IB paper 1 Mechanics 2008
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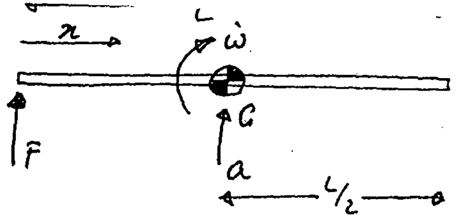
1a)

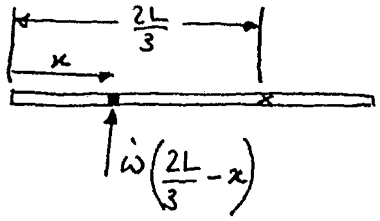


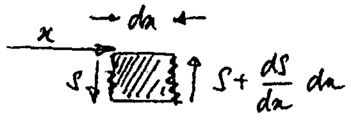
consider c.of mass G.
acc. G = a , angular acc. = $\dot{\omega}$

then: $F = ma$ & $\frac{FL}{2} = I_G \dot{\omega} = \frac{mL^2 \dot{\omega}}{12}$ $\Rightarrow$ $a = \frac{F}{m}$ & $\dot{\omega} = \frac{6F}{mL}$

centre of rotation @ x where $x = \frac{L}{2} + \frac{a}{\dot{\omega}} = \frac{2L}{3}$

b) consider eqm. of an element @ x:





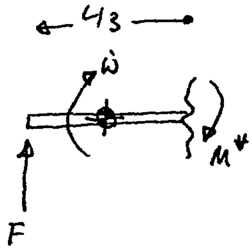
$\frac{dS}{dx} dx = \frac{m}{L} dx \, \dot{\omega}\left(\frac{2L}{3} - x\right)$ $\Rightarrow$ $\frac{dS}{dx} = \frac{m}{L}\dot{\omega}\left(\frac{2L}{3} - x\right)$ $\therefore S = \frac{m}{L}\dot{\omega}\left(\frac{2Lx}{3} - \frac{x^2}{2}\right) + C$

@ $x = 0$, $S = -F$ $\therefore -F = C$ $\Rightarrow$ $S = \frac{m}{L}\dot{\omega}\left(\frac{2Lx}{3} - \frac{x^2}{2}\right) - F$

B.M. max @ $S = 0$ $\therefore \dot{\omega}\left(\frac{2Lx}{3} - \frac{x^2}{2}\right) = \frac{FL}{m}$ $\Rightarrow 6\left(\frac{2Lx}{3} - \frac{x^2}{2}\right) = L^2$

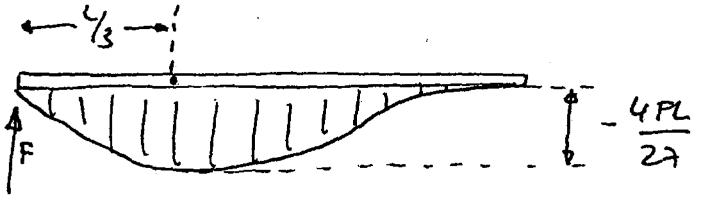
$3x^2 - 4Lx + L^2 = 0$ $\Rightarrow$ $S = 0$ @ $x = \frac{L}{3}$ & $x = L$ $\Rightarrow$ max B.M. @ $x = \frac{L}{3}$

c) let max B.M. $= M^*$ & take moments w. cut @ $x = \frac{L}{3}$ @ centre of free body

