functions of different frequencies, then the response will oscillate on the same frequency. Once the transients have died down, the measured ratio of the output to input amplitudes will give us the magnitude of the transfer function, while the measured phase difference between input and output is the argument of the transfer function for the corresponding frequency.

(b) When $\omega \to 0$:

$$G(j\omega) = \frac{1}{j\omega \times 1} \tag{1}$$

hence the modulus approaches ∞ at an angle of $-\pi/2$.

If ω is small, we can use a binomial series expansion for $G(j\omega)$, as follows:

$$G(j\omega) = \frac{1}{j\omega(j\omega+1)^2} \simeq \frac{1-2j\omega+\ldots}{j\omega} \longrightarrow -j\infty - 2 \tag{2}$$

therefore the real part of the transfer function approaches -2.

When $\omega \to \infty$:

$$G(j\omega) = \frac{1}{j\omega(j\omega)^2} \tag{3}$$

hence the modulus approaches 0 at an angle of $\pi/2$.

For the imaginary part of $G(j\omega)$ to be 0, the imaginary part of the denominator has to be 0 (since the numerator is a real number).

$$\Im(j\omega(1+j\omega)^2) = \Im(j\omega(1-\omega^2+2j\omega)) = 1-\omega^2 \tag{4}$$

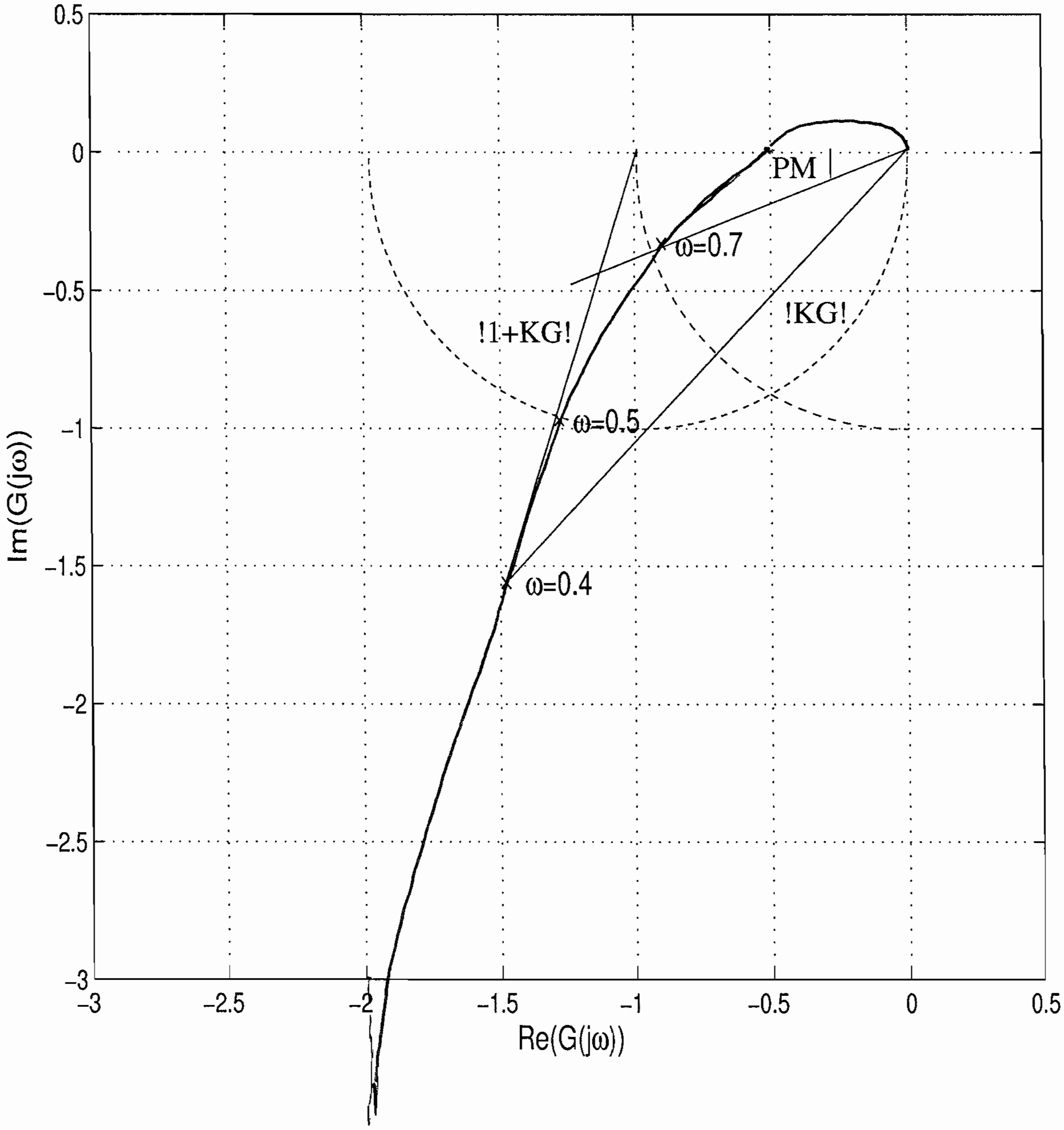
which becomes zero when $\omega = 1$ (the case $\omega = 0$ has already been analysed, and $\omega = -1$ is not an acceptable solution, since the frequency has to be positive). At $\omega = 1$, the modulus of $G(j\omega)$ is 0.5 (by simple substitution) therefore the gain margin is 2.

