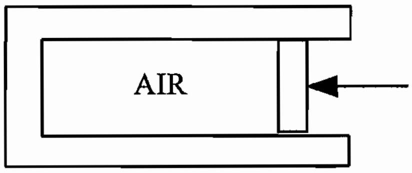
ENGINEERING TRIPOS PART IB 2007

PAPER 4 – THERMOFLUID MECHANICS

SOLUTIONS TO TRIPOS QUESTIONS

T.B. NICKELS & J.B. YOUNG

1. (a)



1 → 2 is reversible adiabatic compression, 2 → 3 is constant volume cooling.

From the data book for air, $R_a = 287$ J/kg K and $c_{va} = 718$ J/kg K.

The compression is isentropic (adiabatic and reversible), so the final air temperature is,

$$T_{a2} = T_{a1}\left(\frac{V_{a1}}{V_{a2}}\right)^{R/c_v} = 300.0\times\left(\frac{0.01}{0.002}\right)^{287/718} = 570.8 \text{ K}$$

During the cooling process, the volume remains constant so $T_a dS_a = dU_a$. Assuming air behaves as a perfect gas, $dU_a = m_a c_{va} dT_a$ and the entropy change of the air is,

$$\Delta S_a = \int_2^3 \frac{dU_a}{T_a} = m_a c_{va}\int_{T_{a2}}^{T_{a3}} \frac{dT_a}{T_a} = m_a c_{va}\ln\left(\frac{T_{a3}}{T_{a2}}\right) = 0.02\times718\times\ln\left(\frac{300.0}{570.8}\right) = -9.237 \text{ J/K}$$

From the First Law, the heat transfer to the air during the cooling process ($W = 0$),

$$Q_a = U_{a3} - U_{a2} = m_a c_{va}(T_{a3} - T_{a2}) = 0.02\times718\times(300.0 - 570.8) = -3888.7 \text{ J}$$

Hence the heat transfer to the cylinder is $Q_c = -Q_a = 3888.7$ J.
The entropy change of the cylinder is,

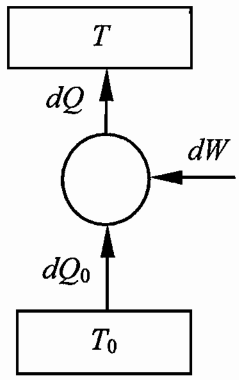
$$\Delta S_c = \frac{Q_c}{T_c} = \frac{3888.7}{300} = 12.962 \text{ J/K}$$

The total entropy change of the air plus cylinder system is,

$$\Delta S = \Delta S_a + \Delta S_c = 12.962 - 9.237 = 3.725 \text{ J/K}$$ [9]

The entropy creation is caused solely by the irreversible heat transfer from the air to the cylinder. [1]

(b) (i) Consider the heat pump at the instant when the house temperature is T. Heat dQ_0 is pumped from the environment, work dW is supplied to the pump, heat dQ is transferred to the house and its temperature increases by dT.



The actual PER is α times the ideal reversible PER. Hence,

$$\text{PER} = \frac{dQ}{dW} = \alpha \frac{T}{(T - T_0)}$$

The heat capacity of the house is C and so $dQ = CdT$. Hence,

$$dW = \frac{C}{\alpha}\frac{(T - T_0)}{T}dT = \frac{C}{\alpha}\left(1 - \frac{T_0}{T}\right)dT$$

Thus,

$$W = \frac{C}{\alpha}\int_{T_0}^{T_1}\left(1 - \frac{T_0}{T}\right)dT = \frac{C}{\alpha}\left[(T_1 - T_0) - T_0 \ln\left(\frac{T_1}{T_0}\right)\right]$$ [6]

(ii) When $T_0 = 275$ K and $T_1 = 293$ K,

$$W = \frac{C}{\alpha}\left[(293 - 275) - 275 \times \ln\left(\frac{293}{275}\right)\right] = \frac{0.5646C}{\alpha}$$

The thermal efficiency of the engine driving the pump is 0.30, so the required heat input is,

