ENGINEERING TRIPOS PART IB

Thursday 7 June 2007 2 to 4

Paper 6

INFORMATION ENGINEERING

SOLUTIONS

SECTION A

*Answer not more than **two** questions from this section.*

1 (a) Explain briefly what is meant by the terms *gain margin* and *phase margin*. [3]

This part is bookwork. GM is amount in dB by which the gain falls short of unity when the phase is 180°. *PM is amount by which phase falls short of* 180° *when gain is unity. The GM and PM of a system provide measures of the margin of stability of the closed loop system.*

(b) A system has a transfer function

$$G(s) = \frac{2}{s(s+1)(s+2)}$$

and a controller connected as shown in Fig. 1. Assuming that $K(s) = 1$, use the Nyquist diagram for $G(s)$ shown in Fig. 2 to estimate:

(i) the gain and phase margins;

(ii) the frequency in radians/sec at which $|1 + K(s)G(s)|$ is minimum and the magnitude of the closed-loop frequency response at that frequency. [5]

The gain and phase margins can be measured directly from Fig. 2.

$$GM = \frac{1}{g_1} = \frac{10}{3.3} = 9.6dB \qquad (\text{exact ans} = 9.54)$$

$$PM = 32^\circ \qquad (\text{exact ans} = 32.6^\circ)$$

Min $|1 + G(s)|$ *occurs at* ≈ 0.9 *rad/sec.*

$$\left|\frac{G(s)}{1+G(s)}\right| = \left|\frac{g_2}{1+g_2}\right| \approx \frac{7.4}{4.2} = 1.76$$

Q1cont.

(c) A phase compensator is now used to replace $K(s)$ such that

$$K(s) = \frac{1+2s}{1+s/4}.$$

Sketch the Nyquist diagram for $K(s)G(s)$ on the copy of Fig. 2 provided. [6]

Simplest approach is to compute the magnitude g and phase θ of $K(s)$ at a few spot frequencies and then map the given points on $G(s)$ by multiplying by g and adding the additional phase θ. Eg.

ω	g	θ	$\lvert G(s)\rvert \times g$	$\arg G(s) + \theta$
3.0	4.9	43.7	0.28	8
2.0	3.7	49.4	0.57	30
1.5	3.0	51.0	0.86	32

Resulting Nyquist plot for $K(s)G(s)$ is shown in Fig. 2 below.

(d) Using your sketch, re-estimate the quantities found in part (b) for the compensated system. Based on these estimates, what can be inferred about the effect of the compensator on the closed-loop step response? [6]

As in part (a), the gain and phase margins can be measured directly from the new plot of $K(s)G(s)$ in Fig. 2.

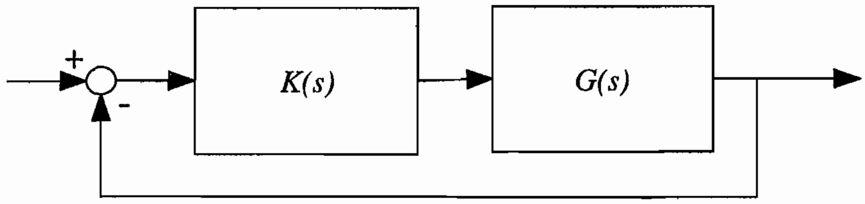
$$GM = \frac{10}{2.3} = 12.7dB \qquad \text{(exact ans = 12.6)}$$

$$PM = 54^\circ \qquad \text{(exact ans} = 53.7^\circ)$$

Min $|1+K(s)G(s)|$ occurs at ≈ 2.1 rad/sec.