For the phase margin, a circle of radius 1 centred in the origin (which becomes an ellipse if the scales are different on the x and y axes) meets the diagram close to the point $\omega = 0.7$, and the phase margin can be estimated as 20 degrees.

For the system to remain stable, the intercept of $KG(j\omega)$ with the imaginary axis has to be to the right of -1, therefore $0.5K < 1$, so $0 < K < 2$.

(c) $|KG|$ is the distance from the point $\omega = 0.4$ to the origin, while $|1+KG|$ is the distance to the (-1,0) point. Reading from the graph, the ratio can be estimated as 1.45. Since the complementary sensitivity is larger than 1, resonance may happen at a frequency close to 0.4 rad/s.

(d) The condition for effective feedback is $\frac{1}{|1+KG|} < 1$. A circle of radius one centred in (-1,0) meets the Nyquist diagram close to the point of frequency $\omega = 0.5$. It can be inferred that the feedback will reduce errors in $\bar{w}(s)$ for frequencies lower than 0.5 rad/s.



Examiner's comment: Most solutions to this question provided reasonably correct diagrams, although not always well explained. Some candidates confused the closed-loop gain with the sensitivity to errors, although most were able to estimate graphically the magnitude of the closed-loop transfer function for a given frequency.

3 (a) PD control is a type a feedback where the reference is compared to a signal which has a component proportional to the output and another proportional to the first derivative of the output. Proportional control has the advantage of increasing the damping, and thus the stability of a second-order system.

(b) Taking the Laplace transform of the ODE:

$$s^2\bar{y}(s) + 4s\bar{y}(s) + \alpha\bar{y}(s) = \bar{x}(s) \rightarrow G(s) = \frac{1}{s^2 + 4s + \alpha} \tag{5}$$

(i)

$$G(s) = \frac{1}{s^2 + 4s + 3} = \frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{s+3}\right) \rightarrow g(t) = \frac{1}{2}\left[\exp(-t) - \exp(-3t)\right] \tag{6}$$

Initial value

$$g(0) = 0 \tag{7}$$

Initial slope

$$\frac{dg}{dt}(0) = 1 \tag{8}$$

Turning points:

$$\frac{dg}{dt} = \frac{1}{2}\left[-\exp(-t) + 3\exp(-3t)\right] = 0 \rightarrow \exp(2t) = 3 \rightarrow t = \frac{\ln(3)}{2} \simeq 0.55 \tag{9}$$

Final value:

$$t \rightarrow \infty \quad \rightarrow \quad g(t) \rightarrow 0 \tag{10}$$

(ii)

$$G(s) = \frac{1}{s^2 + 4s + 4} = \frac{1}{(s+2)^2} \rightarrow g(t) = t\exp(-2t) \tag{11}$$

$g(t)$ starts from 0 with a positive initial slope, has a turning point at $t = 0.5$ and approaches 0 as $t \rightarrow \infty$.