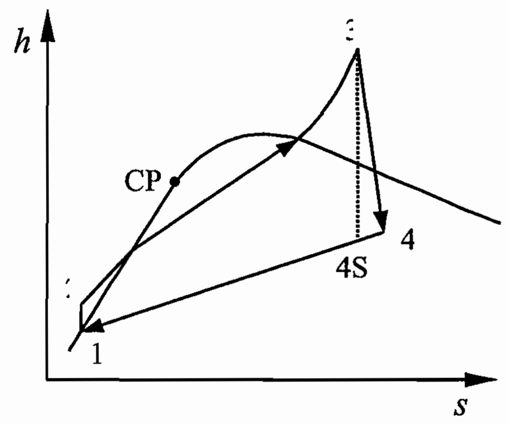
$$Q = \frac{W}{0.30} = \frac{0.5646C}{0.30\alpha}$$

If fuel were burned directly, the heat required would be $C(T_1 - T_0) = (293 - 275)C = 18C$
Hence, if the engine-pump combination is to be thermodynamically more efficient,

$$\frac{0.5646C}{0.30\alpha} < 18C \quad \rightarrow \quad \alpha > \frac{0.5646}{0.30 \times 18} = 0.105$$ [4]

So the engine-pump combination is a thermodynamically attractive system.

2. (a) (i)



[3]

(ii) From the superheated steam tables at p_3 = 150 bar, T_3 = 500 °C :
h_3 = 3310.8 kJ/kg, s_3 = 6.348 kJ/kg K.

From the saturated steam tables at p_4 = 0.04 bar :
h_f = 121.4 kJ/kg, h_g = 2553.7 kJ/kg, s_f = 0.422 kJ/kg K, s_g = 8.473 kJ/kg K.

At state 4S :

$$x_{4S} = \frac{s_{4S} - s_f}{s_g - s_f} = \frac{s_3 - s_f}{s_g - s_f} = \frac{6.348 - 0.422}{8.473 - 0.422} = 0.736$$

$$h_{4S} = h_f + x_{4S}(h_g - h_f) = 121.4 + 0.736\times(2553.7-121.4) = 1911.6 \text{ kJ/kg}$$

At turbine exit :

$$h_4 = h_3 - \eta_T(h_3 - h_{4S}) = 3310.8 - 0.83\times(3310.8 - 1911.6) = 2149.5 \text{ kJ/kg}$$

$$x_4 = \frac{h_4 - h_f}{h_g - h_f} = \frac{2149.5 - 121.4}{2553.7 - 121.4} = 0.834$$

The turbine specific work output is,

$$w_T = (h_3 - h_4) = (3310.8 - 2149.5) = 1161.3 \text{ kJ/kg}$$ [5]

(iii) From the saturated steam tables at 0.04 bar : v_f = 0.001004 m^3/kg.
For an isentropic compression in the feed pump, $Tds = dh - vdp = 0$. Hence,

$$w_{FP} = (h_2 - h_1) \cong v_f(p_2 - p_1)$$

$$= 0.001004\times(150.0\times10^5 - 0.04\times10^5) = 15.1\times10^3 \text{ J/kg} = 15.1 \text{ kJ/kg}$$ [2]

(iv) The specific enthalpy after the feed pump is : $h_2 = 121.4 + 15.1 = 136.5$ kJ/kg

The heat supplied in the boiler per kg of steam generated is,

$$q_B = (h_3 - h_2) = (3310.8 - 136.5) = 3174.3 \text{ kJ/kg}$$

The cycle efficiency is,

$$\eta_{cycle} = \frac{w_T - w_{FP}}{q_B} = \frac{1161.3 - 15.1}{3174.3} = 0.361$$ [2]