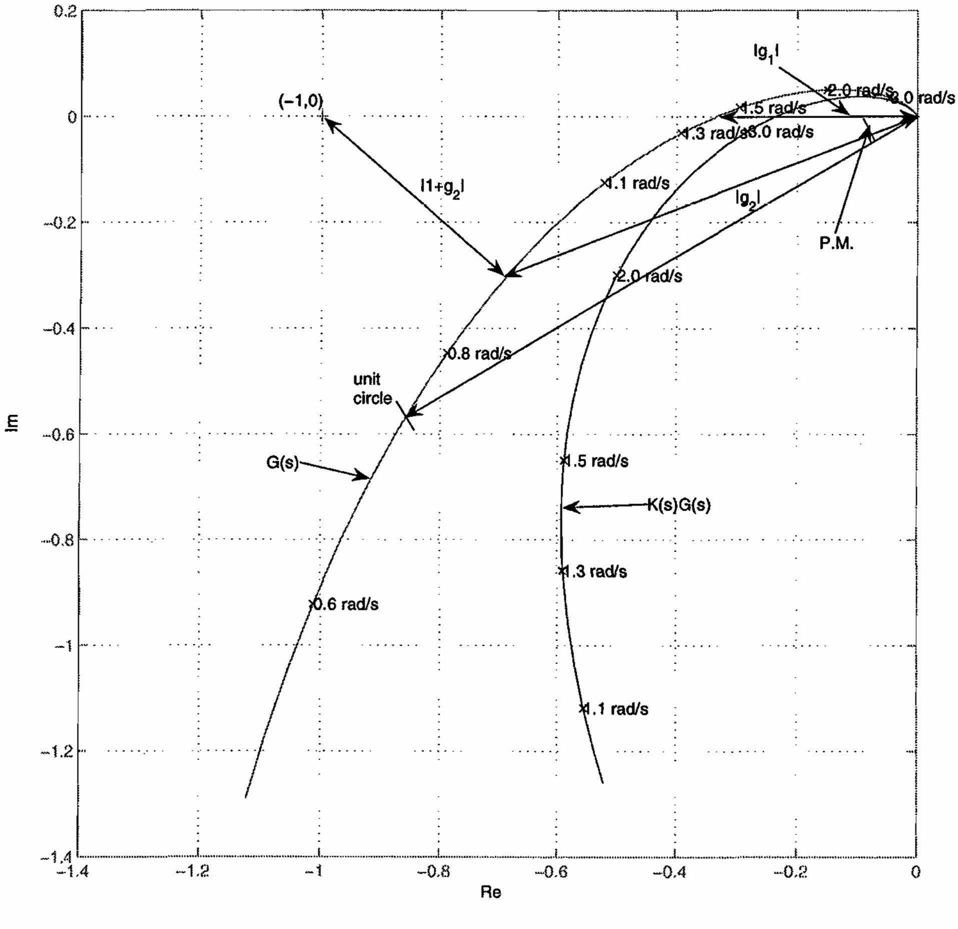
$$\left|\frac{K(s)G(s)}{1+K(s)G(s)}\right| \approx \frac{5.3}{5.8} = 0.91$$

These results show that the addition of the phase compensator has improved the gain and phase margins. Also, the (approximately) peak closed-loop gain has decreased and the frequency of the peak has increased. All this points to a more stable system with a faster response. Thus, the step response should exhibit a faster rise time and reduced overshoot.



+
-
K(s)
G(s)

Fig. 1



(−1,0)
|1+g2|
|g1|
|g2|
P.M.
unit circle
G(s)
K(s)G(s)
2.0 rad/s
3.0 rad/s
1.5 rad/s
1.3 rad/s
1.1 rad/s
0.8 rad/s
0.6 rad/s
Im
Re

Fig. 2

2 Figure 4 shows a feedback control system in which

$$G(s) = \frac{20}{s(s^2+6s+20)}$$

and Fig. 5 shows the magnitude of the frequency response of $G(s)$. The control system is to be designed such that the steady-state response to disturbances is zero and the frequency range over which the gain of the closed-loop system exceeds 0.5 should extend to at least 7 radians/sec. You may assume that the closed loop system is always stable.

(a) Derive an expression for $\bar{y}(s)$ as a function of the reference input $\bar{r}(s)$ and the disturbance $\bar{d}(s)$. [4]

This is bookwork.

$$\begin{aligned} \bar{y}(s) &= K(s)G(s)(\bar{r}(s)-\bar{y}(s)) + G(s)\bar{d}(s) \\ &= \frac{K(s)G(s)}{1+K(s)G(s)}\bar{r}(s) + \frac{G(s)}{1+K(s)G(s)}\bar{d}(s) \end{aligned}$$

(b) For $K(s) = 2$, find the steady state response of $y(t)$ when:

(i) $r(t) = \cos(7t)$ and $d(t) = 0$;

(ii) $r(t) = 0$ and $d(t) = H(t)$;

(iii) $r(t) = \cos(7t)$ and $d(t) = H(t)$,

where $H(t)$ is the unit step function. [5]

System is linear so the responses to $r(t)$ and $d(t)$ can be calculated separately and summed.

For $K(s) = k = 2$,

(i) Response to $r(t) = \cos(7t)$ will be $A\cos(7t+\phi)$ where

$$A = \left|\frac{kG(s)}{1+kG(s)}\right| = \left|\frac{k}{1/G(s)+k}\right| = \left|-\frac{40}{254+203j}\right| = 0.123$$

$$\phi = \tan^{-1}\left(-\frac{40}{254+203j}\right) = 2.47$$

Q2cont.

