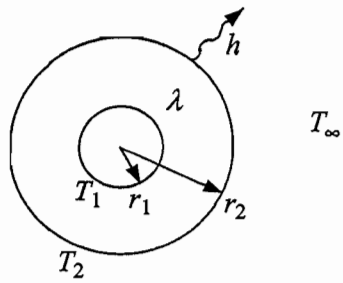
Question 1:

(a) (i) Databook: $\dot{Q} = -\lambda 2\pi r L \dfrac{dT}{dr}$ solve for $T(r)$ with $\dot{Q} = constant$.

Separate variables: $\dfrac{\dot{Q}}{L}\displaystyle\int_{r_1}^{r_2} \frac{1}{2\pi\lambda}\frac{dr}{r} = \int_{T_1}^{T_2}(-1)dT$



Integrate with limits: $\left(\dfrac{\dot{Q}}{L}\right)\dfrac{1}{2\pi\lambda}\ln\left(\dfrac{r_2}{r_1}\right) = T_1 - T_2$

Compare with electrical analogue: "IR=V" gives:

Thermal resistance per unit length = $\underline{\underline{R_{conduction} = \dfrac{1}{2\pi\lambda}\ln\left(\dfrac{r_2}{r_1}\right)}}$

{Majority of candidates did this part very well.}

(ii) Surface convection: $\dot{Q} = hA\Delta T$ with $A = 2\pi r_2 L$

Rearrange: $\left(\dfrac{\dot{Q}}{L}\right)\dfrac{1}{2\pi r_2 h} = T_2 - T_\infty$ so $R_{convection} = \dfrac{1}{2\pi r_2 h}$

So total thermal resistance is: $R_{total} = \left\{\dfrac{1}{2\pi\lambda}\ln\left(\dfrac{r_2}{r_1}\right) + \dfrac{1}{2\pi r_2 h}\right\}$ (ignoring radiation)

$$R_{total} = \frac{1}{2\pi}\left\{\frac{1}{0.03}\ln\left(\frac{0.025}{0.0075}\right) + \frac{1}{0.025\times 6}\right\} = 7.448 \text{ mKW}^{-1}$$

Specified temperature difference: $\Delta T = T_1 - T_\infty = 60\ °\text{C} - 10\ °\text{C} = 50\ \text{K}$

Hence: $\dfrac{\dot{Q}}{L} = \dfrac{\Delta T}{R_{total}} = \dfrac{50}{7.448} = \underline{\underline{6.713\ \text{Wm}^{-1}}}$

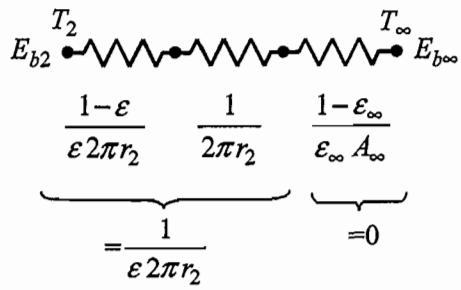
{Common error was to omit one of the two resistance terms.}

(iii) Need surface temperature of insulation to calculate radiation heat transfer:

$$T_2 = T_1 - \frac{\dot{Q}}{L}\frac{1}{2\pi\lambda}\ln\left(\frac{r_2}{r_1}\right) = 60 - 6.713\frac{1}{2\pi\times 0.03}\ln\left(\frac{0.025}{0.0075}\right) = 17.1\ °\text{C} = 290.1\ \text{K}$$

For radiation heat transfer between a grey body at temperature T_2 with emissivity ε and a black body at temperature T_∞ with unity view factor the net heat flux is:

$$\frac{\dot{Q}}{L} = \varepsilon\sigma 2\pi r_2\left(T_2^4 - T_\infty^4\right)$$



Substituting values: $\dfrac{\dot{Q}}{L} = 0.8\times 5.67\times 10^{-8}\times 2\pi\times 0.025\times\left(290.1^4 - 283^4\right) = \underline{\underline{4.76\ \text{Wm}^{-1}}}$

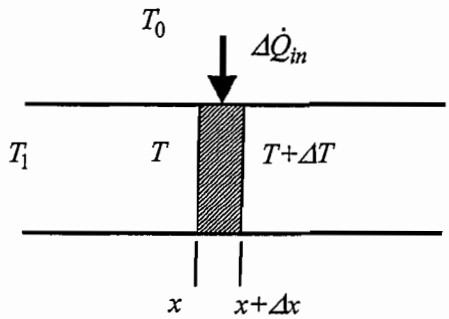
Note that the heat loss due to radiation is 71% of the heat loss due to convection.

{Most candidates forgot to calculate the surface temperature of the insulation.}

Question 1 continued:

(b) (i) Consider a small element of water between distance x and $x+\Delta x$ along the hole. The faces of this small element are at temperatures T and $T+\Delta T$.

The element of water, at temperature $T(x)$, receives an amount $\Delta\dot{Q}_{in}$ of heat (via convection) from the heat sink which is at the uniform temperature T_0.



