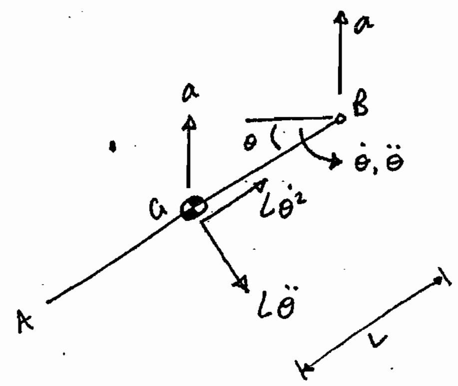
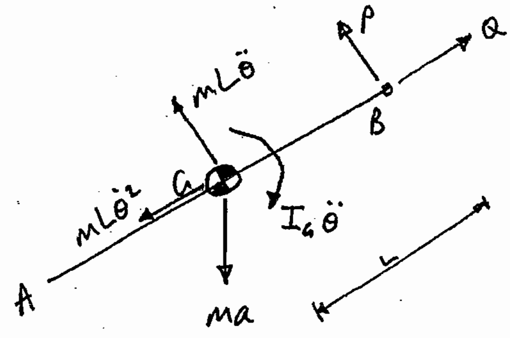
SECTION A

1(a) Consider AB when partially rotated:



ACCELERATIONS



FORCES

M_B ↻ : $mL^2\ddot\theta + I_G\ddot\theta - ma(\cos\theta)L = 0$

$\Rightarrow L^2\ddot\theta + k^2\ddot\theta = aL(\cos\theta) \quad \therefore \quad \ddot\theta = \dfrac{aL\cos\theta}{(k^2+L^2)}$

(b) $\ddot\theta = \dfrac{d\dot\theta}{d\theta}\dfrac{d\theta}{dt} = \dot\theta\dfrac{d\dot\theta}{d\theta} \Rightarrow \dfrac{\dot\theta^2}{2} = \dfrac{aL}{(k^2+L^2)}\sin\theta + \text{Const.}$

B.C.'s $\theta = 0,\ \dot\theta = 0 \Rightarrow \text{const} = 0 \quad \therefore \quad \dot\theta^2 = \dfrac{2aL}{(k^2+L^2)}\sin\theta$

for $\theta = \dfrac{\pi}{2} \qquad \dot\theta = \sqrt{\dfrac{2aL}{(k^2+L^2)}}$

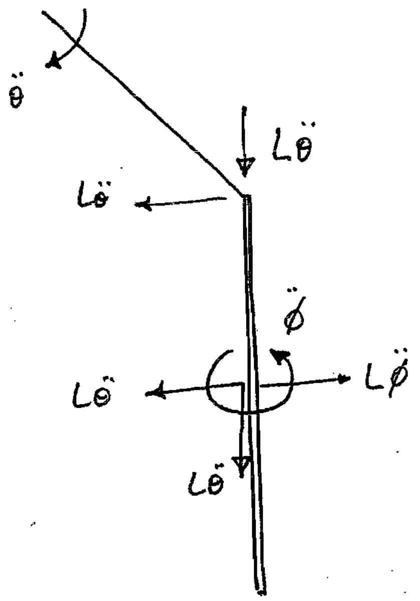
c) $R(/\!/_{AB})$: $Q = mL\dot\theta^2 + ma\sin\theta \qquad R(\perp_{AB})$: $P + mL\ddot\theta = ma\cos\theta$

when $\theta = \dfrac{\pi}{2},\ \ddot\theta = 0 \Rightarrow P = 0,\ Q = \dfrac{2maL^2}{(k^2+L^2)} + ma = \dfrac{ma(k^2+3L^2)}{(k^2+L^2)}$

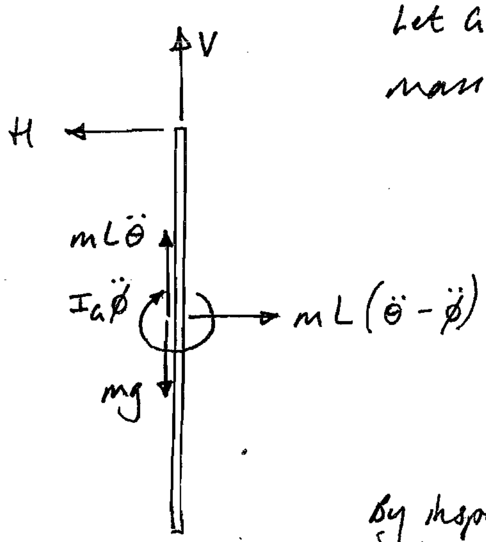
For $\theta = 0,\ \dot\theta = 0$: $Q = 0,\ P = ma - \dfrac{maL^2}{(k^2+L^2)} = \dfrac{mak^2}{(k^2+L^2)}$

i.e. hinge force is greater at point just before AB strikes block than when block first starts to move.

