ENGINEERING TRIPOS PART IB

2007

Paper 7

MATHEMATICAL METHODS

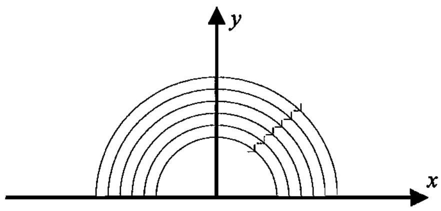
Solutions

SECTION A

1 (a) The double integral is:

$$\int\!\!\int \mathrm{e}^{-(x^2+y^2)}\,dx\,dy = \int_{r=a}^{b}\int_{\theta=0}^{\pi} \mathrm{e}^{-r^2} r\,dr\,d\theta$$
$$= \frac{\pi}{2}(\mathrm{e}^{-a^2}-\mathrm{e}^{-b^2})\ .$$

(b) The field lines of **B** are:



and, since $d\mathbf{l} = r d\theta$, the line integral is:

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \int_0^{\pi} -\frac{1}{2}\mathrm{e}^{-b^2}\,d\theta + \int_{\pi}^{0} -\frac{1}{2}\mathrm{e}^{-a^2}\,d\theta$$
$$= \frac{\pi}{2}(\mathrm{e}^{-a^2}-\mathrm{e}^{-b^2})\ .$$

(c) The curl of **B** is

$$\nabla\times\mathbf{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\hat{\mathbf{e}}_z = \frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{1}{2}\mathrm{e}^{-r^2}\right)\hat{\mathbf{e}}_z$$
$$= \mathrm{e}^{-r^2}\hat{\mathbf{e}}_z\ .$$

The answers to (a) and (b) are the same because, by Stokes' theorem,

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \int\!\!\int(\nabla\times\mathbf{B})\cdot d\mathbf{A} = \int_a^b\int_0^{\pi}\mathrm{e}^{-r^2}\,r\,dr\,d\theta$$
$$= \frac{\pi}{2}(\mathrm{e}^{-a^2}-\mathrm{e}^{-b^2})\ .$$

(d) $\nabla\cdot\mathbf{B} = 0$ by inspection. Therefore **B** does have a vector potential because every solenoidal vector field has a vector potential.

2 (a) The expression for $\mathbf{q}$ is:

$$\begin{aligned}\mathbf{q} &= -D\nabla c \\ &= D\left\{2x(1-y^2)\hat{\mathbf{e}}_x + 2y(1-x^2)\hat{\mathbf{e}}_y\right\} \ .\end{aligned}$$

(b) There is no flow across the top and bottom faces. The four side faces are identical so we evaluate one of the faces and multiply by four. The integral is shown here for the face at $y = 1$, where $d\mathbf{s} = \hat{\mathbf{e}}_y$:

$$\begin{aligned}\int_{x=-1}^{1} \mathbf{q}\cdot d\mathbf{s} &= D\int_{-1}^{1} 2(1-x^2)\,dx \\ &= 8/3 \ \text{ per unit depth in the } z\text{-direction.}\end{aligned}$$

There are four faces, each of depth 2 units in the z-direction so the rate at which methane escapes is $64D/3$.

(c) Methane is produced non-uniformly within the bin. If we consider a small volume element within the bin, with volume dV and surface $d\mathbf{S}$ then, by Gauss' theorem:

$$\int\int\int(\nabla\cdot\mathbf{q})dV = \int\int \mathbf{q}\cdot d\mathbf{S}$$

The right hand side of this equation is the flux out of this volume element. The left hand side is the integral of the rate of production within the volume element. Therefore the (non uniform) rate of production per unit volume is $\nabla\cdot\mathbf{q}$.

$$\begin{aligned}\nabla\cdot\mathbf{q} &= D\left\{2(1-y^2)+2(1-x^2)\right\} \\ &= 2D(2-x^2-y^2) \ .\end{aligned}$$

(d) The rate of production per unit volume must be integrated over the whole bin and is found to be the same as the answer to (b).

$$\begin{aligned}\int\int\int(\nabla\cdot\mathbf{q})\,dV &= 2\int_{-1}^{1}\int_{-1}^{1} 2D(2-x^2-y^2)\,dx\,dy \\ &= 4D\int_{-1}^{1}\left(\frac{10}{3}-2x^2\right)\,dx = 4D\left(\frac{20}{3}-\frac{4}{3}\right) \\ &= \frac{64D}{3} \ .\end{aligned}$$