(ii) response to $d(t) = H(t)$ is

$$y(t)|_{t\to\infty} = \left[s.\frac{1}{s}\left(\frac{G(s)}{1+kG(s)}\right)\right]_{s\to 0} = \left[\frac{1}{1/G(s)+k}\right]_{s\to 0} = \frac{1}{k} = 0.5$$

(iii) System is linear so the responses to $r(t) = cos(7t)$ and $d(t) = H(t)$can be summed.

(c) The proportional controller is now extended by adding integral and derivative terms such that $K(s) = 2+1/s+3s$. By expressing $K(s)$ as the product of a quadratic and the term $1/s$, draw the magnitude response of $K(s)G(s)$ on the additional copy of Fig. 5. [6]

The bode plot for $K(s)G(s)$ can be found by first noting that

$$K(s) = 2+1/s+3s = \frac{1}{s}(3s^2+2s+1)$$

then separately plotting $1/s$ and $3s^2+2s+1$ on the supplied graph, finally adding them to $G(s)$. The first of these terms is a straight line of gradient 20dB/decade intercepting the 0dB axis at 1 rad/s. The second is a standard second order term with a turning point at $1/\sqrt{3} = 0.58$ and then rising at 40 dB/decade. The damping factor is ≈ 0.6 so there is no significant resonance peak. See Fig. 3.

(d) Verify that the control system now meets the design requirements. [5]

Response to $d(t) = H(t)$ is now

$$y(t)|_{t\to\infty} = \left[s.\frac{1}{s}\left(\frac{G(s)}{1+K(s)G(s)}\right)\right]_{s\to 0} = \left[\frac{1}{1/G(s)+K(s)}\right]_{s\to 0} = 0$$

so zero steady-state error to a disturbance.

For the frequency response, at 7 rad/sec $K(s)G(s) \approx 1$ from the Bode plot. Magnitude of frequency response

$$A = \left|\frac{K(s)G(s)}{1+K(s)G(s)}\right|$$

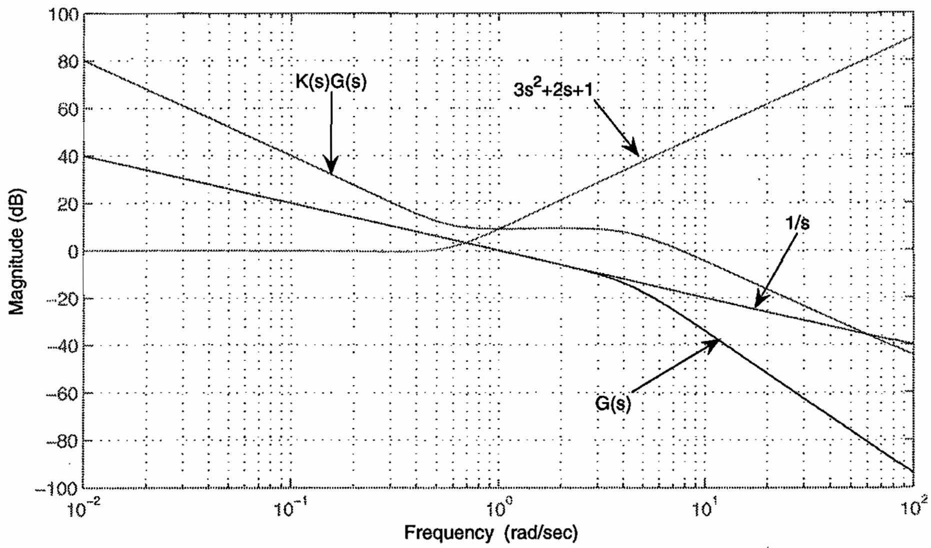


Fig. 3

In the "worst case" of 0 phase, $A \approx 0.5$, for any other phase A will be larger. Hence the gain is > 0.5 at 7 rads/sec meaning that the frequency response requirement is met.