SFEE: $$\Delta\dot{Q}_{in} - 0 = \dot{m}\left[c_p(T+\Delta T) - c_pT\right] = \dot{m}c_p\Delta T = \dot{m}c_p\frac{dT}{dx}\Delta x$$

Convection: $$\Delta\dot{Q}_{in} = h(\pi d\Delta x)(T_0 - T)$$

Combining: $$\Delta\dot{Q}_{in} = \dot{m}c_p\frac{dT}{dx}\Delta x = h(\pi d\Delta x)(T_0 - T)$$

Cancelling Δx and observing that T_0 is a constant allows the following re-arrangement:

$\dot{m}c_p\frac{d}{dx}(T_0 - T) = -\pi dh(T_0 - T)$ Integrating $(T_0 - T) = A\exp\left\{\frac{-\pi hd}{\dot{m}c_p}x\right\}$ A is a constant.

At x=0 $T = T_1$ gives $A = (T_0 - T_1)$ Thus: $$\underline{\underline{T(x) = T_0 - (T_0 - T_1)\exp\left\{\frac{-\pi hd}{\dot{m}c_p}x\right\}}}$$

{Many candidates were very creative in the use of the LMTD to get this result.}

(ii) Area = $\pi\times0.0025^2 = \underline{19.635\times10^{-6}\ \text{m}^2}$

$$V = \frac{\dot{m}}{\rho A} = \frac{0.06}{970\times19.635\times10^{-6}} = \underline{3.150\ \text{ms}^{-1}} \quad \text{so} \quad Re_d = \frac{\rho Vd}{\mu} = \frac{970\times3.150\times0.005}{0.56\times10^{-3}} = \underline{\underline{27281}}$$

Reynolds number ≈27300 so, from databook ($2300 < Re_d < 10^7$), expect <u>turbulent flow</u>.

{Common errors were in the calculation of velocity or not looking in the data book!!}

(iii) Databook for convective heat transfer for turbulent flow: $Nu_d = 0.023\,Re_d^{0.8}\,Pr^{0.4}$

$$Pr = \frac{\mu c_p}{\lambda} = \frac{0.56\times10^{-3}\times4.22\times10^3}{0.65} = \underline{3.636} \quad \text{so} \quad Nu_d = 0.023\times27281^{0.8}\times3.636^{0.4} = \underline{\underline{136.4}}$$

{Many candidates simply used the wrong formulae from data book!!}

(iv) Surface heat transfer coefficient: $$h = \frac{Nu_d\times\lambda}{d} = \frac{136.4\times0.65}{0.005} = \underline{17730\ \text{Wm}^{-2}\text{K}^{-1}}$$

$$\beta = \frac{\pi hdL}{\dot{m}c_p} = \frac{\pi\times17730\times0.005\times0.5}{0.06\times4.22\times10^3} = 0.5500 \quad \text{so} \quad e^{-\beta} = 0.5769$$

Water at 5 bar so $T_g = 151.83\ ^\circ C$ so maximum $T_2 = T_g - 20 = 151.83 - 20 = 131.83\ ^\circ C$

Now $T_2 = T(L) = T_0 - (T_0 - T_1)e^{-\beta}$ so $T_0(1 - e^{-\beta}) = T_2 - T_1e^{-\beta}$

Hence: $$T_0 = \frac{T_2 - T_1e^{-\beta}}{(1 - e^{-\beta})} = \frac{131.83 - 25\times0.5769}{1 - 0.5769} = \underline{\underline{277.5\ ^\circ C}}$$

(Note: above equation is independent of origin of temperature.)

Question 2:

(a) At low partial pressures, typical of those in the superheated region applicable to air-conditioning plants, water vapour can be modelled as a perfect gas so h(t) and c_p is a constant.

{Generally reasonably well understood.}

(b) $$R_s = \frac{\overline{R}}{M} = \frac{8.3143\ \text{kJ kmol}^{-1}\text{K}^{-1}}{18\ \text{kg kmol}^{-1}} = \underline{\underline{0.462\ \text{kJ kg}^{-1}\text{K}^{-1}}}$$

(Here ignoring the slightly more accurate molar mass for water of $18.016\ \text{kg kmol}^{-1}$)

(c) At 30 °C $p_g = 4247$ Pa so at 79% relative humidity: $p_{ps1} = 0.79 \times 4247 = \underline{3355\ \text{Pa}}$

For the (dry) air: $p_{pa1} = p_{atmos} - p_{ps1} = 101325 - 3355 = \underline{97970\ \text{Pa}}$

For the water vapour: $p_{ps1}\dot{V} = \dot{m}_{s1} R_s T$ and for the (dry) air: $p_{pa1}\dot{V} = \dot{m}_a R_a T$

Dividing these two yields: $\dfrac{\dot{m}_{s1}}{\dot{m}_a} = \dfrac{p_{ps1}}{p_{pa1}}\dfrac{R_a}{R_s} = \dfrac{3355}{97970} \times \dfrac{287}{462} = 0.021274$

$\dot{m}_a + \dot{m}_{s1} = \dot{m}_a\left(1 + \dfrac{\dot{m}_{s1}}{\dot{m}_a}\right) = \dot{m}_a(1 + 0.021274) = 15\ \text{kg s}^{-1}$ so $\underline{\dot{m}_a = 14.6875\ \text{kg s}^{-1}}$