2a)



ACCELERATIONS



FORCES

Let G be centre of mass of slab.

By inspection V = H because $\theta = 45°$.

$R(\uparrow)$: $V = mg - mL\ddot\theta$ $\qquad$ $R(\rightarrow)$: $H = mL(\ddot\theta - \ddot\phi)$ $\qquad$ M_G ↻: $I_G\ddot\phi = HL$

$\Rightarrow I_G = \dfrac{mL^2}{3}$ $\qquad \Rightarrow \dfrac{mL^2}{3}\ddot\phi = mL^2(\ddot\theta - \ddot\phi)$

$$\frac{4mL^2}{3}\ddot\phi = mL^2\ddot\theta \quad\Rightarrow\quad \ddot\theta = \frac{4}{3}\ddot\phi$$

Subs: $\dfrac{mL^2}{3}\ddot\phi = mgL - mgL^2\ddot\theta$

$$= mgL - \frac{4mgL^2}{3}\ddot\phi$$

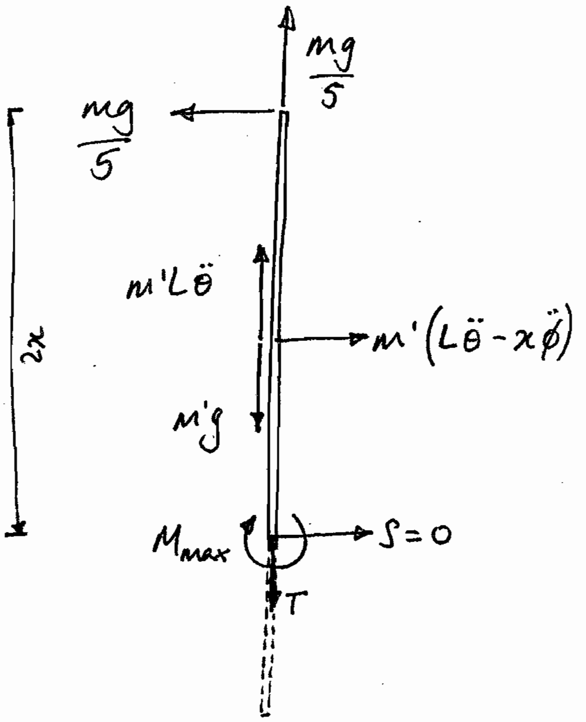
$$\frac{5mL^2}{3}\ddot\phi = mgL \quad\Rightarrow\quad \underline{\ddot\phi = \frac{3g}{5L}} \qquad \therefore\ \ddot\theta = \frac{4g}{5L}$$

b) $V = mg - mL\left(\dfrac{4g}{5L}\right) = \dfrac{mg}{5} \quad\Rightarrow\quad T = \dfrac{\sqrt{2}}{5}mg$

2c) Consider a slab of length $2x$ extending to the point where $S = 0$ i.e. where $dM/dx = 0$:

Let mass of this bar be m' where $m' = m\left(\frac{x}{L}\right)$

and $I_{G'} = \frac{m'}{3}x^2 = \frac{mx^3}{3L}$



FORCES

$$R(\rightarrow): \quad S + m'(L\ddot{\theta} - x\ddot{\phi}) = \frac{mg}{5}$$

$$\frac{mx}{L}(L\ddot{\theta} - x\ddot{\phi}) = \frac{mg}{5}$$

$$\frac{x}{L}\left(\frac{4Lg}{5L} - x\frac{3g}{5L}\right) = \frac{g}{5}$$

$$4\left(\frac{x}{L}\right) - 3\left(\frac{x}{L}\right)^2 = 1$$

$$\Rightarrow 3\left(\frac{x}{L}\right)^2 - 4\left(\frac{x}{L}\right) + 1 = 0$$

$$\therefore \left(\frac{x}{L}\right) = \frac{4 \pm \sqrt{16-12}}{6} \Rightarrow \left(\frac{x}{L}\right) = 1 \text{ or } \frac{1}{3}$$