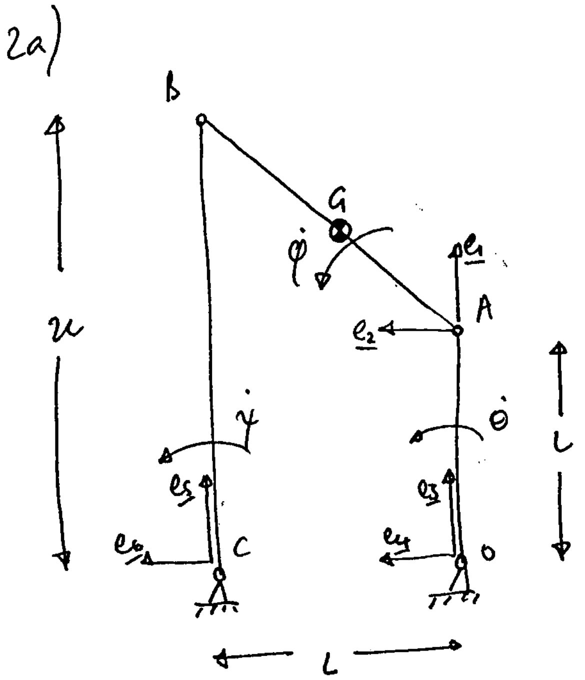


$I^* = \frac{1}{12}\left(\frac{m}{3}\right)\left(\frac{L}{3}\right)^2 = \frac{mL^2}{324}$

$M+) \Rightarrow$ $M^* + \frac{FL}{6} - \frac{mL^2\dot{\omega}}{324} = 0$ $\Rightarrow M^* = -FL\left(\frac{1}{6} - \frac{1}{54}\right) = \frac{-4F}{27}$



2a)



$$\underline{v_B} = \underline{v_A} + \underline{\omega_{AB}} \times \underline{AB}$$

$$2L\dot\psi\,\underline{e_6} = L\dot\theta\,\underline{e_4} + L\dot\varphi\,\underline{e_2} - L\dot\varphi\,\underline{e_1}$$

at instant shown,

$$\underline{e_6} = \underline{e_4} = \underline{e_2}$$

$$\left.\begin{aligned} \therefore\ 2\dot\psi &= \dot\theta + \dot\varphi \\ 0 &= -\dot\varphi \end{aligned}\right\} \Rightarrow \begin{aligned} \dot\psi &= \dot\theta/2 \\ \dot\varphi &= 0 \end{aligned}$$

b) $\underline{a_B} = \underline{a_A} + \underline{\omega_{AB}} \times \underline{\omega_{AB}} \times \underline{AB} + \underline{\dot\omega_{AB}} \times \underline{AB}$ but now $\underline{\omega_{AB}} = \dot\varphi = 0$

$$\therefore\ 2L\ddot\psi\,\underline{e_6} - 2L\dot\psi^2\,\underline{e_5} = -L\dot\theta^2\,\underline{e_3} - \cancelto{0}{L\dot\varphi^2}(\underline{e_1}+\underline{e_2}) + L\ddot\varphi(\underline{e_2}-\underline{e_1})$$

at instant shown $\underline{e_6} = \underline{e_4} = \underline{e_2}$ and $\underline{e_1} = \underline{e_3} = \underline{e_5}$

$$\left.\begin{aligned} \therefore\ \text{in } \underline{e_1} \text{ direction: } & -2\dot\psi^2 = -\dot\theta^2 - \ddot\varphi \\ \text{in } \underline{e_2} \text{ direction: } & 2\ddot\psi = \ddot\varphi \end{aligned}\right\} \Rightarrow \ddot\varphi = 2\ddot\psi = 2\dot\psi^2 - \dot\theta^2 = \frac{\dot\theta^2}{2} - \dot\theta^2 = \frac{-\dot\theta^2}{2}$$

c) Centre of mass of bar AB (let equal G) @ $\frac{L}{2}(\underline{e_1}+\underline{e_2})$ from A

$$\underline{a_G} = \underline{a_A} + \underline{\omega_{AB}} \times \underline{\omega_{AB}} \times \underline{AG} + \underline{\dot\omega_{AB}} \times \underline{AG}$$

$$= -L\dot\theta^2\,\underline{e_3} - \frac{L}{2}\cancelto{0}{\dot\varphi^2}(\underline{e_1}+\underline{e_2}) + \frac{L}{2}\ddot\varphi\,(\underline{e_2}-\underline{e_1})$$