Hence: $\dot{m}_{s1} = 15 - 14.6875 = \underline{\underline{0.3125\ \text{kg s}^{-1}}}$

{Common error was to assume 15 ks/s of DRY air or 1 bar pressure.}

(d) At 20 °C $p_g = 2339$ Pa so at 40% relative humidity: $p_{ps4} = 0.40 \times 2339 = \underline{936\ \text{Pa}}$

For the (dry) air: $p_{pa4} = p_{atmos} - p_{ps4} = 101325 - 936 = \underline{100389\ \text{Pa}}$

$$\frac{\dot{m}_{s4}}{\dot{m}_a} = \frac{p_{ps4}}{p_{pa4}}\frac{R_a}{R_s} = \frac{936}{100389} \times \frac{287}{462} = 0.005792$$

So, at exit, water vapour: $\dot{m}_{s4} = 0.005792 \times \dot{m}_a = 0.005792 \times 14.6875 = \underline{\underline{0.0851\ \text{kg s}^{-1}}}$

To determine the temperature T_3 it is necessary to observer that both the mass flow rate of water vapour and (dry) air are unchanged across the heater: $\dfrac{\dot{m}_{s3}}{\dot{m}_a} = \dfrac{\dot{m}_{s4}}{\dot{m}_a}$. Thus: $\dfrac{p_{ps3}}{p_{pa3}} = \dfrac{p_{ps4}}{p_{pa4}}$.

Since $p_{ps3} + p_{pa3} = p_3 = p_4 = p_{ps4} + p_{pa4}$ the individual partial pressures are also unchanged. Therefore $p_{ps3} = p_{ps4} = 936$ Pa which is almost the saturation pressure at 6 °C ($p_g = 935$ Pa).

{Very few candidates fully justified why the partial pressures at "3" = "4".}

(e) SFEE across the heater: $\dot{Q}_{heater} - 0 = (\dot{m}_a h_{a4} + \dot{m}_{s4} h_{s4}) - (\dot{m}_a h_{a3} + \dot{m}_{s3} h_{s3})$

Rewrite, observing $\dot{m}_{s3} = \dot{m}_{s4}$, gives: $\dot{Q}_{heater} = \dot{m}_a c_{pa}(T_4 - T_3) + \dot{m}_{s4} c_{ps}(T_4 - T_3)$

$\dot{Q}_{heater} = 14.6875 \times 1.005 \times (20 - 6) + 0.0851 \times 1.86 \times (20 - 6) = 206.65 + 2.22 = \underline{\underline{208.87\ \text{kJ s}^{-1}}}$

{Done well.}

(f) $\dot{m}_{w2} = \dot{m}_{s1} - \dot{m}_{s3} = 0.3125 - 0.0851 = \underline{\underline{0.2274\ \text{kg s}^{-1}}}$

(g) SFEE across cooler: $\left(-\dot{Q}_{cooler}\right)-0=\left(\dot{m}_a h_{a3}+\dot{m}_{s3} h_{s3}+\dot{m}_{w2} h_{w2}\right)-\left(\dot{m}_a h_{a1}+\dot{m}_{s1} h_{s1}\right)$

Rearrange: $\left(-\dot{Q}_{cooler}\right)=\dot{m}_a c_{pa}\left(T_3-T_1\right)+\left(\dot{m}_{w2} h_{w2}\right)+\left(\dot{m}_{s3} h_{s3}-\dot{m}_{s1} h_{s1}\right)$ (*)

h_{w2} : 6 °C saturated liquid: from tables $h_{w2}=\underline{25.2\ \text{kJkg}^{-1}}$

h_{s3} : 6 °C saturated vapour: from tables $h_{s3}=\underline{2511.9\ \text{kJkg}^{-1}}$

h_{s1} : 30 °C and 40% relative humidity:

$$h_{s1}=h_g\big|_{3355\,\text{Pa}}+c_{ps}\left(30\ ^\circ\text{C}-T_g\big|_{3355\,\text{Pa}}\right)$$

At 3355 Pa saturation, tables: $T_g\approx 26\ ^\circ\text{C}$ and $h_g\approx 2548.3\ \text{kJkg}^{-1}$

$$h_{s1}=2548.3+1.86\times(30-26)=\underline{2555.7\ \text{kJkg}^{-1}}$$

ALT: $h_{s1}=h_g\big|_{30\,^\circ\text{C}}=\underline{2555.5\ \text{kJkg}^{-1}}$ (virtually identical to above value)

Substituting these values yields:

$$\dot{m}_a c_{pa}\left(T_3-T_1\right)=14.6875\times1.005\times(6-30)=-354.226\ \text{kJ s}^{-1}$$

$$\left(\dot{m}_{w2} h_{w2}\right)=0.2274\times25.2=5.73\ \text{kJ s}^{-1}$$

$$\left(\dot{m}_{s3} h_{s3}-\dot{m}_{s1} h_{s1}\right)=0.0851\times2511.9-0.3125\times2555.7=213.76-798.66=-584.90\ \text{kJs}^{-1}$$