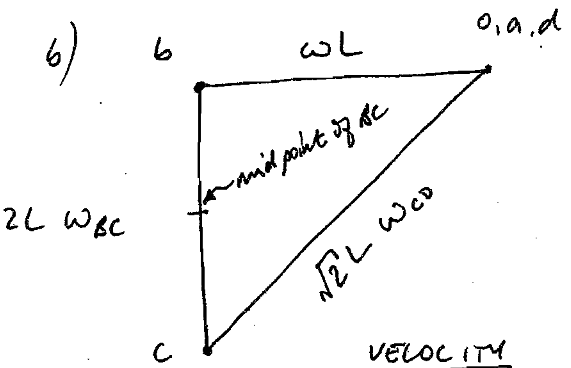
Hence $S = 0$ in slab either at $2L$ or $2L/3$ from top $\rightarrow$ $2L$ is the free end of the slab, hence M_{max} occurs $2L/3$ from top.

$$M_{G'} \circlearrowleft: \quad M_{max} + I_{G'}\ddot{\phi} - \frac{mgx}{5} = 0 \Rightarrow M_{max} = \frac{mgx}{5} - \frac{mx^3}{3L}\left(\frac{3g}{5L}\right)$$

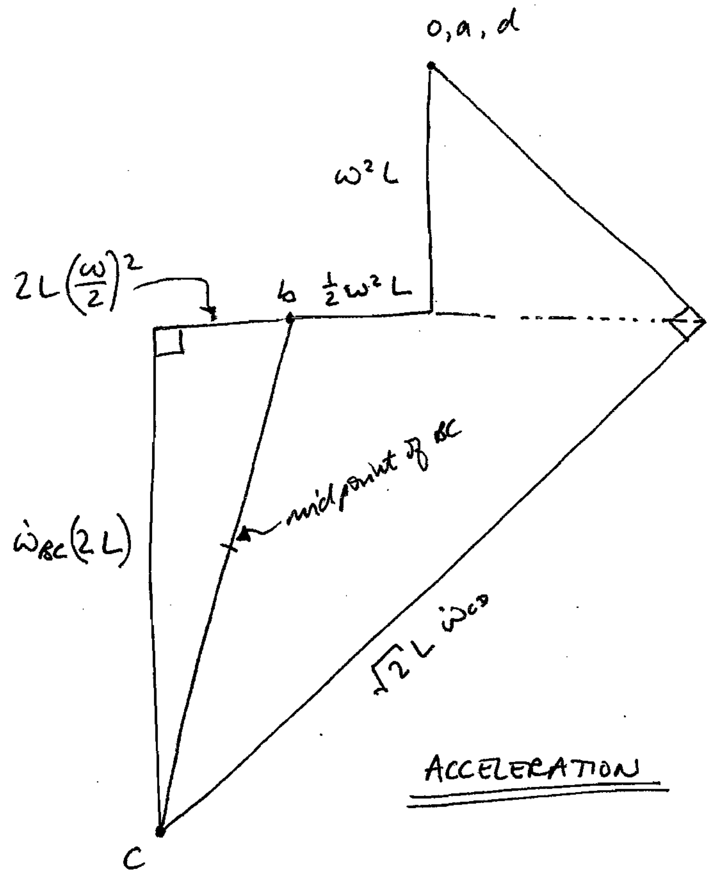
for $x = \frac{L}{3}$, $\quad M_{max} = \frac{8mgL}{135}$

3a) From databook, I_{zz} for disc $= \frac{1}{2}(8m)\left(\frac{3L}{2}\right)^2 = 9mL^2$

b)