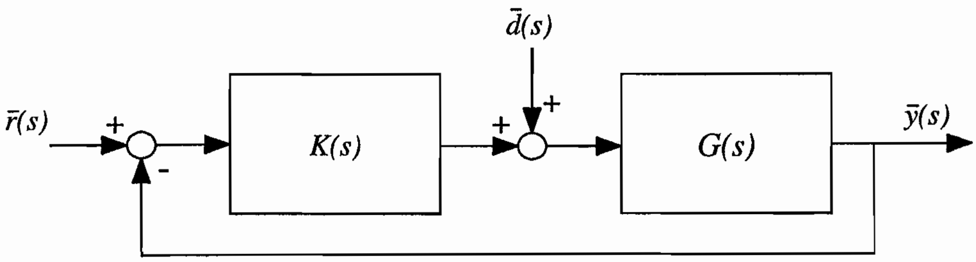
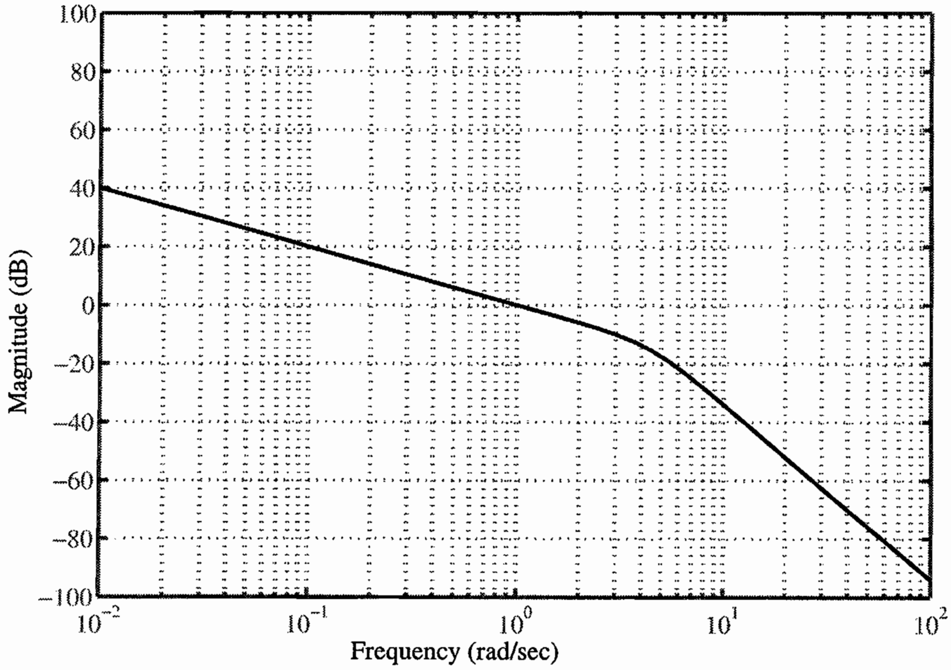


Fig. 4



100
80
60
40
20
0
−20
−40
−60
−80
−100
Magnitude (dB)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Fig. 5

3 (a) For any system with a proper rational transfer function, state the conditions in terms of pole positions for the system to be unstable. [4]

There are two conditions:

(i) if any pole has a +ve real part, system will be unstable;

(ii) if any repeated pole lies on the imaginary axis, system will be unstable.

(b) The feedback control system shown in Fig. 7 has

$$G(s) = \frac{1}{(s-1)(s+3)}\,.$$

If $K(s) = k$, show that the poles of the closed-loop system are given by

$$s = -1 \pm \sqrt{4-k}\,.$$

Hence, determine the range of k for which the feedback system is stable. [6]

In closed loop system, the transfer function is $\frac{K(s)G(s)}{1+K(s)G(s)}$*. Poles are therefore given by the solution to* $1+K(s)G(s) = 0$*. Hence,*

$$(s+1)(s+3) = -k; \quad \textit{therefore} \quad s^2+2s+(k-3) = 0$$

From which the result $s = -1 \pm \sqrt{4-k}$ *follows.*

This result shows that when k is very small, the closed loop poles are real and one of them lies in the +ve half-plane. Most positive root is $s = \sqrt{4-k} - 1$*.*
When $k < 3$, $s > 0$ *hence system is unstable.*
When $3 \leq k < 4$, $s \leq 0$*, hence system is stable.*
When $4 < k$*, poles become complex with real part -ve, hence system is stable.*

Overall, system is stable for $3 \leq k < \infty$*.*

(c) When $k = 20$, compute the damping factor of the closed-loop system and hence sketch the system step response. Your sketch should show the peak overshoot and the approximate time at which the peak occurs. [5]

Q3cont.