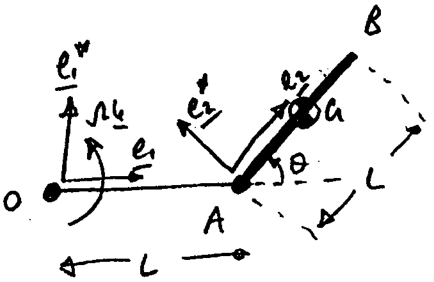
at instant shown, $\underline{e_1} = \underline{e_3} = \underline{e_5}$ and $\underline{e_2} = \underline{e_4} = \underline{e_6}$

$$\therefore\ \underline{a_G} = L\left(\frac{\dot\theta^2}{4} - \dot\theta^2\right)\underline{e_1} - \frac{L}{4}\dot\theta^2\,\underline{e_2}$$

With torque T applied @ O, horizontal force @ A $= T/L$

$$\therefore\ \text{in } \underline{e_2} \text{ direction } \frac{T}{L} = \frac{-mL\dot\theta^2}{4} \Rightarrow T = \frac{-mL^2\dot\theta^2}{4}$$

3a) Consider general position θ & apply unit vectors. Define G @ c. of mass AB.



$$\underline{r}_G = L\underline{e}_1 + \frac{L}{2}\underline{e}_2$$

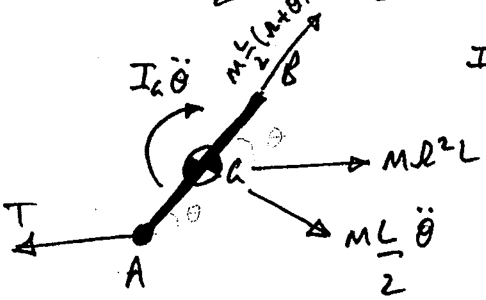
$$\dot{\underline{e}} = \underline{\omega}\times\underline{e} \Rightarrow \dot{\underline{e}}_1 = \Omega\underline{k}\times\underline{e}_1 = \Omega\underline{e}_1^*$$

$$\dot{\underline{e}}_2 = (\Omega+\dot\theta)\underline{k}\times\underline{e}_2 = (\Omega+\dot\theta)\underline{e}_2^*$$

$$\therefore \dot{\underline{r}}_G = L\dot{\underline{e}}_1 + \frac{L}{2}\dot{\underline{e}}_2 = L\Omega\underline{e}_1^* + \frac{L}{2}(\Omega+\dot\theta)\underline{e}_2^*$$

$$\ddot{\underline{r}}_G = L\Omega\dot{\underline{e}}_1^* + \frac{L}{2}(\ddot\theta)\underline{e}_2^* + \frac{L}{2}(\Omega+\dot\theta)\dot{\underline{e}}_2^* = -L\Omega^2\underline{e}_1 + \frac{L\ddot\theta}{2}\underline{e}_2^* - \frac{L}{2}(\Omega+\dot\theta)^2\underline{e}_2$$

Consider d'Alembert forces:



$$I_G = \frac{mL^2}{12}$$

$$M_A: \; I_G\ddot\theta + \frac{L}{2}\left(m\frac{L}{2}\ddot\theta\right) + \frac{L}{2}(m\Omega^2L)\sin\theta = 0$$

$$\therefore \; \ddot\theta\left(\frac{1}{12}+\frac{1}{4}\right) + \frac{\Omega^2}{2}\sin\theta = 0$$

