ENGINEERING TRIPOS PART IB

Friday 8 June 2007 9 to 11

Paper 7

MATHEMATICAL METHODS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

*All questions carry the same number of marks.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

*Answers to questions in each section should be tied together and handed in separately.*

*There are no attachments.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

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## SECTION A

*Answer not more than **two** questions from this section.*

1 (a) By a suitable change of variable, evaluate the double integral

$$\int\int e^{-(x^2+y^2)}\ dx\,dy\ ,$$

where the domain of integration is the area bounded by the $x$-axis and the semi-circles of radius $a$ and $b$ shown in Fig. 1. [4]

(b) Consider the vector field in two-dimensional polar coordinates

$$\mathbf{B} = -\frac{1}{2r}\, e^{-r^2}\, \hat{\mathbf{e}}_\theta\ .$$

Sketch the field lines and evaluate the line integral

$$\oint_C \mathbf{B}\cdot d\mathbf{l}\ ,$$

where $C$ is the curve that bounds the shaded area in Fig. 1. [6]

(c) Calculate $\nabla\times\mathbf{B}$. Why are the answers to (a) and (b) the same? [8]

(d) Calculate $\nabla\cdot\mathbf{B}$. Does $\mathbf{B}$ have a vector potential? [2]

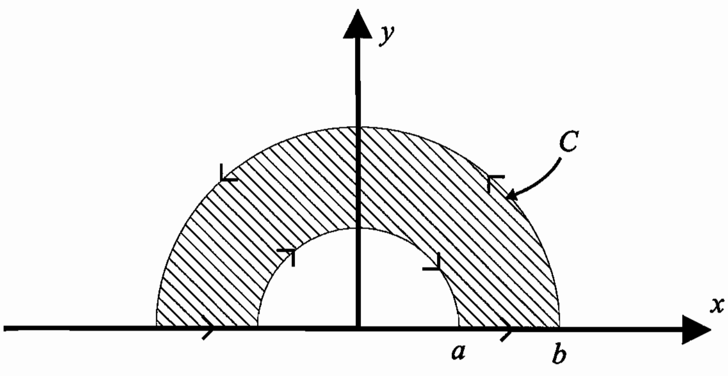


Fig. 1

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2 A container for decaying garden waste with dimensions $2 \times 2 \times 2$ units is sealed on its top and bottom faces but has porous side faces. Methane is produced inside the bin and diffuses according to $\mathbf{q} = -D\,\nabla c$, where $\mathbf{q}$ is the flux of methane, $D$ is the diffusivity and $c$ is the concentration of methane. The diffusivity is constant within the bin.

The coordinates $x$, $y$ and $z$ are measured from the centre of the bin. The faces at $z = -1$ and $z = 1$ are sealed. The methane concentration in the bin is given by

$$c = (1 - x^2)(1 - y^2) \ .$$

(a) Derive an expression for the flux of methane, $\mathbf{q}$. [4]

(b) By evaluating a suitable integral over the sides of the bin, calculate the rate at which methane escapes from the bin. [6]

(c) By considering a small volume element within the bin, derive an expression for the (non-uniform) rate of production of methane per unit volume within the bin. Explain your reasoning. [6]

(d) Verify the results of (b) and (c) using Gauss' theorem. [4]

3 Laplace's equation in spherical polar coordinates may be written as:

$$\nabla^2 T(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} = 0$$

(a) Assume that the solution can be obtained by separation of the variables. Show that the radial component of the solution, $R(r)$ , satisfies: [8]

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - kR = 0\,,$$

where $k$ is a constant.

(b) Write down the differential equations satisfied by the $\theta$ component and the $\phi$ component of the solution. [2]

(c) Show that radial solutions of the form

$$R_n(r) = A_n r^n + B_n r^{-(n+1)}$$

are permitted, where $A_n$ and $B_n$ are constants. Find $k$ in terms of $n$ . [10]

**SECTION B**

*Answer not more than **two** questions from this section.*

4 Consider the following linear system of three equations written in matrix form:

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ a \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 0 \end{bmatrix}.$$

(a) Find a value of $a$ which produces a unique solution for $\begin{bmatrix} x \\ y \end{bmatrix}$. [3]

(b) Compute the QR factorisation of $A$ using the Gram-Schmidt orthogonalisation process. [6]

(c) Find the least squares solution for $\begin{bmatrix} x \\ y \end{bmatrix}$ when $a = 0$. [6]

(d) Let $\bar{\mathbf{x}}$ be the least squares solution to $A\mathbf{x} = \mathbf{b}$, where $A$ is an $m \times n$ matrix, and let $C$ be an $m \times m$ matrix. What conditions on $C$ will make $\bar{\mathbf{x}}$ the least squares solution to

$$CA\mathbf{x} = C\mathbf{b}\ ?$$

[5]

5 Let $A$ be an $n \times n$ matrix, with elements $a_{ij}$. Assume that each element $a_{ij}$ is an independent binary random variable taking the value $a_{ij} = 1$ with probability $p$ and the value $a_{ij} = 0$ with probability $1 - p$. Similarly, let $\mathbf{b}$ be an $n \times 1$ vector with elements $b_i$, where each $b_i$ is also an independent binary variable taking the value 1 with probability $p$.

(a) Compute the expected value of the random variable

$$y = \mathbf{b}^t \mathbf{b} = \sum_{i=1}^{n} b_i^2$$

in terms of $n$ and $p$. [4]

(b) What is the distribution of $y$? Explain why. [4]

(c) Compute the expectation of

$$z = \mathbf{b}^t A \mathbf{b} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_i b_j \ .$$

[6]

(d) Now assume that each element $b_i$ is independent and normally distributed with mean 0 and standard deviation 1. Compute the new expectation of $z$. [6]

6 The Erlang distribution is a continuous probability distribution often used in queuing theory. Its probability density function is

$$f(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}$$

where $k$ is an integer greater than 0, and $\lambda > 0$. Notice that when $k = 1$, this is the exponential probability density function, $f(t) = \lambda e^{-\lambda t}$.

(a) Recall the definition of the moment generating function

$$g(s) = \int e^{-st} f(t) dt.$$

Derive the moment generating function for the Erlang distribution. [6]

(b) Show that the sum of $k$ independent and identically distributed exponential waiting times follows an Erlang distribution. [8]

(c) Compute the mean and variance of the Erlang distribution. [6]

**END OF PAPER**