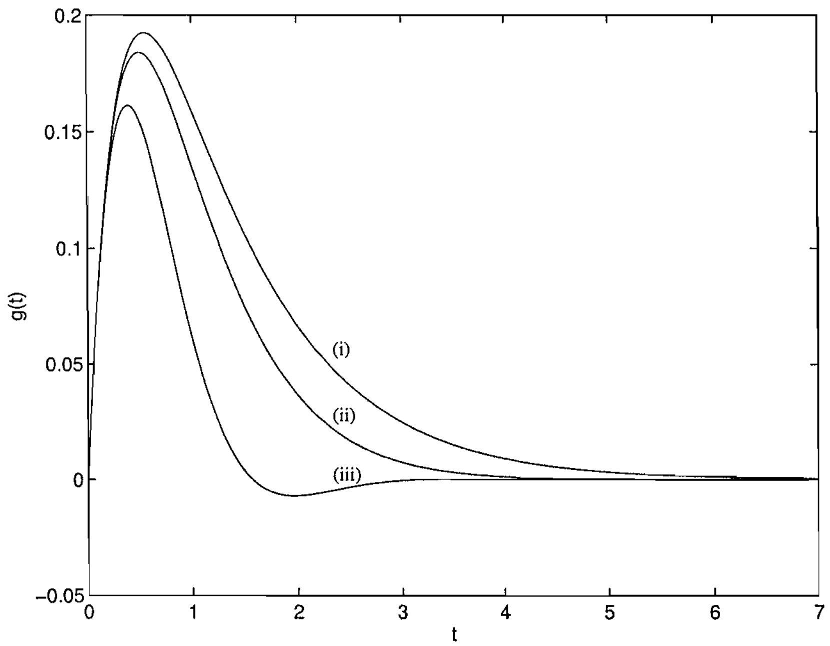
(iii)

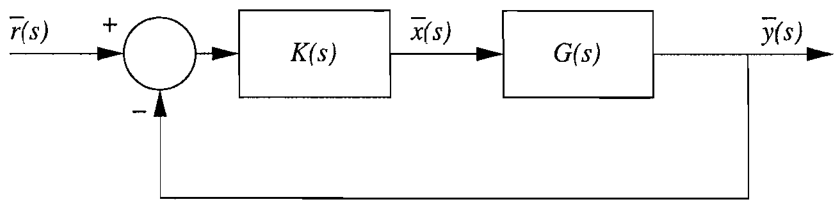
$$G(s) = \frac{1}{s^2 + 4s + 8} = \frac{1}{(s+2)^2 + 4} \rightarrow g(t) = \frac{1}{2}\exp(-2t)\sin(2t) \tag{12}$$

which is an attenuated cos function.

(d) See figure.

$$G_{new}(s) = \frac{k_1 + sk_2}{s^2 + (4 + k_2)s + (\alpha + k_1)} \tag{13}$$





therefore increasing k_1 will increase the bandwidth (the system will be able to follow signals that change more rapidly) and increasing k_2 leads to an increased damping, therefore to a more stable system, where oscillations due to a disturbance will die down more quickly. K_1 also decreases damping (slightly).

Examiner's comment: The most challenging part of this question was to sketch the impulse response. Many candidates calculated the inverse Laplace transforms and plotted them correctly, and many others noted the differences between overdamped and underdamped systems. Almost all solutions provided correct block diagrams and transfer functions for the closed-loop system, but very few followed them with a complete analysis of the effects of PD feedback.

4 (a)

Spectrum: a representation which provides the amplitude and phase of a signal as a function of frequency. 2

Periodic → coef of fourier series 1

Non-periodic → Fourier Transform 1

+ 1 some extra detail given. $\boxed{5}$

(b) Direct Solution

$$F_1(\omega) = \int_{-T}^{0} -e^{-j\omega t}\,dt + \int_{0}^{T} e^{-j\omega t}\,dt$$

$$= \left[\frac{e^{-j\omega t}}{j\omega}\right]_{-T}^{0} - \left[\frac{e^{-j\omega t}}{j\omega}\right]_{0}^{T}$$

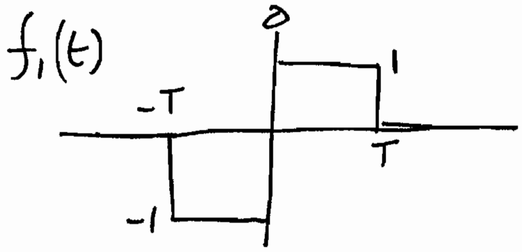
$$= \frac{1}{j\omega}\left[2 - \left(e^{j\omega T} + e^{-j\omega T}\right)\right]$$

$$= \frac{2}{j\omega}\left[1 - \cos\omega T\right]$$

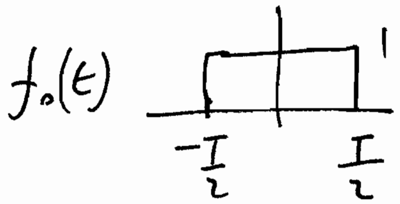
$$= -\frac{4j}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$$

OR

4 cont



Consider



$$F_0(\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 . e^{-j\omega t}\, dt = \frac{1}{j\omega}\left[e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}\right]$$

$$= \frac{2}{\omega} \sin\left(\omega\frac{T}{2}\right)$$

$\therefore$ by shift theorem

$$F_1(\omega) = \frac{2}{\omega}\sin\left(\frac{\omega T}{2}\right)\left[e^{-j\omega\frac{T}{2}} - e^{+j\omega\frac{T}{2}}\right]$$

$$= -\frac{4j}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$$

$\boxed{5}$

(c) Using (b) + (c)