VELOCITY



ACCELERATION

$$\Rightarrow \omega_{BC} = \frac{\omega}{2}$$

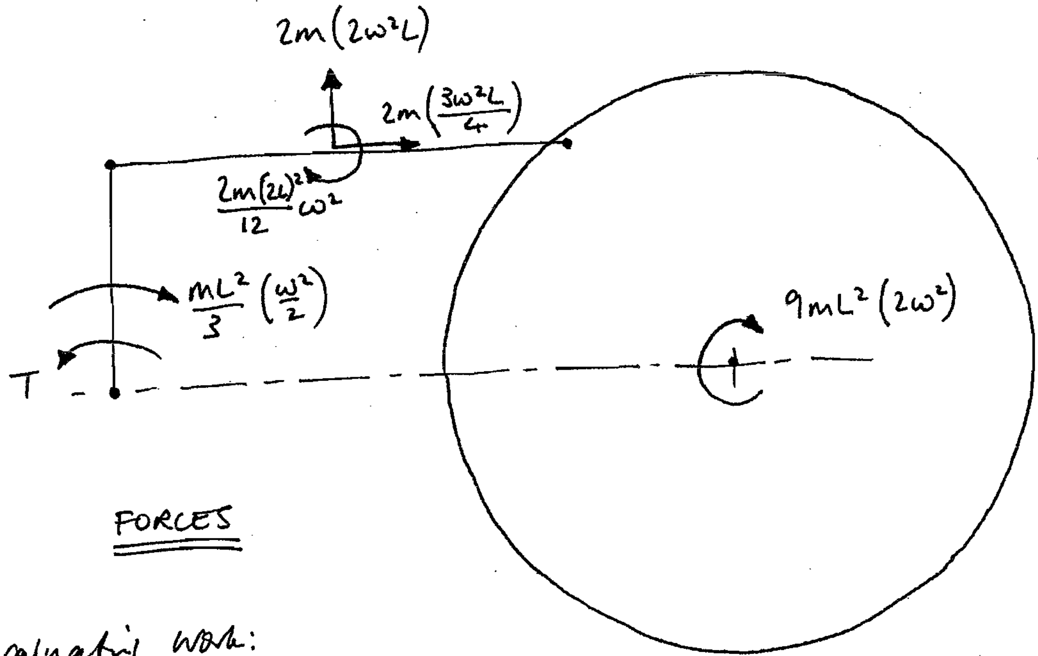
$$\omega_{CD} = \omega$$

$$\dot{\omega}_{BC} = \omega^2$$

$$\dot{\omega}_{CD} = 2\omega^2$$

$$a_c = \sqrt{10}\,\omega^2 L$$

c)



FORCES

Evaluating work:

$$T\omega = \frac{mL^2}{6}\omega^3 + 4m\omega^2 L\left(\frac{\omega L}{2}\right) + \frac{3mL}{2}\omega^2(\omega L) + \frac{2mL^2}{3}\omega^2\left(\frac{\omega}{2}\right) + 18mL^2\omega^3$$

$$\Rightarrow T = 22\,mL^2\omega^2$$

SECTION B

4) (a) (i) $\overrightarrow{OP} = a\cos\theta\, \underset{\sim}{e}_1 + a\sin\theta\, \underset{\sim}{k}$

(ii) $\underset{\sim}{v}_p = \frac{d}{dt}(\overrightarrow{OP}) = -a\dot{\theta}\sin\theta\, \underset{\sim}{e}_1 + a\cos\theta\,\dot{\phi}\, \underset{\sim}{e}_2$

$+ a\dot{\theta}\cos\theta\, \underset{\sim}{k}$

Note: $\frac{d}{dt}\underset{\sim}{e}_1 = \dot{\phi}\underset{\sim}{e}_2 \qquad \frac{d}{dt}\underset{\sim}{e}_2 = -\dot{\phi}\underset{\sim}{e}_1 \qquad \frac{d}{dt}\underset{\sim}{k} = 0$

(iii) $\underset{\sim}{a}_p = \frac{d}{dt}(\underset{\sim}{v}_p) = -a\ddot{\theta}\sin\theta\, \underset{\sim}{e}_1 - a\dot{\theta}^2\cos\theta\, \underset{\sim}{e}_1 - a\dot{\theta}\sin\theta \times \dot{\phi}\underset{\sim}{e}_2$

$- a\dot{\theta}\sin\theta . \dot{\phi}\underset{\sim}{e}_2 + a\cos\theta . \ddot{\phi}\underset{\sim}{e}_2 - a\cos\theta . \dot{\phi}^2\underset{\sim}{e}_1$

$- a\dot{\theta}^2\sin\theta\, \underset{\sim}{k} + a\cos\theta . \ddot{\theta}\underset{\sim}{k}$

$= -a\left(\ddot{\theta}\sin\theta + (\dot{\theta}^2 + \dot{\phi}^2)\cos\theta\right)\underset{\sim}{e}_1$

$+ a\left(\ddot{\phi}\cos\theta - 2\dot{\theta}\dot{\phi}\sin\theta\right)\underset{\sim}{e}_2$

$+ a\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\underset{\sim}{k}$

(b)(i) Kinetic energy $\quad T = \frac{1}{2}m\,|\underset{\sim}{v}_p|^2$

$\therefore T = \frac{1}{2}m\left[(a\dot{\theta}\sin\theta)^2 + (a\dot{\phi}\cos\theta)^2 + (a\dot{\theta}\cos\theta)^2\right]$

$= \frac{1}{2}ma^2\left[\dot{\theta}^2(\sin^2\theta + \cos^2\theta) + \dot{\phi}^2\cos^2\theta\right]$

$= \frac{1}{2}ma^2\left(\dot{\theta}^2 + \dot{\phi}^2\cos^2\theta\right)$

4) (b) (cont.)