When $k = 20$, closed loop poles are at $s = -1 \pm 4j$. Therefore

$$\omega_n^2 = 1^2 + 4^2, \qquad \textit{hence} \qquad w_n = \sqrt{17}$$

$$c = \frac{1}{\omega_n} = \frac{1}{\sqrt{17}} = 0.243$$

Step response can now be "copied" from Mechanics Data Book - see Fig. 6

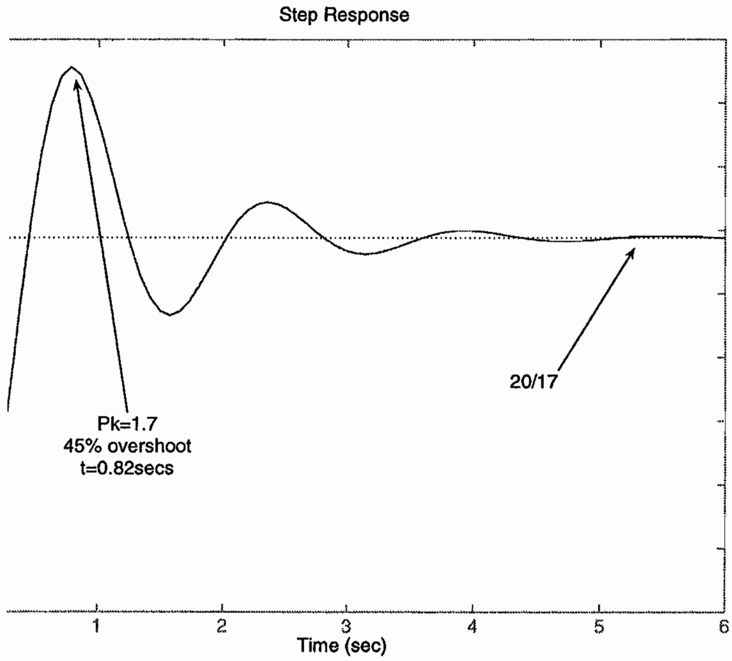


Fig. 6

(d) Show that adding a derivative term to the controller so that $K(s) = k + k_d s$ can increase damping. Using the same value $k = 20$, find the value of k_d required to give a damping factor of 0.5. [5]

With derivative control added

$$1 + K(s)G(s) \quad \Rightarrow \quad 1 + \frac{k + k_d s}{(s^2 + 2s - 3)}$$

therefore, the characteristic equation becomes

$$s^2 + (2 + k_d)s + 17 = 0$$

Hence, the derivative gain term adds directly to the damping.

$$2c\omega_n \equiv 2 + k_d$$

To achieve a damping factor of $c = 0.5$, $k_d = 2c\omega_n - 2 = \sqrt{17} - 2 = 2.12$.

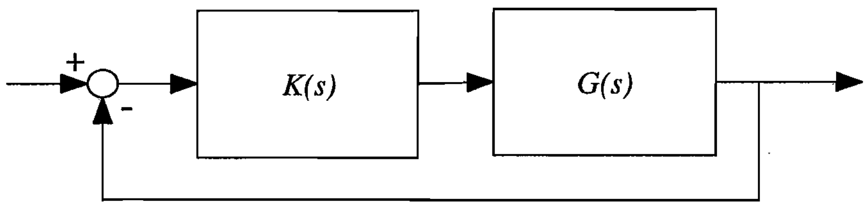


Fig. 7

SECTION B

*Answer not more than **two** questions from this section.*

4 The signal $x(t)$ shown in Fig. 9 is a periodic zero-mean triangular waveform with repetition frequency 1 kHz and peak amplitude 1 volt.

(a) The signal $x(t)$ is low-pass filtered with cut-off frequency 10 kHz and sampled at 20 kHz. Sketch the resulting spectrum. [5]

The spectrum of a periodic signal is a line spectrum where each line corresponds to a single frequency component. The amplitudes and phases of each spectral line are given by the Fourier Series coefficients, which for the triangle shown are in the databook. The filtering will limit the spectrum to 5 components at frequencies 1, 3, 5, 7 and 9 kHz plus the corresponding -ve components at -1, -3, -5, -7, -9 kHz. The effect of the sampling is then to repeat this spectrum centred at multiples of the sampling frequency. Fig. 8 illustrates.