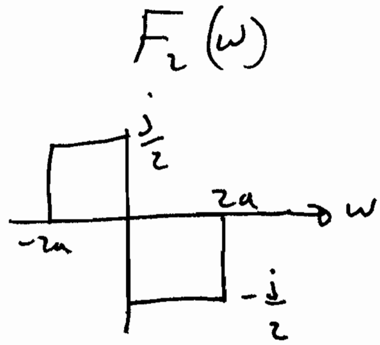
if $\quad f_1(t) \rightleftharpoons -\frac{4j}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$

then by duality $\quad -\frac{4j}{t}\sin^2\left(\frac{tT}{2}\right) \rightleftharpoons 2\pi f_1(-\omega)$

Let $a = \frac{T}{2}$

$$\therefore \quad \frac{\sin^2(at)}{\pi t} \rightleftharpoons -\frac{1}{2j} f_1(-\omega)$$

$$\rightleftharpoons \frac{j}{2} f_1(-\omega)$$



$$\therefore \quad F_2(\omega) = \begin{cases} \frac{j}{2} & -2a < \omega < 0 \\ -\frac{j}{2} & 0 < \omega < 2a \end{cases}$$

$\boxed{5}$

4 (d) By Parseval
cont

$$\int \left(\frac{\sin^2 t}{\pi t}\right)^2 dt = \frac{1}{2\pi} \int \left|F_2(\omega)\right|^2 d\omega$$

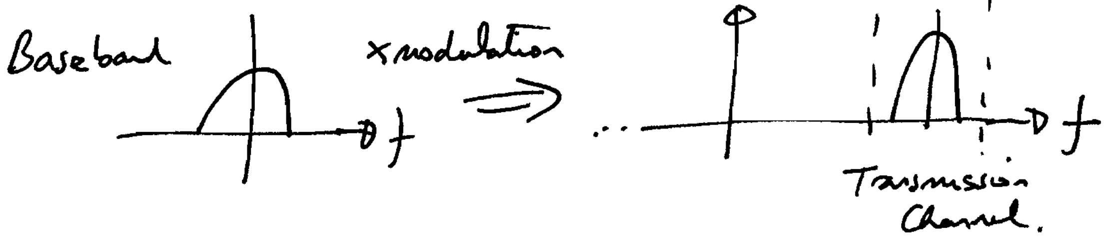
$$\therefore \int \left(\frac{\sin^2 t}{t}\right)^2 dt = \frac{\pi}{2} \left[\frac{1}{4} \cdot 2a \cdot 2\right] \quad \left(\text{from diagram of } F_2(\omega)\right)$$

$(a=1)$

$$= \underline{\underline{\frac{\pi}{2}}}$$

$\boxed{5}$

Q5 (a) Lecture material – direct transmission infeasible



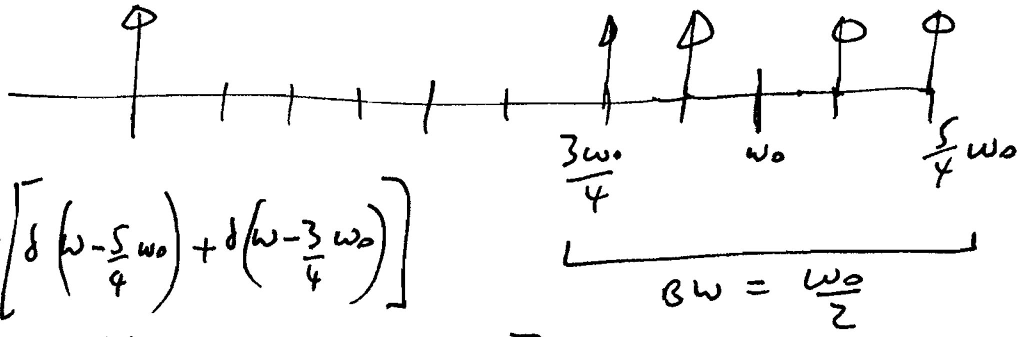
(b) $$u(t) = g_1(t)\cos\omega_0 t + g_2(t)\sin\omega_0 t$$

if $g_1(t) = \cos\dfrac{\omega_0 t}{4}$ and $g_2(t) = \cos\left(\dfrac{\omega_0 t}{8}\right)$

$$U(t) = \cos\left(\frac{\omega_0 t}{4}\right)\cos(\omega_0 t) + \cos\left(\frac{\omega_0 t}{8}\right)\sin(\omega_0 t)$$

$$= \frac{1}{2}\left[\cos\left(\frac{5\omega_0 t}{4}\right) + \cos\left(\frac{3\omega_0 t}{4}\right) + \sin\left(\frac{9\omega_0 t}{8}\right) + \sin\left(\frac{7\omega_0 t}{8}\right)\right]$$