Combining: $\left(-\dot{Q}_{cooler}\right)=-354.26+5.73+(-584.90)=-933.43\ \text{kJ s}^{-1}$

Hence: $\underline{\underline{\dot{Q}_{cooler}=933.43\ \text{kJ s}^{-1}}}$

ALT: The expression (*) can be rearranged using $\dot{m}_{s3}=\dot{m}_{s1}-\dot{m}_{w2}$ to give:

$$\left(-\dot{Q}_{cooler}\right)=\dot{m}_a c_{pa}\left(T_3-T_1\right)+\left(\dot{m}_{w2} h_{w2}\right)+\left(\left[\dot{m}_{s1}-\dot{m}_{w2}\right]h_{s3}-\dot{m}_{s1} h_{s1}\right)$$

$$\left(-\dot{Q}_{cooler}\right)=\dot{m}_a c_{pa}\left(T_3-T_1\right)+\dot{m}_{w2}\left(h_{w2}-h_{s3}\right)+\dot{m}_{s1}\left(h_{s3}-h_{s1}\right)$$

$$\left(-\dot{Q}_{cooler}\right)=\dot{m}_a c_{pa}\left(T_3-T_1\right)+\dot{m}_{s1} c_{ps}\left(T_3-T_1\right)-\dot{m}_{w2}\left(h_{s3}-h_{w2}\right) \quad (**)$$

Where: $h_{s3}-h_{w2}=h_{fg}\big|_{6\,^\circ\text{C}}=2486.7\ \text{kJkg}^{-1}$

Substituting values yield:

$$\dot{m}_a c_{pa}\left(T_3-T_1\right)=14.6875\times1.005\times(6-30)=-354.226\ \text{kJ s}^{-1}$$

$$\dot{m}_{s1} c_{ps}\left(T_3-T_1\right)=0.3125\times1.86\times(6-30)=-13.95\ \text{kJ s}^{-1}$$

$$\dot{m}_{w2}\left(h_{s3}-h_{w2}\right)=0.2274\times2486.7=565.48\ \text{kJ s}^{-1}$$

Combining: $\left(-\dot{Q}_{cooler}\right)=-354.26-13.95-565.48=-933.69\ \text{kJ s}^{-1}$

Hence: $\underline{\underline{\dot{Q}_{cooler}=933.69\ \text{kJ s}^{-1}}}$

Expression (**) has the advantage that mass conservation for each phase/species is explicitly satisfied and the constants associated with the consistent definition for the origin of enthalpy for air and water cancel out in the individual terms.

{Most candidates had a good idea of which terms to evaluate and how.}

Question 3:

(a) $$COP = \frac{get}{pay} = \frac{\dot{Q}_{cold}}{\dot{Q}_{hot} - \dot{Q}_{cold}}$$

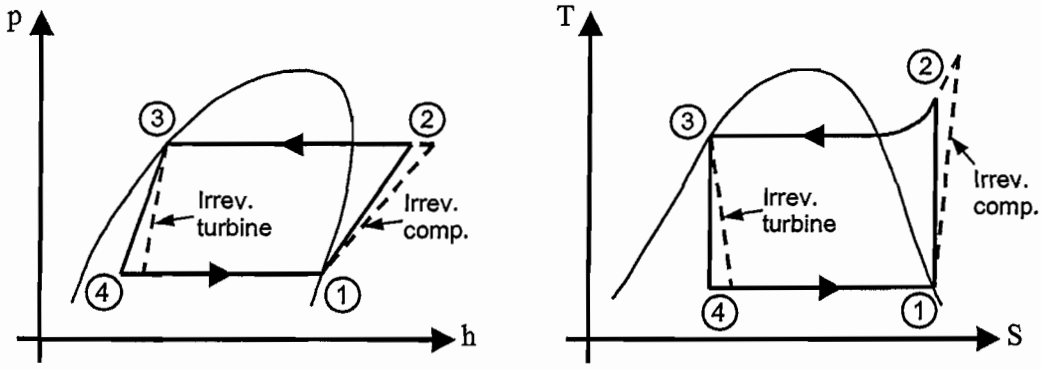
$$COP\big|_{rev} = \frac{T_{cold}}{T_{hot} - T_{cold}} = \frac{(273-15)}{(273+50)-(273-15)} = \frac{258}{65} = \underline{\underline{3.97}}$$

{Many candidates just quoted "heat engine" result - little real understanding!}

(b) Liquid water entering a turbine is likely to cause damage. Also turbines are large and expensive whilst a throttle is very simple (and reliable).

{Done very well!}

(c)



{Many candidates confused refrigeration and Rankine cycles. Many candidates had one or both of the cycles going the wrong way round. Few candidates appreciated that a turbine extracts work from the flow so that the enthalpy decreases!}

(d) At compressor entry, $T_1 = -15\ ^\circ\text{C}$, saturated vapour, refrigerant R-134a.