$$\Rightarrow \ddot\theta = -\frac{3}{2}\Omega^2\sin\theta$$

Initial boundary conditions: $t=0$, $\theta = \frac{\pi}{2}$, $\dot\theta = 0$

$$\ddot\theta = \frac{d\dot\theta}{d\theta}\frac{d\theta}{dt} = \frac{d\dot\theta}{d\theta}\dot\theta = -\frac{3}{2}\Omega^2\sin\theta$$

$$\Rightarrow \int_0^{\dot\theta}\dot\theta\,d\dot\theta = -\frac{3}{2}\Omega^2\int_{\pi/2}^{\theta}\sin\theta\,d\theta \Rightarrow \dot\theta = \Omega\sqrt{3\cos\theta}$$

b) by inspection T is a maximum when $\theta = 0$

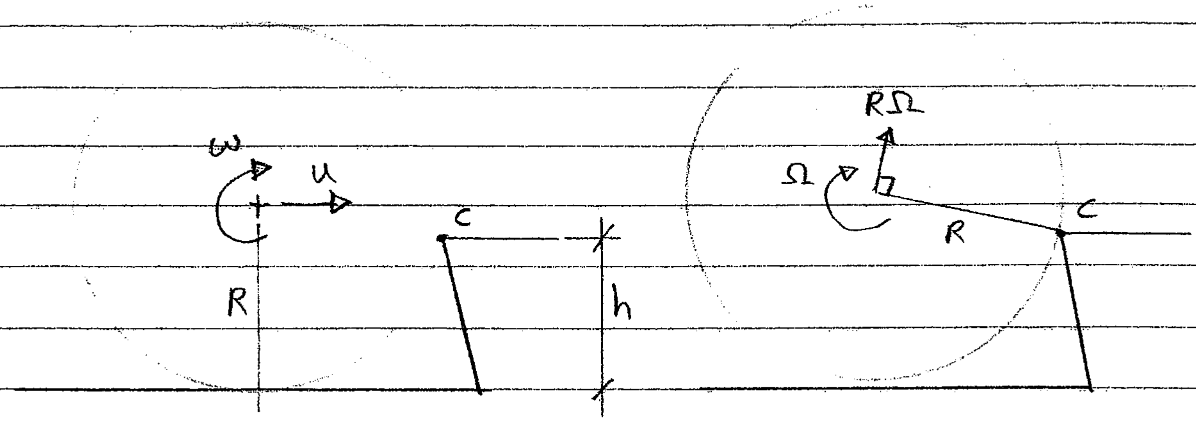
$$R_{\parallel OA}: \; T = m\Omega^2L + m\frac{L}{2}(\Omega+\dot\theta)^2$$

$$= m\Omega^2L + m\frac{L}{2}(\Omega+\sqrt{3}\Omega)^2 = 4.73\,m\Omega L^2$$

4) SECTION B

Moment of inertia of solid sphere $J = \frac{2}{5} m R^2$

Rolling without slipping $\therefore$ $u = R\omega$



Before impact

After impact (sphere sticks to corner c)

(a) Moment of momentum is conserved about c

$$mu(R-h) + J\omega = m \times R\Omega \times R + J\Omega$$

$$mR\omega(R-h) + \tfrac{2}{5}mR^2\omega = mR^2\Omega + \tfrac{2}{5}mR^2\Omega$$

$$\frac{7}{5}mR^2\omega - mRh\omega = \frac{7}{5}mR^2\Omega$$

$$\therefore \quad \Omega = \left(1 - \frac{5h}{7R}\right)\omega$$

Ω is only +ve providing that $h/R < 7/5$

hence if $h > 7R/5$ the sphere will not climb the step at all and will not come out of contact with the ground

4) (cont.)

(b) After the impact the sphere rotates about C. The sphere will just reach the top of the step if all of the kinetic energy just after the impact is equal to the potential energy gained climbing the step.