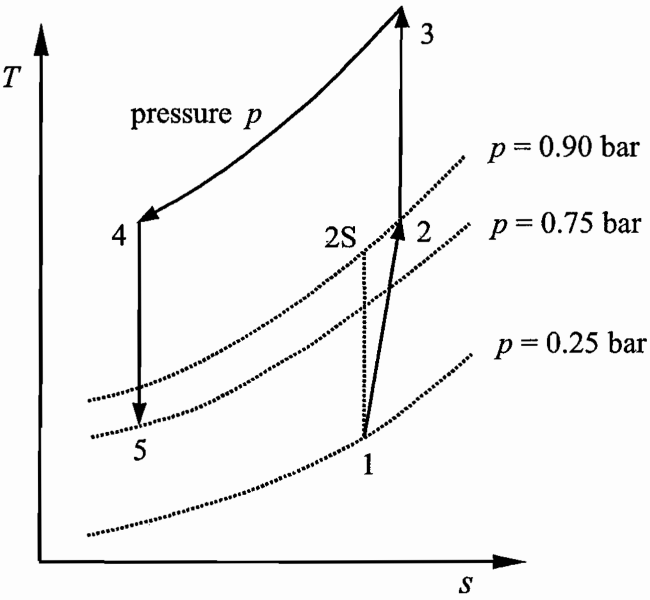
(b) (i) Dryness fraction $x_4 = 0.834$ is too low. The excessive wetness in the final stages of the turbine will result in (i) reduced LP turbine efficiency and (ii) erosion of the blades. [2]

(ii) The remedy is to expand the steam in a HP turbine to around 40 bar and then reheat in the boiler to 500 °C before expanding in a LP turbine to 0.04 bar. This will move state 4 to the right on the (T-s) diagram thus increasing the dryness fraction (ideally to around 0.92). [2]

(iii) Introducing reheat will improve the cycle efficiency because (i) extra heat is added at high temperature (which is thermodynamically beneficial) and (ii) the LP turbine efficiency will improve because the steam will be drier. [2]

(iv) The cooling water temperature at inlet to the condenser tubes is fixed by the water supply temperature (river or cooling tower exit), say 15 °C. To obtain the required heat transfer rate in the condenser, the steam temperature must be a few degrees higher, say 29 °C. Because wet steam is an equilibrium mixture of liquid and vapour, the steam-side pressure must be the saturated vapour pressure corresponding to this temperature, $p_{sat}(29\ °C) = 0.04$ bar. This is the back pressure on the turbine. No moving parts are required to maintain the vacuum (although a pump is always fitted to extract the air which inevitably leaks in). [2]

3. (a)



[5]

(b) For the jet engine compressor,

$$T_{2S} = T_1\left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} = 220.0\times\left(\frac{0.90}{0.25}\right)^{0.4/1.4} = 317.2\,\text{K}$$

$$T_2 = T_1 + \frac{(T_{2S} - T_1)}{\eta_{comp}} = 220.0 + \frac{(317.2 - 220.0)}{0.88} = 330.4\,\text{K}$$

[2]

(c) To find the operating pressure p_3 we note that the work output of the turbine equals the work input to the compressor. These work terms are given by,

$$w_C = (h_3 - h_2) = c_p(T_3 - T_2) = c_pT_2\left(\frac{T_3}{T_2} - 1\right) = c_pT_2\left[\left(\frac{p_3}{p_2}\right)^{(\gamma-1)/\gamma} - 1\right]$$

$$w_T = (h_4 - h_5) = c_p(T_4 - T_5) = c_pT_5\left(\frac{T_4}{T_5} - 1\right) = c_pT_5\left[\left(\frac{p_4}{p_5}\right)^{(\gamma-1)/\gamma} - 1\right]$$

Noting that $p_4 = p_3$ we have (with p_3 in bar) :

$$330.4\times\left[\left(\frac{p_3}{0.90}\right)^{0.4/1.4} - 1\right] = 293.2\times\left[\left(\frac{p_3}{0.75}\right)^{0.4/1.4} - 1\right]$$

$$\left[\frac{330.4}{0.9^{(0.4/1.4)}} - \frac{293.2}{0.75^{(0.4/1.4)}}\right]p_3^{(0.4/1.4)} = 330.4 - 293.2$$

$$22.18\,p_3^{(0.4/1.4)} = 37.2 \qquad \rightarrow \qquad p_3 = 6.15\text{ bar}$$

[8]

(d) From the SFEE, the heat transferred per kg of cabin air supplied is,

$$q = (h_3 - h_4) = c_p(T_3 - T_4)$$

From $w_C = w_T$ we have $(T_3 - T_4) = (T_2 - T_5)$. Hence,

$$q = 1.005\times(330.4 - 293.2) = 37.4 \text{ kJ/kg of cabin air}$$ [2]