Tables: $h_1 = 389.6\ \text{kJ kg}^{-1}$ and $s_1 = 1.7368\ \text{kJ kg}^{-1}\,\text{K}^{-1}$

Compressor is reversible and adiabatic so the process "1" to "2" is isentropic: $s_2 = s_1$

Pressure corresponding to a saturation temperature of $50\ ^\circ\text{C}$ is $13.18\ \text{bar}$. Need to interpolate in the superheated region, at this pressure, to obtain the required entropy: $s_2 = s_1 = 1.7368\ \text{kJ kg}^{-1}\,\text{K}^{-1}$

Tables ($50\ ^\circ\text{C}$): $s\big|_{13.18\ \text{bar},\ sat.vapour} = 1.7073\ \text{kJ kg}^{-1}\,\text{K}^{-1}$ $\quad h = 423.4\ \text{kJ kg}^{-1}$

$s\big|_{13.18\ \text{bar},\ 20\ \text{K}\ superheat} = 1.7772\ \text{kJ kg}^{-1}\,\text{K}^{-1}$ $\quad h = 446.7\ \text{kJ kg}^{-1}$

Interpolation: $\dfrac{1.7368 - 1.7073}{1.7772 - 1.7073} = 0.4220$

Superheat at "2" is: $20 \times 0.4220 = \underline{\underline{8.44\ \text{K}}}$

Later will need enthalpy at "2": $h_2 = 423.4 + 0.4220 \times (446.7 - 423.4) = \underline{433.2\ \text{kJ kg}^{-1}}$

{many candidates would have got the correct superheat temperature had it not been given in the question!}

(e) At turbine entry, $T_3 = 50\ ^\circ\text{C}$ saturated liquid.

Tables: $h_3 = 271.6\ \text{kJ kg}^{-1}$ and $s_3 = 1.2374\ \text{kJ kg}^{-1}\ \text{K}^{-1}$

Turbine is reversible and adiabatic so process "3" to "4" is isentropic: $s_4 = s_3$

Need to find the dryness fraction in the two-phase region along the $T = -15\ ^\circ\text{C}$ to get state "4".

Tables ($-15\ ^\circ\text{C}$): $s\big|_{-15\ ^\circ\text{C},\ sat.liquid} = 0.9253\ \text{kJ kg}^{-1}\ \text{K}^{-1}$ $\quad h = 180.1\ \text{kJ kg}^{-1}$

$s\big|_{-15\ ^\circ\text{C},\ sat.vapour} = 1.7368\ \text{kJ kg}^{-1}\ \text{K}^{-1}$ $\quad h = 389.6\ \text{kJ kg}^{-1}$

Interpolation: $\dfrac{1.2374 - 0.9253}{1.7368 - 0.9253} = 0.3846$ (38.46% dry)

Hence: $h_4 = 180.1 + 0.3846 \times (389.6 - 180.1) = \underline{\underline{260.7\ \text{kJ kg}^{-1}}}$

{Most candidates made reasonable attempts at this part.}

(f) $\dot{Q}_{cold} = h_1 - h_4 = 389.6 - 260.7 = 128.9\ \text{kJ kg}^{-1}$

$\dot{Q}_{hot} = h_2 - h_3 = 433.2 - 271.6 = 161.6\ \text{kJ kg}^{-1}$

$$COP\big|_{actual} = \frac{\dot{Q}_{cold}}{\dot{Q}_{hot} - \dot{Q}_{cold}} = \frac{128.9}{161.6 - 128.9} = \underline{\underline{3.94}}$$

In part (a) it was calculated that $COP\big|_{rev} = 3.97$ which is the maximum possible COP for a reversible refrigerator operating between $-15\ ^\circ\text{C}$ and $50\ ^\circ\text{C}$. In the actual cycle, although both the compressor and turbine are reversible and the heat is extracted from the cold space at a constant temperature ($-15\ ^\circ\text{C}$), the heat is rejected across a range of temperatures between $50\ ^\circ\text{C}$ and $(50 + 8.44)\ ^\circ\text{C}$. Thus the mean temperature of heat rejection is slightly large than $50\ ^\circ\text{C}$. Hence the actual cycle is operating across a larger temperature difference (but same T_{cold}) which means that the maximum possible COP is slightly reduced.

[Although not required in the exam, the mean temperature of heat rejection, $\overline{T}_{hot}$, can be calculated:

$\dot{Q}_{hot} / \overline{T}_{hot} = s_2 - s_3$

$\overline{T}_{hot} = 161.6/(1.7368 - 1.2374) = 323.59\ \text{K}$ $\quad (\overline{T}_{hot} = 50.59\ ^\circ\text{C})$

$$COP\big|_{rev} = \frac{T_{cold}}{\overline{T}_{hot} - T_{cold}} = \frac{(273 - 15)}{323.59 - (273 - 15)} = \underline{\underline{3.93}}$$ (which is almost 3.94)]

{Very few candidates understood the idea of the mean temperature of heat rejection and consequently were unable to explain the lower COP.}

Question 4:

(a) Force per unit length, so: $C_D = \dfrac{F}{\frac{1}{2}\rho V^2 d}$ and $Re = \dfrac{\rho V d}{\mu}$ $\left[= \dfrac{Vd}{\nu}\right]$

{Common error was to wrong "area" (often $\pi d^2/4$) and so get wrong C_d lengthscale.}

(b) Within a fluid, the force per unit volume produced by the pressure gradient (dp/dx) either provides the necessary inertial (acceleration) force ($\rho V\, dV/dx$, per unit volume) or balances the viscous force ($\mu\, d^2V/dx^2$, per unit volume). The Reynolds number characterises the ratio between these two forces, thus:

$$Re = \frac{Inertial\ force}{Viscous\ force} = \frac{\rho V\, dV/dx}{\mu\, d^2V/dx^2} = \frac{\rho V\, \Delta x}{\mu}$$

(In the above it has been assumed that the characteristic lengthscales of the inertial terms and the viscous terms are both Δx.)