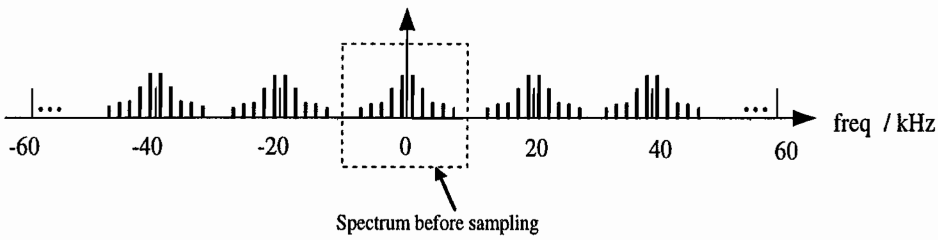


Fig. 8

(b) Show that the rms noise voltage introduced by an ideal uniform quantiser with a step size δV is given by $\delta V/\sqrt{12}$. [5]

This is standard bookwork. The key assumption is that the difference v between the sampled value and the true value will be distributed uniformly across the range $-\delta V/2$

to $\delta V/2$. Hence, the mean square noise is just the expected value of v^2, i.e.

$$v_{rms}^2 = \int_{-\delta v/2}^{\delta v/2} p(v) v^2 dv$$

where $p(v)$, the probability of v, is $1/\delta v$. The result $v_{rms} = \delta v/\sqrt{12}$ follows immediately from this.

(c) If the quantiser is 8-bit and the range is $\pm$ 5 volts, calculate the signal-to-noise power ratio (SNR) of the quantised signal in decibels. How could this SNR be improved without increasing the number of quantisation levels? [5]

For the given sampler,

$$\delta v = 10/256 = 0.0391. \qquad \textit{Hence,} \quad v_{rms} = 0.0113$$

We also require the rms voltage V_{rms} of the signal.

$$V_{rms}^2 = \frac{4}{T} \int_0^{T/4} \left(\frac{4t}{T}\right)^2 dt = 1/3$$

Hence, $V_{rms} = 0.5774$.
The required signal to noise ratio is then $20\log(V_{rms}/v_{rms}) = 34.19dB$.

This SNR could be improved simply by scaling the input signal by a factor of 5 so that all of the available quanisation range is used. This would increase the SNR by $20\log 5 = 14dB$.

(d) Calculate the bit rate of the signal sampled in part (a) and quantised in part (c). Determine the minimum bandwidth that a communications channel should have to reliably support this bit rate at an SNR of 5dB. [5]

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*Bit rate = $f_s N$ where f_s is the sample rate and N is the number of bits per sample. Hence, for the given case, bit rate = 20,000 * 8 = 160kbits/sec.*

According to Shannon's formula,

$$C = B\log_2(1 + S/N)$$

Hence, $B = \frac{160000}{\log_2(1+3.1623)} = \frac{160000}{2.0574} = 77.8$ kHz.

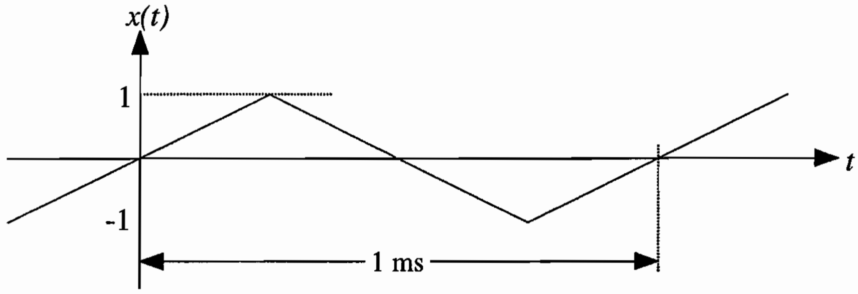

Fig. 9

5 (a) With regard to amplitude modulation schemes, explain what is meant by *single side-band (SSB)* and *double side-band (DSB)* operation and briefly discuss their relative merits. [4]

Bookwork. Amplitude modulation involves shifting the baseband spectrum (centred on zero frequency) so that it is centred on a frequency which can be transmitted. Since the baseband spectrum has both +ve and -ve frequencies, when it is shifted both components become real resulting in double side-bands (DSB). If one of these side bands is suppressed, the result is a single-side band (SSB) modulation scheme. See Fig. 10.