(e) For a relative humidity of 60 % in the cabin at 20 °C, the partial pressure of the water vapour must be,

$$p_v = 0.60\times p_{sat}(T_5) = 0.60\times0.02339 = 0.0140 \text{ bar}$$

The partial pressure of the air is therefore,

$$p_a = p_5 - p_v = 0.75 - 0.0140 = 0.7360 \text{ bar}$$

Writing the ideal gas equation for the water vapour and the air, we have,

$$\frac{p_v V}{p_a V} = \frac{m_v R_v T_5}{m_a R_a T_5}$$

Hence, if M_v and M_a are the molar masses of water vapour and air,

$$\frac{m_v}{m_a} = \frac{p_v R_a}{p_a R_v} = \frac{p_v M_v}{p_a M_a} = \frac{0.0140\times18}{0.7360\times29} = 0.0118 \text{ kg } H_2O \text{ per kg air.}$$ [3]

Section B

4 (a)

$$f = \phi_1(V, d, \mu, \rho) = 0 \tag{1}$$

where ϕ_1 is just some unknown function of the variables. There are five variables which can be expressed in terms of three fundamental quantities (M, L, T) and so from Buckingham's theorem 5-3=2 - there are two non-dimensional groups. There are a few equivalent possibilities but the most obvious are Strouhal number and Reynolds number. Since there are only two groups that describe the problem then one must be simply a function of the other.

$$\frac{fd}{V} = \phi_2\left(\frac{\rho V d}{\mu}\right) \tag{2}$$

where ϕ_2 is some unknown function. So if we fix the Reynolds number then the Strouhal number is also fixed.

(b) For full dynamic similarity then we must ensure that the Reynolds number in the wind-tunnel (WT) is the same as that in the ocean and so

$$\frac{\rho_{\text{ocean}} V_{\text{ocean}} d_{\text{ocean}}}{\mu_{\text{ocean}}} = \frac{\rho_{\text{WT}} V_{\text{WT}} d_{\text{WT}}}{\mu_{\text{WT}}} \tag{3}$$

$$V_{\text{WT}} = \frac{\rho_{\text{ocean}}}{\rho_{\text{WT}}} \frac{\mu_{\text{WT}}}{\mu_{\text{ocean}}} \frac{d_{\text{ocean}}}{d_{\text{WT}}} V_{\text{ocean}} = 60m/s \tag{4}$$

(c) Since we have now made the Reynolds number the same by the above choice then the Strouhal number is the same

$$\frac{f_{\text{WT}} d_{\text{WT}}}{V_{\text{WT}}} = \frac{f_{\text{ocean}} d_{\text{ocean}}}{V_{\text{ocean}}} \tag{5}$$

$$f_{\text{ocean}} = 2Hz \tag{6}$$

(d) The peak force is

$$F_p = \phi_3(V, d, \rho, \mu) \tag{7}$$

and so we have again 5 variables and 3 fundamental quantities and hence two non-dimensional numbers.

$$\frac{F_p}{\rho V^2 d} = \phi_4\left(\frac{\rho V d}{\mu}\right) \tag{8}$$

and since we have matched the Reynolds number then this new non-dimensional force must also be matched between the ocean and the wind-tunnel. This gives

$$\frac{F_{p\ \text{ocean}}}{F_{p\ \text{WT}}} = \frac{\rho_{\text{ocean}} V_{\text{ocean}}^2 d_{\text{ocean}}}{\rho_{\text{WT}} V_{\text{WT}}^2 d_{\text{WT}}} = 0.926 \tag{9}$$

(e) The most obvious problem is due to Mach number effects. In order to avoid the effects of compressibility then the Mach number should be < 0.3 which in the wind-tunnel would be about 100 m/s. This corresponds to a velocity in the ocean of only about 1.67 m/s which would be the highest velocity for which we could achieve full dynamic similarity. For the exam question it is not necessary to calculate the numbers but just to realise that Mach number effects limit the maximum speed for which full dynamic similarity can be achieved. Another problem noted by some students is the problem with approaching the natural frequency of the cylinder - note this is another non-dimensional parameter related to the material properties and dimensions of the cylinder which may be simply written as f/f_o where f_o is the natural frequency of transverse vibrations of the cylinder. Note in both these situations what happens is that full dynamic similarity fails when an additional parameter becomes important. When choosing the original parameters we implicitly assumed some terms in the equations of motion were small (eg. ones associated with compressibility). The truth is that we never have "complete" dynamic similarity since there are always extra very small effects that we ignore (eg. the effect of the earth's rotation in wind-tunnel experiments leads to Coriolis forces that are very tiny).