Hence: K.E. = P.E. gained

$$\tfrac{1}{2}m(R\Omega)^2 + \tfrac{1}{2}J\Omega^2 = mgh$$

$$\tfrac{1}{2}mR^2\Omega^2 + \tfrac{1}{2}\cdot\tfrac{2}{5}mR^2\Omega^2 = mgh$$

$$\tfrac{7}{10}R^2\Omega^2 = gh$$

$$\therefore \quad \frac{7}{10}R^2\left(1-\frac{5h}{7R}\right)^2\omega^2 = gh \qquad \text{but } u = R\omega$$

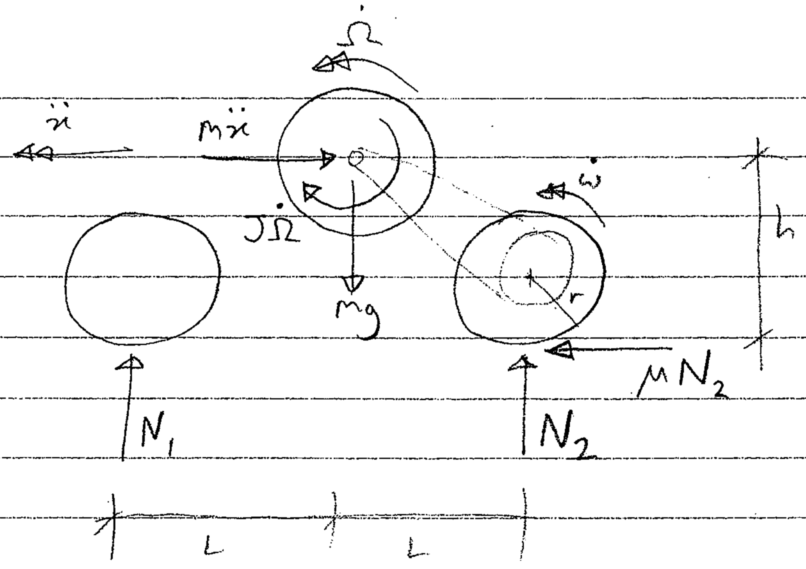
$$\therefore \quad u^2 = \frac{10gh}{7\left(1-\frac{5h}{7R}\right)^2}$$

Minimum value of u to climb step:

$$u_{min} = \frac{1}{\left(1-\frac{5h}{7R}\right)}\sqrt{\frac{10gh}{7}}$$

SECTION B (cont.)

5)



Accelerations and d'Alembert forces and torques

(a) If the car is 'pulling a wheelie' then $N_1 = 0$

$\therefore$ $N_2 = mg$ and $m\ddot{x} = \mu N_2$

Hence $\ddot{x} = \mu g$ this is the maximum possible horizontal acceleration

(b) The torque acting on the rear wheels is $\mu N_2 r$
The gear ratio is G $\Omega = G\omega$
$\therefore$ the angular acceleration of the flywheel $\dot{\Omega}$ is given by

$$J\dot{\Omega} = -\frac{\mu N_2 r}{G} \quad (1)$$ i.e. a deceleration

(c)

Resolve horizontally $m\ddot{x} = \mu N_2$ (2)

Resolve vertically $N_1 + N_2 = mg$ (3)

Moments about rear wheel contact point: $\circlearrowleft +$

$$2N_1 L + J\dot{\Omega} + m\ddot{x}h = mgL \quad (4)$$

5) (c) cont.

Sub. 1 and 2 into 4:

$$2N_1L - \frac{\mu N_2 r}{G} + \mu N_2 h = mgL$$

$$\therefore 2N_1L - \mu N_2 \left(r/G - h \right) = mgL \quad \text{Sub. in } 3$$

$$\therefore 2N_1L - \mu \left(mg - N_1 \right) \left(r/G - h \right) = mgL$$

$$N_1 \left\{ 2L + \mu \left(r/G - h \right) \right\} = mgL + \mu mg \left(r/G - h \right)$$

$$\therefore \quad N_1 = \frac{mg\left[L + \mu \left(r/G - h \right) \right]}{2L + \mu \left(r/G - h \right)}$$

For the front wheels to remain on the ground
$N_1 \geqslant 0$

$$\therefore \quad L + \mu \left(r/G - h \right) \geqslant 0$$

$$\text{or} \quad h \leqslant L/\mu + r/G$$

SECTION B (cont.)