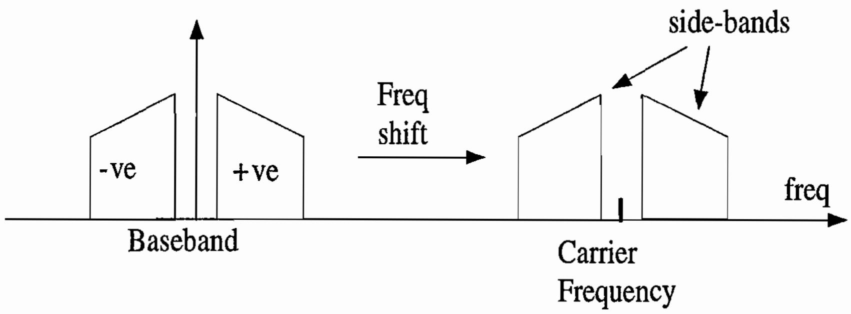

Fig. 10

(b) By considering the case of $x(t) = \cos(\omega t)$, show that simple multiplication of $x(t)$ by a carrier wave $\cos(\omega_c t)$ where $\omega_c \gg \omega$ leads to double side-bands. [4]

Let $x(t) = \cos \omega t$ *and let* $x(t) \rightleftharpoons X(\omega)$ *be a transform pair, then*

$$\begin{aligned} x_m(t) &= \cos \omega t \cos \omega_c t \\ &= \frac{1}{2} \cos \omega t \left[e^{j\omega_c t} + e^{-j\omega_c t} \right] \end{aligned}$$

Hence,

$$x_m(t) \rightleftharpoons \frac{1}{2} X(\omega - \omega_c) + \frac{1}{2} X(\omega + \omega_c)$$

Since $X(\omega)$ *has components at* $\pm\omega$*, the shifted version* $X(\omega + \omega_c)$ *will have the same components at* $\omega_c \pm \omega$*. Hence, the modulated signal has double side-bands.*

Q5cont.

(c) By considering the transmission of speech with a bandwidth of 300 Hz to 3 kHz, explain why it is not practical to obtain SSB modulation by directly filtering $x(t)\cos(\omega_c t)$ to suppress one of the side-bands. [4]

Let ω_o be the lowest required frequency in the baseband signal (eg for speech this would be about 30 Hz for reasonable quality or 300 Hz for telephone speech). In the modulated signal, this would leave a gap of $2\omega_o$ between the two bands. In order to suppress one of the side-bands, a bandpass filter would be needed with a very sharp cut-off to achieve useful suppression. In fact, too sharp to be practical.

For example, suppose that the unwanted side-band must be at least 30dB lower than the retained side-band. Even for low quality telephone speech, this would require 30dB to be achieved over a frequency range of just 600Hz. Suppose that the carrier is at 1MHz, then the cut-off rate would need to be about 10^5 dB's per decade!

(d) Fig. 11 shows a block diagram of an alternative amplitude modulation scheme. The modulation signals are $m_1(t) = 2\cos(Bt)$ and $m_2(t) = \cos((\omega_c - B)t)$, where B is the cut-off frequency of the lowpass filter, and ω_c is the carrier frequency. If $x(t) = \cos(\omega t)$ and $\omega < B \ll \omega_c$, derive an expression for the output $y(t)$ and hence demonstrate that this is an SSB scheme. [8]

As shown in Fig. 11, let the upper signal path be denoted by subscript u and the lower by subscript l. Then

$$
\begin{aligned}
q_u(t) &= 2\cos Bt\cos\omega t \\
&= \cos(B-\omega)t + \cos(B+\omega)t \\
q_l(t) &= 2\sin Bt\cos\omega t \\
&= \sin(B-\omega)t + \sin(B+\omega)t
\end{aligned}
$$

After filtering these become

$$
\begin{aligned}
r_u(t) &= \cos(B-\omega)t \\
r_l(t) &= \sin(B-\omega)t
\end{aligned}
$$

Modulating again gives

$$
s_u(t) = \cos(\omega_c - B)t\cos(B-\omega)t
$$

Q5cont.

$$s_l(t) = \sin(\omega_c - B)t \sin(B - \omega)t \tag{1}$$

Hence,

$$\begin{aligned} y(t) &= s_u(t) - s_l(t) \\ &= \cos(\omega_c - B)t \cos(B - \omega)t - \sin(\omega_c - B)t \sin(B - \omega)t \\ &= \cos(\omega_c - B + B - \omega)t \\ &= \cos(\omega_c - \omega)t \end{aligned}$$

Hence, $y(t)$ consists of just the lower side-band of the modulated signal.