Spectrum is



$$U(\omega) = \frac{\pi}{2}\left[\delta\left(\omega - \frac{5}{4}\omega_0\right) + \delta\left(\omega - \frac{3}{4}\omega_0\right)\right] + \frac{\pi}{2j}\left[\delta\left(\omega - \frac{9}{8}\omega_0\right) + \delta\left(\omega - \frac{7}{8}\omega_0\right)\right]$$

(c) $$u(t)\cos(\omega_0 t + \phi)$$

$$= g_1(t)\cos\omega_0 t\cos(\omega_0 t + \phi) + g_2(t)\sin\omega_0 t\cos(\omega_0 t + \phi)$$

5 cont $= \frac{g_1(t)}{2}\left[\cos(2\omega_0 t + \phi) + \cos\phi\right] +$

$\frac{g_2(t)}{2}\left[\sin(2\omega_0 t + \phi) + \sin\phi\right]$

Low Pass filter to remove high freq comps

$\Rightarrow \quad v_f(t) = \frac{1}{2}\left[g_1(t)\cos\phi + g_2(t)\sin\phi\right]$

$\therefore$ let $\cos\phi = 1 \quad \Rightarrow g_1(t) \qquad$ ie $\phi = 0$

$\sin\phi = 1 \quad \Rightarrow g_2(t) \qquad$ ie $\phi = \frac{\pi}{2}$.

6(a) Bookwork – Nyquist, cyclic repetition at period NT

(b) $$x(t) = a\cos\left(\frac{2\pi q}{NT}\right)t$$

$$\Rightarrow \quad x_n = a\cos\frac{2\pi q n}{N} \qquad (t \to nT)$$

$$= \frac{a}{2}\left[e^{j\frac{2\pi q n}{N}} + e^{-j\frac{2\pi q n}{N}}\right] \qquad (1)$$

By defn,

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k\, e^{j\frac{2\pi k n}{N}} \qquad (2)$$

Hence, comparing coef in (1) & (2)

$$\frac{1}{N}X_k = \frac{a}{2} \quad \text{if } q = k \text{ or } q = N-k$$

$$= 0 \quad \text{otherwise}$$

(c) $$X_k = \sum_{n=0}^{N-1} x_{1n}\, x_{2n}\, e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{m=0}^{N-1} X_{1m}\, e^{j\frac{2\pi m n}{N}}\right) x_{2n}\, e^{-j\frac{2\pi k n}{N}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_{1m}\sum_{n=0}^{N-1} x_{2n}\, e^{-j\frac{2\pi (k-m) n}{N}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_{1m}\, X_{2|k-m|}$$

6 (d) cont Let $t \to nT$

$$y_n = \underbrace{\frac{1}{8}\left(1+3\cos\frac{2\pi}{N}n\right)}_{x_{1n}\ (q=1)}\ \underbrace{\cos\frac{4\pi}{N}n}_{x_{2n}\ (q=2)}$$

From (b)

$$X_{1m} = \left[1, \tfrac{3}{2}, 0, 0, 0, 0, 0, \tfrac{3}{2}\right] \times$$

$$X_{2m} = [0, 0, 4, 0, 0, 0, 4, 0] \Rightarrow 0$$

$$X_{2|1-m|} = [0, 0, 0, 4, 0, 0, 0, 4] \Rightarrow 6$$

$$X_{2|2-m|} = [4, 0, 0, 0, 4, 0, 0, 0] \Rightarrow 4$$

$$X_{2|3-m|} = [0, 4, 0, 0, 0, 4, 0, 0] \Rightarrow 6$$

$$X_{2|4-m|} = [0, 0, 4, 0, 0, 0, 4, 0] \Rightarrow 0$$

$$X_{2|5-m|} = [0, 0, 0, 4, 0, 0, 0, 4] \Rightarrow 6$$

$$X_{2|6-m|} = [4, 0, 0, 0, 4, 0, 0, 0] \Rightarrow 4$$

$$X_{2|7-m|} = [0, 4, 0, 0, 0, 4, 0, 0] \Rightarrow 6$$

$$\therefore Y_k = \frac{1}{8}[0, 6, 4, 6, 0, 6, 4, 6]$$