(ii) Moment of momentum about $\underset{\sim}{k}$ axis

$$= m \times a\dot{\phi}\cos\theta \times a\cos\theta \qquad = ma^2\dot{\phi}\cos^2\theta$$

(iii) Kinetic energy is conserved $\quad T = \text{constant}$

$$\therefore \frac{d}{dt}T = 0 \qquad \therefore \frac{d}{dt}\left(\dot{\theta}^2 + \dot{\phi}^2\cos^2\theta\right) = 0$$

$$2\dot{\theta}\ddot{\theta} + 2\dot{\phi}\ddot{\phi}\cos^2\theta - 2\dot{\phi}^2\dot{\theta}\sin\theta\cos\theta = 0$$

$$\therefore \dot{\theta}\ddot{\theta} + \dot{\phi}\ddot{\phi}\cos^2\theta - \dot{\phi}^2\dot{\theta}\sin\theta\cos\theta = 0 \qquad (1)$$

Moment of momentum about $\underset{\sim}{k}$ axis is conserved

$$\therefore \frac{d}{dt}\left(ma^2\dot{\phi}\cos^2\theta\right) = 0 \qquad \frac{d}{dt}\left(\dot{\phi}\cos^2\theta\right) = 0$$

$$\ddot{\phi}\cos^2\theta - 2\dot{\phi}\dot{\theta}\sin\theta\cos\theta = 0$$

$$\therefore \ddot{\phi}\cos\theta - 2\dot{\phi}\dot{\theta}\sin\theta = 0 \qquad (2)$$

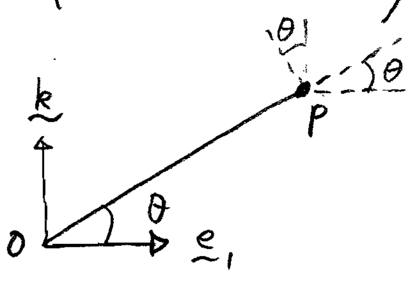
Now consider the acceleration components. P can have no acceleration perpendicular to OP.

Hence the component of $\underset{\sim}{a}_P$ in the $\underset{\sim}{e}_2$ direction must be zero:

$$\ddot{\phi}\cos\theta - 2\dot{\theta}\dot{\phi}\sin\theta = 0 \qquad (3)$$

And the component of $\underset{\sim}{a}_P$ in the $\underset{\sim}{e}_1, \underset{\sim}{k}$ plane perpendicular to $\overrightarrow{OP}$ must also be zero:

$$a\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\cos\theta \;-\; -a\left(\ddot{\theta}\sin\theta + \left(\dot{\theta}^2 + \dot{\phi}^2\right)\cos\theta\right)\sin\theta = 0$$



4) (b) (iii) (cont.)

$$\ddot{\theta}\cos^2\theta - \dot{\theta}^2\sin\theta\cos\theta + \ddot{\theta}\sin^2\theta + \left(\dot{\theta}^2 + \dot{\phi}^2\right)\cos\theta\sin\theta = 0$$

$$\therefore \quad \ddot{\theta} + \dot{\phi}^2\cos\theta\sin\theta = 0 \qquad (4)$$

Substitute (2) into (1)

$$\dot{\theta}\ddot{\theta} + 2\dot{\theta}\dot{\phi}\sin\theta \,.\, \dot{\phi}\cos\theta - \dot{\phi}^2\dot{\theta}\sin\theta\cos\theta = 0$$

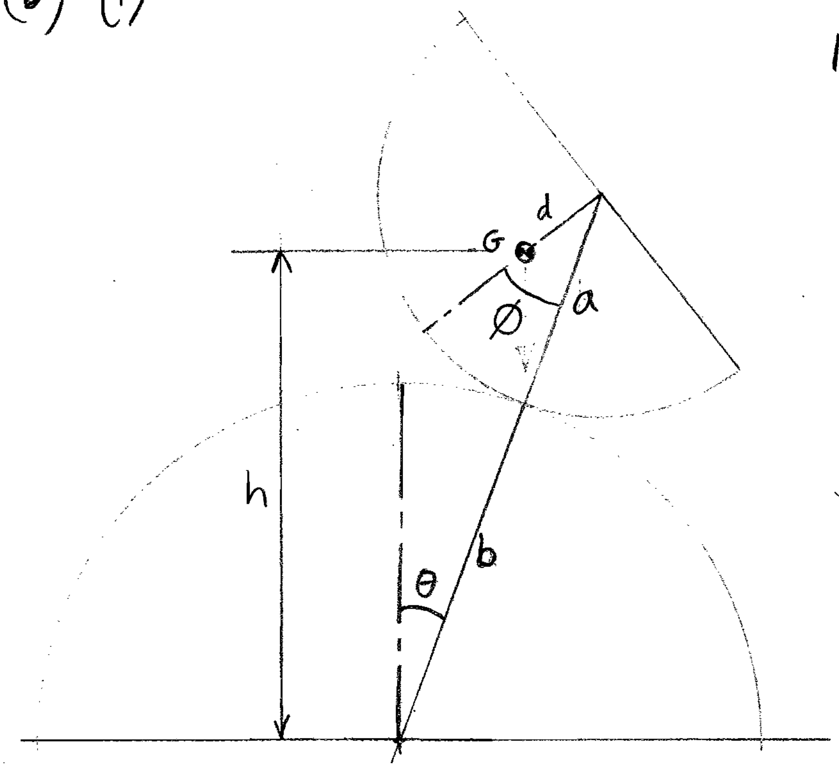
$$\therefore \quad \dot{\theta}\ddot{\theta} + \dot{\theta}\dot{\phi}^2\sin\theta\cos\theta = 0$$