6) (a) $Q = A\dot{r}$ (and $\therefore \dot{Q} = A\ddot{r}$)

(b) Velocity diagram.

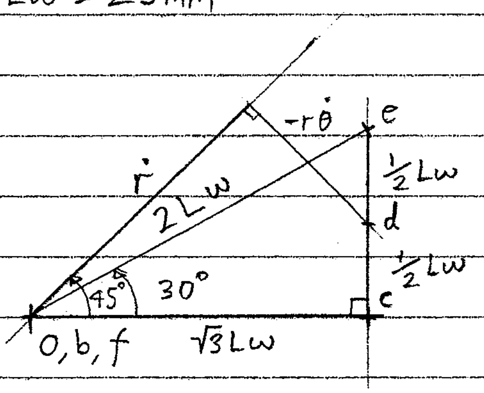
Note that angle EFD = 30°

Scale: $L\omega = 25\text{mm}$

At this instant

$\theta = 45° = \pi/4$

and $r = \sqrt{2}L$



$$\frac{\dot{r}}{\sqrt{2}} + \frac{(-r\dot{\theta})}{\sqrt{2}} = \sqrt{3}L\omega \quad \text{and} \quad \frac{\dot{r}}{\sqrt{2}} - \frac{(-r\dot{\theta})}{\sqrt{2}} = \frac{1}{2}L\omega$$

$$\therefore \frac{2\dot{r}}{\sqrt{2}} = \sqrt{3}L\omega + \frac{1}{2}L\omega \quad \therefore \dot{r} = \frac{(\sqrt{3} + \frac{1}{2})L\omega}{\sqrt{2}}$$

$$= 1.578\,L\omega$$

$$\frac{2(-r\dot{\theta})}{\sqrt{2}} = \sqrt{3}L\omega - \frac{1}{2}L\omega$$

$$\therefore \dot{\theta} = \frac{(\sqrt{3} - \frac{1}{2})L\omega}{\sqrt{2} \times -\sqrt{2}L} = -\left(\frac{\sqrt{3}}{2} - \frac{1}{4}\right)\omega = -0.616\,\omega$$

(i.e. BD clockwise)