(f) If the cylinder is near the surface then flow over it will generate waves. The appropriate number here is the Froude number (Fr) which measures the ratio of inertia forces to gravity forces. This may be written in different forms but for this case it would be $\text{Fr} = V/\sqrt{gd}$ where d is the cylinder diameter. The other potential effect is that of surface tension which involves the Weber number, this is often small but it depends on the precise situation.

5 (a) Consider the forces on a thin ring of fluid between the two cylinders located at a radius r and with length dx and radial extent dr. The forces acting are shear forces on the inner and outer faces of the element and pressure forces on the up- and down-stream ends. The area of the ends (normal to the $x-$direction) are the same and equal to $2\pi r dr$ - the circumference of the ring times its thickness in the radial direction. The area of the inside face of the ring is $2\pi r dx$ and the area of the outside face is slightly bigger at $2\pi(r + dr)dx$. Balancing the forces in the $x-$direction:

$$p \times 2\pi r dr - (p + \frac{dp}{dx}dx)2\pi r dr - \tau \times 2\pi r dx + (\tau + \frac{d\tau}{dr}dr)2\pi(r + dr)dx = 0 \tag{1}$$

which becomes

$$-(\frac{dp}{dx})2\pi r dr dx = -2\pi\tau dr dx - \frac{d\tau}{dr}2\pi r dr dx - \frac{d\tau}{dr}2\pi(dr)^2 dx \tag{2}$$

and we drop the last term since it is third order in the small quantities whereas the others are only second order (so as we take $dr \to 0$ and $dx \to 0$ then the last term quickly becomes negligible compared with the other terms). If we divide through by $drdx$ then we are left with

$$\frac{dp}{dx}r = \tau + \frac{d\tau}{dr}r = \frac{d(\tau r)}{dr} \tag{3}$$

and dividing through by r we get the required results

$$\frac{1}{r}\frac{d(\tau r)}{dr} = \frac{dp}{dx} \tag{4}$$

(b) Now in order to find the shear stress (from which we can then find the velocity since we know the fluid is Newtonian) we integrate this equation with respect to r and find

$$\tau = \frac{r}{2}\frac{dp}{dx} + \frac{C}{r} \tag{5}$$

where C is a constant of integration. We do not have any known boundary conditions on the shear stress, only on the velocity and hence we need to integrate this again using the fact that the fluid is Newtonian i.e.

$$\tau = \mu\frac{du}{dr} = \beta r\frac{du}{dr} \tag{6}$$

Note that the fact that the viscosity varies with the radius does not change the fact that it is Newtonian - the shear stress is still independent of the shear rate (i.e. the velocity gradient). We substitute this back into our equation for the stress (5) and it turns out not to be too hard to integrate and we find

$$u = \frac{1}{2\beta}\frac{dp}{dx}r - \frac{C}{\beta r} + D \tag{7}$$

where D is a constant of integration. We now have two unknown constants and we also have two boundary conditions which are that the velocity is zero at $r = R_1$ and $r = R_2$, the inner and outer radii.

$$0 = \frac{1}{2\beta}\frac{dp}{dx}R_1 - \frac{C}{\beta R_1} + D \tag{8}$$

$$0 = \frac{1}{2\beta}\frac{dp}{dx}R_2 - \frac{C}{\beta R_2} + D \tag{9}$$

These are just simultaneous equations which can be solved in a variety of ways. One simple approach is to subtract (8) from (9) to eliminate D and then solve for C. This can then be substituted back to find D.

The details are not very important and the solutions for the constants are

$$C = -\frac{1}{2}\frac{dp}{dx}R_1R_2 \tag{10}$$

and

$$D = -\frac{1}{2\beta}\frac{dp}{dx}(R_1 + R_2) \tag{11}$$

Substituting these back into (7) and collecting terms gives

$$u = \frac{1}{2\beta}\frac{dp}{dx}\left(r + \frac{R_1R_2}{r} - (R_1 + R_2)\right) \tag{12}$$

We can finally check this by making sure it satisfies the boundary conditions - which it does ($u = 0$ at $r = R_1$ and $u = 0$ at $r = R_2$.