{Generally well understood.}

(c) In low Reynolds number flow the inertial forces are much smaller than the viscous forces and so the flowfield is primarily set by a balance between the pressure gradient and the viscous force. Therefore, at low *Re*, the density of the fluid does not have a significant effect in determining the flowfield. Thus in a low Reynolds number characterisation of the flowfield, $C_D = fnc(Re)$, either the density should not be included in any non-dimensional analysis or the density must "cancel-out" in any low Reynolds number expression:

$$C_D = \frac{F}{\frac{1}{2}\rho V^2 d} \propto \frac{1}{\rho} \propto \frac{\mu}{\rho V d} = \frac{1}{Re}$$

Both C_D and $(Re)^{-1}$ have density in the denominator, so it "cancels", yielding: $C_D \propto (Re)^{-1}$

An, often quoted, alternative argument is:

$$Drag\,/\,length \approx (area\,/\,length) \times \tau \propto d \times \mu\, dV/dx$$

If the characteristic lengthscale of the boundary layer, Δx, is the same as the characteristic size of the object, *d*, then the above can be re-arranged to give:

$$Drag\,/\,length \propto \mu V$$

Substitution into the definition for C_D yields:

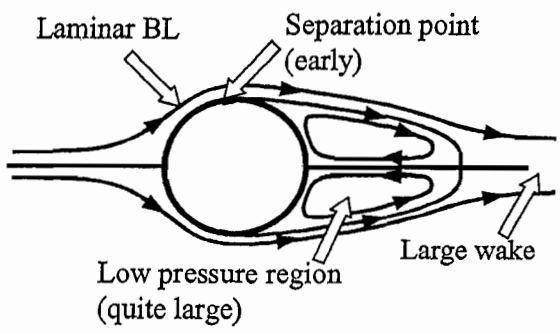
$$C_D = \frac{Drag\,/\,length}{\frac{1}{2}\rho V^2 d} = \frac{\mu V}{\frac{1}{2}\rho V^2 d} \propto \frac{\mu}{\rho V d} = \frac{1}{Re}$$

The weakness in this argument is that the thickness of a laminar boundary grows $\propto \sqrt{length}$ on a flat plate whilst the size of a laminar wake downstream of a cylinder is $\propto diameter$.

{Most candidates used poorly the latter argument.}

(d) At low Reynolds number ($Re < 10$) viscous forces dominate. In the range $10^2 < Re < 10^5$ inertial forces dominate and the boundary layer on the cylinder remains laminar. Consequently, in this range, the flowfield around the cylinder is largely independent of *Re* so the drag coefficient is almost constant. Between points "A" and "B", the boundary layer on the cylinder undergoes transition from initially laminar around the front of the cylinder to turbulent. A turbulent boundary layer can withstand a larger adverse pressure gradient before separation and so the size of the downstream wake is much smaller. Although the turbulent boundary layer has a higher skin friction the reduction in the wake dominates yielding a decrease in the overall drag coefficient by approximately a factor of four. The following table lists the key points:

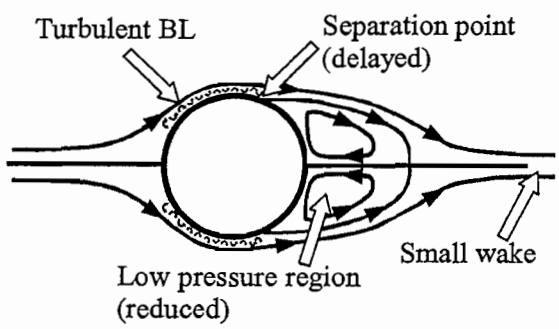
A



Boundary layer wholly laminar:
Low shear stress at wall,
"Easy, early" separates,
Early separation produces large wake.
⇒ Low "skin-friction drag".
⇒ High "form drag".

(Have large low pressure region downstream of cylinder which causes large 'form drag'.)

B



Boundary layer transitions to turbulent:
High shear stress at wall,
"Harder, later" separation,
Later separation yields small wake.
⇒ High "skin-friction drag".
⇒ Low "form drag".

(Have small low pressure region downstream of cylinder which causes small 'form drag'.)

Note these diagrams represent the "time-average" streamlines. Actual flow will have unsteady vortex shedding.