$$\ddot{\theta} + \dot{\phi}^2\sin\theta\cos\theta \qquad (5)$$

(5) is identical to (4) and (3) is identical to (2) and hence the acceleration components obtained are consistent with conservation of kinetic energy and moment of momentum.

(cont.)

5) (b) (i)



No slip hence

$$b\theta = a\emptyset$$

$$\therefore \emptyset = \frac{b}{a}\theta$$

(ii) Distance d to centre of mass of upper half-cylinder

From mechanics data book:

$$d = \frac{2}{3}\frac{a}{\pi/2} = \frac{4a}{3\pi}$$

Potential energy

$$V = mgh = mg\left[(a+b)\cos\theta - d\cos(\theta+\emptyset)\right]$$

$$= mg\left[(a+b)\cos\theta - \frac{4a}{3\pi}\cos\left(\theta\left\{1+\frac{b}{a}\right\}\right)\right]$$

$$\frac{dV}{d\theta} = mg\left[-(a+b)\sin\theta + \frac{4b}{3\pi}\left\{1+\frac{b}{a}\right\}\sin\left(\theta\left\{1+\frac{b}{a}\right\}\right)\right]$$

$\frac{dV}{d\theta} = 0$ when $\theta = 0$ $\therefore$ $\theta = 0$ is a point of equilibrium

5) (cont.)

(b)(iii)

$$\frac{d^2V}{d\theta^2} = mg\left[-(a+b)\cos\theta + \frac{4a}{3\pi}\left\{1+\frac{b}{a}\right\}^2 \cos\left(\theta\left\{1+\frac{b}{a}\right\}\right)\right]$$

@ $\theta = 0$ $$\frac{d^2V}{d\theta^2} = mg\left[-(a+b) + \frac{4a}{3\pi}\left\{1+\frac{b}{a}\right\}^2\right]$$

$$\frac{d^2V}{d\theta^2} > 0 \quad \text{if} \quad \frac{4a}{3\pi}\left\{1+\frac{b}{a}\right\}^2 > (a+b)$$

$$\frac{(a+b)^2}{a^2} > \frac{3\pi}{4a}(a+b)$$

$$\frac{a+b}{a} > \frac{3\pi}{4}$$

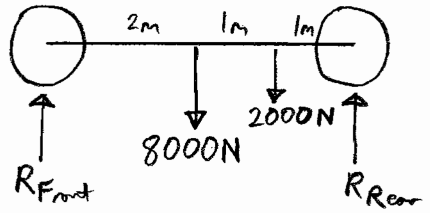
$$\frac{b}{a} > \frac{3\pi}{4} - 1$$

$$\frac{b}{a} > 1.356$$

This is the condition for the potential energy V to be a minimum @ $\theta = 0$ and hence this is the condition for stability.

6) Assume that the car's suspension acts to share the load symmetrically between the four wheels.