(c) This part is fairly straightforward. The drag force per unit length of either cylinder is just the shear stress acting on the surface multiplied by the area per unit length. The area of the outer pipe per unit length is $-2\pi R_2$ and for the inner is just $2\pi R_1$. The minus sign in the first one comes from the fact that the normal to the surface is in the opposite direction. This can be looked at in other ways. The velocity gradient on the top surface is actually negative in the direction normal to the wall and hence the shear stress is in the positive direction. The value of the shear stress on each surface may be found from (5) after inserting the value for C which we found in the last part so (5) becomes

$$\tau = \frac{r}{2}\frac{dp}{dx} - \frac{1}{2r}\frac{dp}{dx}R_1R_2 \tag{13}$$

and collecting terms,

$$\tau = \frac{dp}{dx}\left(\frac{r}{2} - \frac{R_1R_2}{2r}\right) \tag{14}$$

This leads to

$$\frac{F_2}{F_1} = \frac{-R_2\,(R_2 - R_1)}{R_1\,(R_1 - R_2)} = \frac{R_2}{R_1} \tag{15}$$

Note that this particular form arises due to the chosen variation of the viscosity which is convenient mathematically although unlikely in a real flow where it might vary approximately linearly between one value on the inner face and another on the outer face but would not necessarily be zero at $r = 0$. This would make the algebra more complicated but the method would be exactly the same.

6 **(a)** Simply apply continuity (or conservation of mass which is equivalent in this incompressible flow)between station 1 and station 2

$$9U_1h + U_13h = U_23h \tag{1}$$

$$U_2 = 4U_1 \tag{2}$$

(b) Apply continuity again between station 1 and station 3

$$9U_1h + U_13h = U_34h \tag{3}$$

$$U_3 = 3U_1 \tag{4}$$

(c) Apply the momentum equation for a control volume between station 1 and station 2

$$\Sigma F_x = \text{Momentum out} - \text{Momentum in} \tag{5}$$

The forces are due to the pressures acting on the faces of the control volume.

$$P_13h + P_1h - P_24h = \rho U_2^2 3h - (\rho U_1^2 3h + \rho(9U_1)^2 h) \tag{6}$$

$$\begin{aligned} 4P_1h - 4P_2h &= \rho 16U_1^2 3h - \rho U_1^2 3h - \rho 81 U_1^2 h \\ &= -36\rho U_1^2 \end{aligned}$$

$$P_2 = P_1 + 9\rho U_1^2$$

(d) Now apply the momentum equation between station 2 and station 3. Note that we now have an extra force since the plate exerts a force on the fluid in the control volume.

$$P_24h - P_34h + F = \rho U_3^2 4h - \rho U_2^2 3h \tag{7}$$

$$\begin{aligned} F = -\rho 12U_1^2 h + P_34h - P_24h = \\ \rho 3U_1^2 4h - P_34h + (P_1 + 9\rho U_1^2)4h \end{aligned}$$

$$F = -4h(P_1 - P_3 + 12\rho U_1^2)$$

We can check that it makes sense by considering what happens as we change the size of the different terms. For example if we increase P_1 keeping everything else constant then the force the plate exerts on the fluid increases in order to balance the increases force on the LHS of the control volume. Similar reasoning applies to changing P_3. If we increase U_1 then the force also increases to balance the larger change in momentum through the control volume. Note that the sign depends on how you define F (as positive in the positive x direction or alternatively as pointing upstream). The examiner was generous to the candidates with regard to the sign.

(e) The friction on the duct walls acts on the fluid in the direction opposite to the fluid motion and hence the pressure at station 3 would be reduced. If you think in terms of forces then the result is fairly obvious. On the actual exam there were a large number of suggestions, mostly from fuzzy thinking and trying to relate pressure and velocity through Bernoulli. Also suggestions that friction makes the flow slow down since that is what friction does in every day life (to bicycles, for example) . In fact it does slow the flow down near the wall to zero but can cause it to speed up near the centre so as to maintain the same flow rate. This is another way to look at the effect of boundary layers- they lead to an acceleration of the central flow which leads to a drop in the pressure - although there are some subtleties involved.