(TURN OVER

3 (a) Assume that $\nabla^2 T(r,\theta,\phi) = R(r)F(\theta)P(\phi)$. If $\nabla^2 T = 0$ then, by substitution and differentiation:

$$\frac{FP}{r^2}(r^2 R'' + 2rR') + \frac{RP}{r^2 \sin\theta}(F' \cos\theta + F'' \sin\theta) + \frac{RF}{r^2 \sin^2\theta} P'' = 0 \ .$$

Multiply by $r^2/(FPR)$

$$\frac{r^2 R'' + 2rR'}{R} + \frac{F' \cot\theta + F''}{F} + \frac{P''}{\sin^2\theta P} = 0 \ .$$

These three terms must independently be equal to constant values. Taking the radial solution to be equal to k and re-arranging gives:

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - kR = 0 \ . \tag{1}$$

(b) Taking the other two solutions to be equal to k_2 and k_3 gives:

$$\frac{F' \cot\theta + F''}{F} = k_2 \ ,$$

$$\frac{P''}{P} = k_3 \sin^2\theta \ .$$

(c) Differentiating the given expression for $R_n(r)$ gives:

$$\begin{aligned} R_n(r) &= A_n r^n + B_n r^{-(n+1)} \\ \Rightarrow \ R_n'(r) &= A_n n r^{(n-1)} - (n+1)B_n r^{-(n+2)} \\ \Rightarrow \ R_n''(r) &= A_n n(n-1) r^{(n-2)} - (n+1)(n+2)B_n r^{-(n+3)} \ . \end{aligned}$$

Substituting into (1) and re-arranging gives:

$$A_n r^n \left[n(n-1) + 2n - k\right] + B_n r^{-(n+1)} \left[(n+1)(n+2) - 2(n+1) - k\right] = 0 \ .$$

Both expressions in the square brackets are zero if $k = n(n+1)$ so the differential equation is satisfied.

4 (a) $a = 2$ can be obtained in several different ways, e.g. by Gaussian elimination, solving the 2×2 system for x and y, and using $2x = a$.

(b)

$$Q = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ 2/3 & -2/3 \end{bmatrix}$$

and

$$R = \begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}$$

(c) First compute $Q^t\mathbf{b}$.

$$Q^t\mathbf{b} = \begin{bmatrix} 17/3 \\ 16/3 \end{bmatrix}$$

Then solve $R\mathbf{x} = Q^t\mathbf{b}$ by substitution:

$$y = 8/3$$

and

$$x = 1/9$$

(d) In general, the solution to this least squares problem is:

$$x' = (A^tC^tCA)^{-1}A^tC^tCb$$

the solution x' will be identical to $\bar{x}$ when $C^tC = I$. Otherwise, in general they will differ.

5 (a) This is the sum of n independent binary random variables. The mean is np.

(b) The sum of n independent binary variables has a Binomial(n, p) distribution.

(c) Each term in the sum (where $i \neq j$) has probability p^3 of being 1, otherwise it is 0. When $i = j$, the probability of the term being 1 is p^2. Combining terms:

$$p^3(n^2 - n) + p^2 n$$

(d) Again, we divide into two cases. Where $i \neq j$ the expectation is 0. Where $i = j$ the expectation is p. Combining terms:

$$pn$$

6 (a)

$$\begin{aligned}
g(s) &= \int \frac{e^{-st}\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} dt \\
g(s) &= \frac{\lambda^k}{(k-1)!}\int e^{-(s+\lambda)t} t^{k-1} dt \\
&= \frac{\lambda^k}{(k-1)!}\frac{(k-1)!}{(s+\lambda)^k} \\
&= \left(\frac{\lambda}{s+\lambda}\right)^k
\end{aligned}$$

(b) Several ways to show this. The easiest way is to show that the m.g.f. for the exponential distribution is

$$\frac{\lambda}{s+\lambda}.$$

Since the sum of independent random variables corresponds to the product of m.g.f.s we obtain the above result.

(c) Several ways to do this. Starting from the m.g.f. we can compute the mean by evaluating $-g'(0)$.

$$g'(s) = -k\lambda^k(s+\lambda)^{-k-1}$$

$$E[t] = -g'(0) = \frac{k}{\lambda}$$

Similarly for the variance:

$$g''(s) = -k\lambda^k(-k-1)(s+\lambda)^{-k-2}$$

$$E[t^2] = g''(0) = -k\lambda^k(-k-1)(\lambda)^{-k-2} = \frac{k(k+1)}{\lambda^2}$$

$$Var[t] = \frac{k(k+1)}{\lambda^2} - \left(\frac{k}{\lambda}\right)^2 = \frac{k}{\lambda^2}$$

END OF PAPER