(a) No acceleration



Side elevation

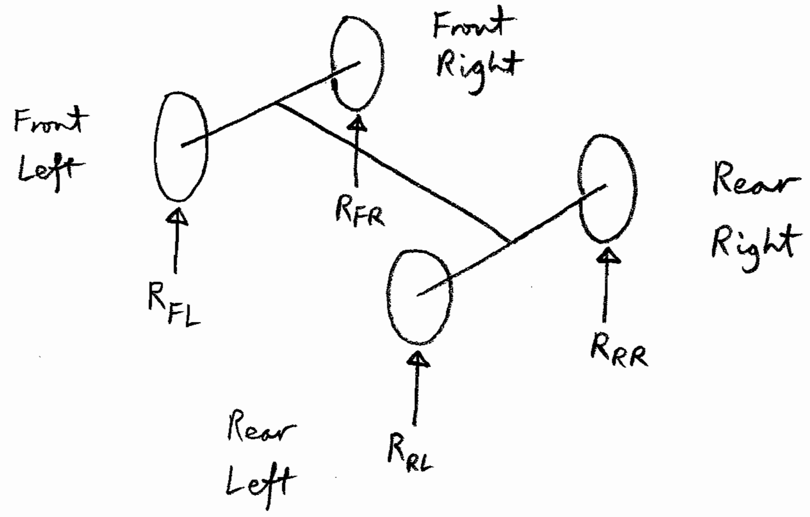
$(g = 10 m/s^2)$

$$R_{Front} + R_{Rear} = 8000 + 2000$$
$$= 10{,}000\ N$$

$$4R_{Front} = 8000 \times 2 + 2000 \times 1$$
$$\therefore R_{Front} = 4500N$$
$$R_{Rear} = 5500N$$

Assume load is shared equally left & right

Hence:



$R_{FL} = R_{FR} = 2250\ N$ $\quad$ $R_{RL} = R_{RR} = 2750N$

(b) Accelerating at $5m/s^2$ in a straight line.

Assume car is in a fixed gear.

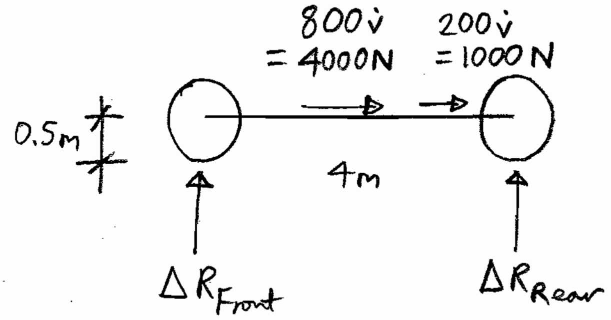
Engine speed $\omega = 2000$ rad/s when $v = 40m/s$

$\therefore$ $\omega = \frac{2000}{40} v = 50v$ $\quad$ $\therefore$ $\dot{\omega} = 50\dot{v}$

$$= 50 \times 5$$
$$= 250\ rad/s^2$$

6) (b) (cont.)

Consider changes in reaction forces due to d'Alembert forces and torques

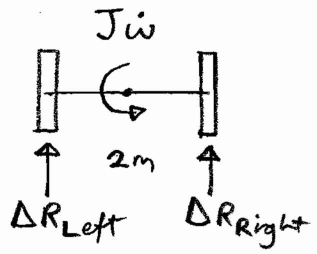


Side elevation

$$4\Delta R_{Rear} = (4000 + 1000) \times 0.5$$

$$\Delta R_{Front} + \Delta R_{Rear} = 0$$

$$\therefore \Delta R_{Rear} = 625\,N$$
$$\Delta R_{Front} = -625\,N$$



Rear view

$$J = mr^2 = 200 \times 0.1^2$$
$$= 2\ \text{kg}\,\text{m}^2$$

$$2\Delta R_L = J\dot{\omega} = 2 \times 250$$

$$\therefore \Delta R_{Left} = 250\,N$$
$$\Delta R_{Right} = -250\,N$$

Hence

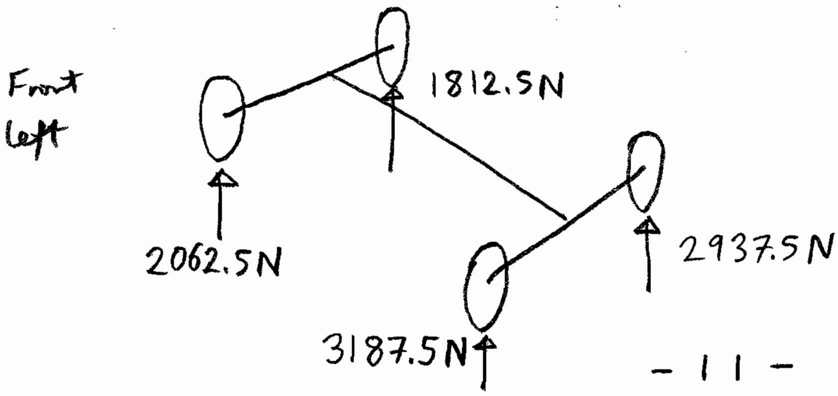
$\Delta R_{FL} = -625/2 + 250/2 = -187.5\,N$ Front left