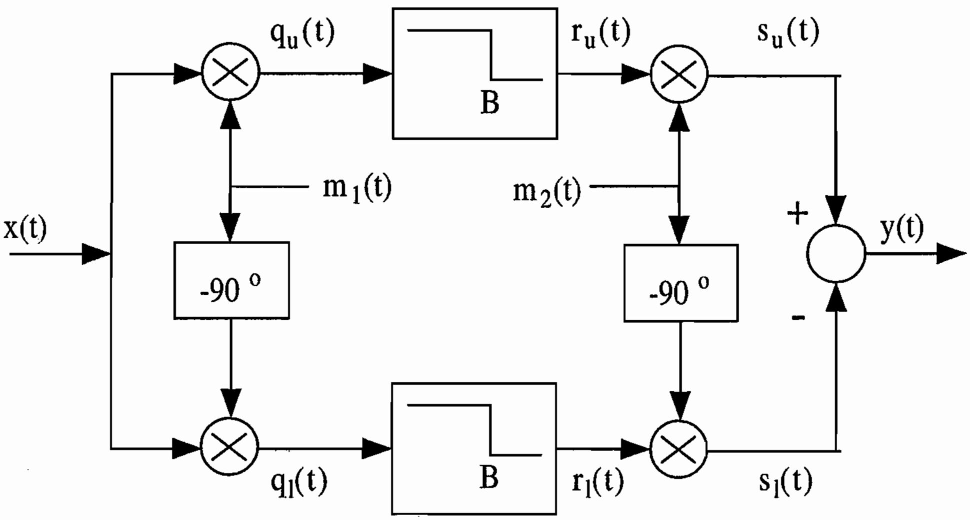

Fig. 11

6 (a) (i) Show that if $x(t) \rightleftharpoons X(\omega)$ where $X(\omega)$ is the Fourier Transform of $x(t)$, then

$$x(t-t_o) \rightleftharpoons e^{-j\omega t_o} X(\omega)\,.$$

[3]

Let $x(t-t_o) \rightleftharpoons X'(\omega)$ then using the substitution $\tau = t - t_o$

$$\begin{aligned}
X'(\omega) &= \int_{-\infty}^{+\infty} x(t-t_o) e^{-j\omega t} dt \\
&= \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} e^{-j\omega t_o} d\tau \\
&= e^{-j\omega t_o} \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} \\
&= e^{-j\omega t_o} X(\omega)
\end{aligned}$$

(ii) Show that the Fourier Transform of $\sin(\omega_o t)$ is

$$\frac{\pi}{j}\left[\delta(\omega-\omega_o) - \delta(\omega+\omega_o)\right]$$

and find the Fourier Transform of $\sin(\omega_o t + \phi)$. [5]

To show that $\sin(\omega_o t) = \frac{\pi}{j}\left[\delta(\omega-\omega_o) - \delta(\omega+\omega_o)\right]$, substitute the RHS into the formula for $x(t)$

$$\begin{aligned}
x(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\pi}{j}\left[\delta(\omega-\omega_o) - \delta(\omega+\omega_o)\right] e^{j\omega t} d\omega \\
&= \frac{1}{2j}\left[e^{j\omega_o t} - e^{-j\omega_o t}\right] = \sin(\omega_o t)
\end{aligned}$$

$sin(\omega_o t + \phi) = sin(\omega_o(t + \frac{\phi}{\omega_o}))$ hence using (i) above the transform is

$$\frac{\pi}{j}\left[\delta(\omega-\omega_o) - \delta(\omega+\omega_o)\right] e^{j\phi\omega/\omega_o}$$

(b) (i) The signal $x(t) = \sin(2\pi f_o t)$ is sampled at times $t = 0$ ms, $t = 0.25$ ms, $t = 0.5$ ms and $t = 0.75$ ms to produce the sequence $x_n = \{0, 1, 0, -1\}$. What is the sampling frequency f_s and what are the possible values of f_o for $f_o > 0$? [5]

The sampling frequency $f_s = 1/T_s$ where $T_s = 0.25msecs$ is the sampling period. Hence, $f_s = 4$ kHz. The lowest possible frequency f_o is clearly 1 kHz. But note that sine waves at frequency 5 kHz, 9 kHz, 13kHz, ... also meet the constraints.

(ii) Assuming that $f_o \leq \frac{1}{2} f_s$, calculate the discrete Fourier Transform (DFT) of x_n. To what frequencies do the individual terms in the DFT correspond? [5]

This is a simple application of the formula $X_k = \sum_0^3 x_n e^{-kn\frac{\pi}{2}j}$ giving $X_k = \{0, -2j, 0, +2j\}$.

(iii) Explain why there is more than one non-zero term in the DFT even though $x(t)$ is a pure sinusoid. [2]

Sampling causes the baseband spectrum to be replicated at multiples of the sampling frequency. The DFT spectrum contains the upper (i.e. positive frequency) half of the baseband spectrum plus the lower half of the first repetition. Thus, the 1 kHz component also appears at $f_s - 1$, ie 3 kHz.

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