{Most candidates understood that a turbulent boundary layer was the cause.}

(e) (i) Properties of gases at sea level conditions for air:

$$\rho = 1.225 \ \text{kg m}^{-3}$$

$$\mu = 17.9\times10^{-6} \ \text{kg m}^{-1}\,\text{s}^{-1}$$

These yield: $Re = \dfrac{\rho V d}{\mu} = \dfrac{1.225\times60\times0.025}{17.9\times10^{-6}} = 103000$ Graph: $C_D = \dfrac{F}{\frac{1}{2}\rho V^2 d} \approx 1$

Thus: $\dfrac{Force}{length} = F = \frac{1}{2}\rho V^2 d \times C_D = \frac{1}{2}\times1.225\times60^2\times0.025\times1 = 55.1 \ \text{N m}^{-1}$

(ii) Properties of liquids at sea level conditions for water:

$$\rho = 1000 \ \text{kg m}^{-3}$$

$$\mu = 1.14\times10^{-3} \ \text{kg m}^{-1}\,\text{s}^{-1}$$

These yield: $Re = \dfrac{\rho V d}{\mu} = \dfrac{1000\times2\times0.6\times10^{-3}}{1.14\times10^{-3}} = 1050$ Graph: $C_D = \dfrac{F}{\frac{1}{2}\rho V^2 d} \approx 1$

Thus: $\dfrac{Force}{length} = F = \frac{1}{2}\rho V^2 d \times C_D = \frac{1}{2}\times1000\times2^2\times0.6\times10^{-3}\times1 = 1.2 \ \text{N m}^{-1}$

{Many candidates got correct C_d but then slipped up.}

Question 5:

(a) The streamline separating the free-stream jet and the separated region is straight. This mean that there is no pressure gradient across it. The streamlines outside the separated region are straight, which means that the pressure is uniform in this region. The pressure within a re-circulating region is approximately uniform, otherwise the flow could not circulate. Therefore the pressure in the free-stream jet and the separated region are the same.

{Very few candidates knew pressure in re-circulation region was uniform.}

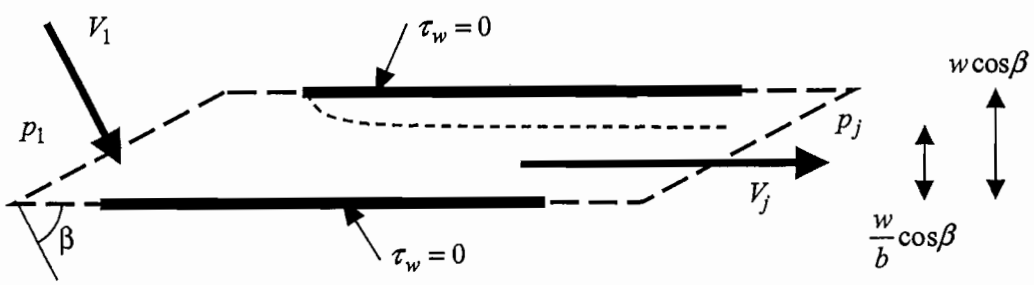
(b) Mass conservation at constant density (consider flow between one pair of plates):

Volume flow rate in $= V_1 w$

Volume flow rate out $= V_j\left(\frac{w}{b}\cos\beta\right)$ (note projected area)

Combine: $V_j\left(\frac{w}{b}\cos\beta\right) = V_1 w$ thus: $\underline{\underline{V_j = \frac{bV_1}{\cos\beta}}}$

(c)



Applying momentum conservation parallel to the plates:

"Total Force on Control Volume" = "Change in Momentum Flux"

$$p_1(w\cos\beta) - p_j(w\cos\beta) = \dot{m}V_j - \dot{m}V_1\cos\beta$$

Force on LH face. Force on RH face. Mom. flux leaving. Mom. flux entering.

Rearranging yields: $\underline{\underline{p_1 w\cos\beta + \dot{m}V_1\cos\beta = p_j w\cos\beta + \dot{m}V_j}}$

{Most candidates could obtain the stated formulae.}

(d) The fluid is incompressible, there is no mixing between the free-stream jet and the separated region (isentropic jet-flow) and it is steady. Thus can apply Bernoulli along a streamline from far upstream into the free-stream jet. Thus:

$$\underline{\underline{p_1 + \tfrac{1}{2}\rho V_1^2 = p_j + \tfrac{1}{2}\rho V_j^2}}$$

Mom. eqn: $(p_1 - p_j)w\cos\beta = \dot{m}(V_j - V_1\cos\beta) = \rho V_1 w(V_j - V_1\cos\beta)$

Hence: $\frac{1}{2}\rho(V_j^2 - V_1^2)w\cos\beta = \rho V_1 w(V_j - V_1\cos\beta)$

$$\frac{1}{2}\left(\frac{b^2V_1^2}{\cos^2\beta} - V_1^2\right)\cos\beta = V_1\left(\frac{bV_1}{\cos\beta} - V_1\cos\beta\right)$$

$$\left(\frac{b^2}{\cos^2\beta}-1\right)\cos\beta = 2\left(\frac{b}{\cos\beta}-\cos\beta\right)$$