$\Delta R_{FR} = -625/2 - 250/2 = -437.5\,N$ Front right

$\Delta R_{RL} = 625/2 + 250/2 = 437.5\,N$ Rear left

$\Delta R_{RR} = 625/2 - 250/2 = 187.5\,N$ Rear right

Add these changes to original load distribution (a):



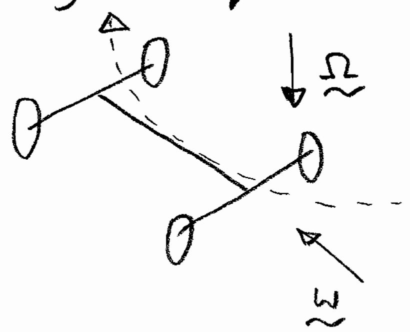
6) (c) Travelling around a right-hand corner of radius 80m at a constant speed of 40 m/s

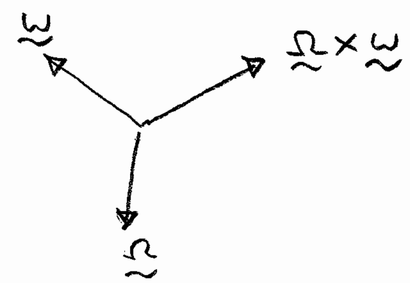
The rotating engine will produce a gyroscopic reaction torque of magnitude $J\Omega\omega$

Where $\omega = 2000$ rad/s is the engine speed

$$\Omega = \frac{V}{R} = \frac{40}{80} = 0.5 \text{ rad/s}$$ is the angular velocity of the car

$\therefore$ Gyro torque $= 2 \times 2000 \times 0.5 = 2000$ Nm

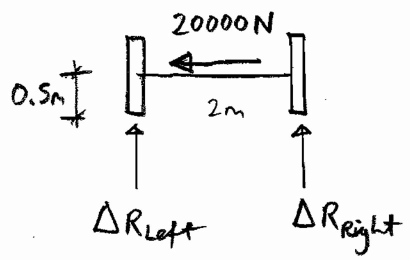




Direction of gyroscopic torque is given by $\underset{\sim}{\Omega} \times \underset{\sim}{\omega}$, hence the load on the front wheels is increased, and the load on the rear wheels is decreased by:

$$\frac{2000}{4} = 500\text{N} = \Delta R_{Front} = -\Delta R_{Rear}$$

We must also consider the d'Alembert force due to the centripetal acceleration $= \frac{V^2}{R} = \frac{40^2}{80} = 20 \text{ m/s}^2$



thus a d'Alembert force of

$$(800 + 200) \times 20 = 20{,}000\text{ N}$$

$$\therefore 2\Delta R_{Left} = 0.5 \times 20{,}000$$

$$\Delta R_{Left} = 5{,}000\text{N} \qquad \Delta R_{Right} = -5000\text{N}$$

6) (c) (contd.)

Hence

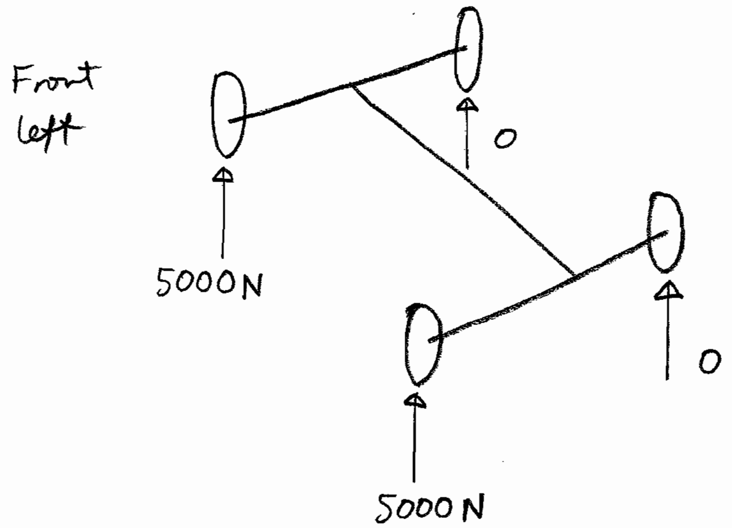
$$\Delta R_{FL} = 500/2 + 5000/2 = 2750N \quad \text{Front left}$$

$$\Delta R_{FR} = 500/2 - 5000/2 = -2250N \quad \text{Front right}$$

$$\Delta R_{RL} = -500/2 + 5000/2 = 2250N \quad \text{Rear left}$$

$$\Delta R_{RR} = -500/2 - 5000/2 = -2750N \quad \text{Rear right}$$

Add these changes to the original load distribution (a):



i.e. entire weight of car is now on left hand wheels and the car is on the point of tipping over