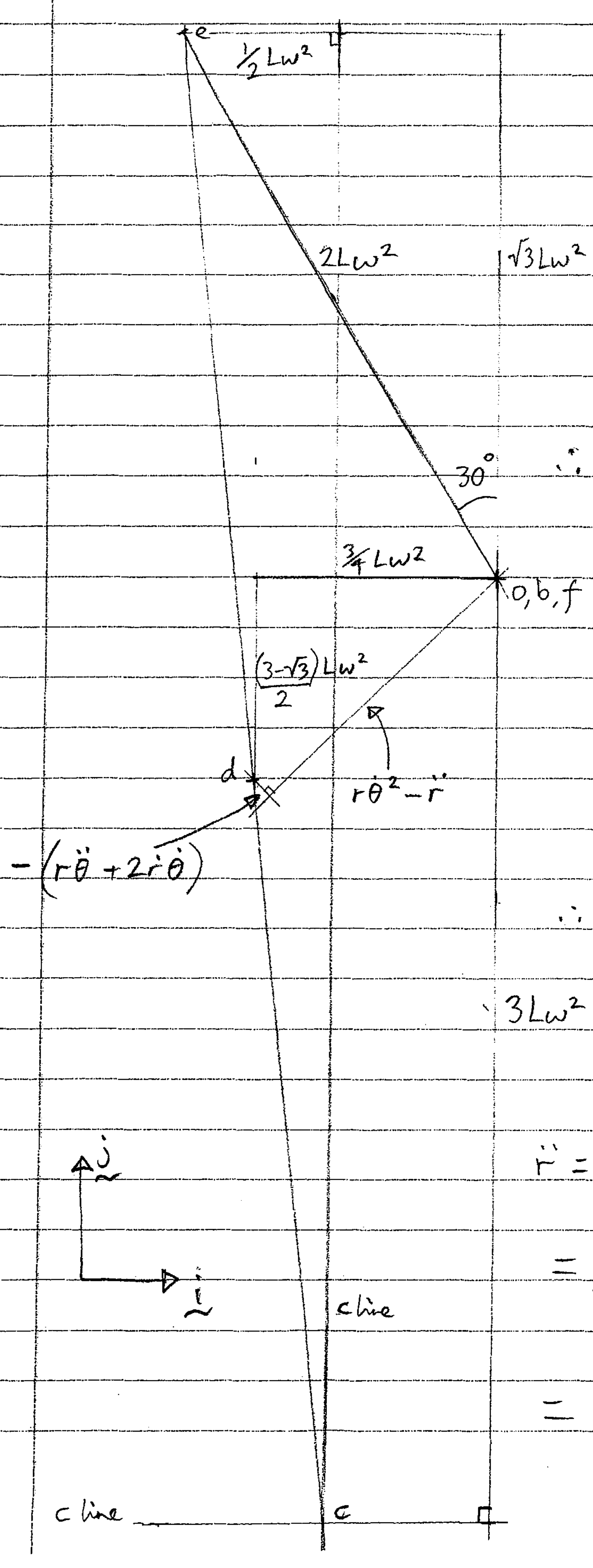
$$\omega_{CDE} = \frac{L\omega}{2L} = \frac{\omega}{2} \quad \text{anti-clockwise}$$

$$\omega_{BC} = \frac{\sqrt{3}L\omega}{L} = \sqrt{3}\omega \quad \text{clockwise}$$

Required flow rate at this instant: $Q = 1.578\,AL\omega$

6) (cont.)

(c) Acceleration diagram scale: $L\omega^2 = 50\text{mm}$



$$CE \times \omega_{CDE}^2 = 2L\left(\frac{\omega}{2}\right)^2 = \frac{1}{2}L$$

$$BC \times \omega_{BC}^2 = L(\sqrt{3}\omega)^2 = 3L\omega$$

Find d by image theorem at mid-point of ce

$$\therefore \underset{\sim}{a}_D = -\frac{3}{4}L\omega^2 \underset{\sim}{i} - \frac{(3-\sqrt{3})}{2}L\omega^2 \underset{\sim}{j}$$

This acceleration can be decomposed into components // and ⊥ to BD

hence

$$r\dot{\theta}^2 - \ddot{r} = \frac{L\omega^2}{\sqrt{2}}\left[\frac{3}{4} + \frac{(3-\sqrt{3})}{2}\right]$$

$$\therefore \ddot{r} = \sqrt{2}L\left(\frac{\sqrt{3}}{2} - \frac{1}{4}\right)^2\omega^2 - \frac{L\omega^2}{\sqrt{2}}\left[\frac{3}{4} + \frac{(3-\sqrt{3})}{2}\right]$$

$$\ddot{r} = L\omega^2\left[\sqrt{2}\left(\frac{13}{16} - \frac{\sqrt{3}}{4}\right) - \frac{1}{\sqrt{2}}\left(\frac{9}{4} - \frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{L\omega^2}{\sqrt{2}}\left[\frac{13}{8} - \frac{\sqrt{3}}{2} - \frac{9}{4} + \frac{\sqrt{3}}{2}\right]$$

$$= \frac{-5L\omega^2}{8\sqrt{2}} = -0.442\,L\omega^2$$

$$\therefore \dot{Q} = -0.442\,AL\omega^2$$