$$b^2-\cos^2\beta = 2b-2\cos^2\beta$$

$$b^2-2b+\cos^2\beta = 0$$

$$b^2-2b+1-\sin^2\beta = 0$$

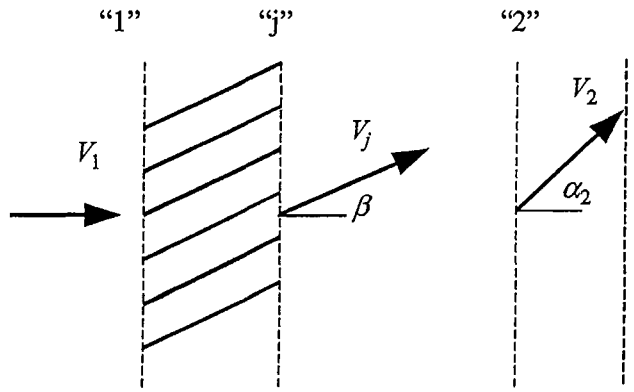
$$(b-1)^2 = \sin^2\beta$$

Taking the square root and remembering that $b \geq 1$ gives: $b-1=|\sin\beta|$

Rearrange: $\underline{\underline{b=1+|\sin\beta|}}$

{Many candidates had problems with the later parts of the algebra.}

(e)



Flow mixes-out fully between the end of the plates (location "j") and far downstream (location "2"). Apply conservation of tangential momentum and conservation of mass.

Conservation of tangential momentum:

$$\dot{m}V_j\sin\beta = \dot{m}V_2\sin\alpha_2 \qquad \text{so} \qquad \underline{V_j\sin\beta = V_2\sin\alpha_2}$$

Conservation of mass:

$$\rho V_1 w = \rho V_j \cos\beta\left(\frac{w}{b}\right) = \rho V_2\cos\alpha_2 w \qquad \text{so} \qquad \underline{V_1 = V_2\cos\alpha_2}$$

Dividing these two expressions yield:

$$\tan\alpha_2 = \frac{V_2\sin\alpha_2}{V_2\cos\alpha_2} = \frac{V_j\sin\beta}{V_1} = \frac{\left(\frac{bV_1}{\cos\beta}\right)\sin\beta}{V_1} = b\tan\beta$$

Hence: $\underline{\underline{\tan\alpha_2 = b\tan\beta}}$

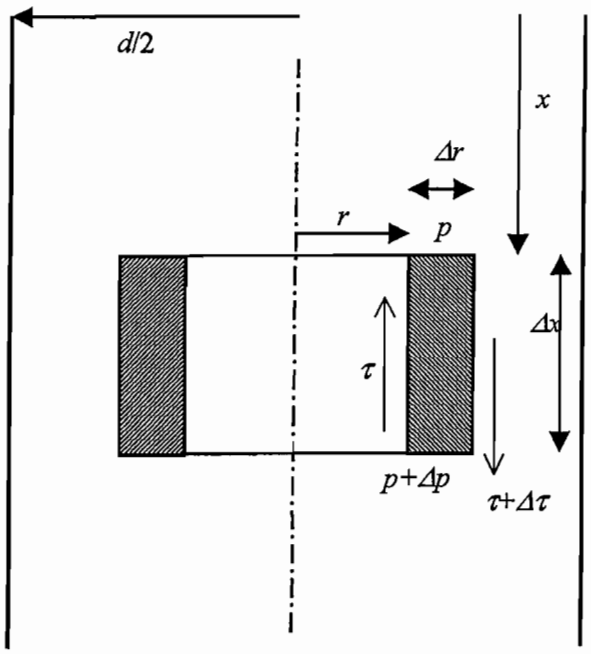
{Few candidates could apply conservation of tangential momentum.}

Question 6:

(a) In the reservoir the velocity will be much smaller than in the pipe (because $D >> d$). Also, the lengthscale over which the velocity changes in the reservoir ($\sim D$) will be much larger than in the pipe ($\sim d$). Consequently, the viscous forces ($\mu\, dV/dy$) in the reservoir will be expected to be much smaller than in the pipe.

{Lot of confusion over this part.}

(b) Consider a small annular element as shown in cross-section on the right.



R(↓):

$$p(2\pi r\,\Delta r)-(p+\Delta p)(2\pi r\,\Delta r)$$
$$+[2\pi(r+\Delta r)(\tau+\Delta\tau)\Delta x]-[2\pi r\,\tau\,\Delta x]$$
$$+\rho g\, 2\pi r\,\Delta r\,\Delta x=0$$

$$-\Delta p\, r\,\Delta r$$
$$+\Delta x[(r+\Delta r)(\tau+\Delta\tau)-r\tau]$$
$$+\rho g\, r\,\Delta r\,\Delta x=0$$

$$\frac{1}{r}\frac{(r+\Delta r)(\tau+\Delta\tau)-r\tau}{\Delta r}=\frac{\Delta p}{\Delta x}-\rho g$$

$$\underline{\underline{\frac{1}{r}\frac{\partial(r\tau)}{\partial r}=\frac{\partial p}{\partial x}-\rho g}}$$

{Common mistake was to forget that force=shear stress × area so did not get ($r\tau$).}

(c) Once the flow has reached steady-state (i.e. Poiseuille flow) then τ is no longer a function of x so, at most, $\tau(r)$. Parallel streamlines in the (small diameter) pipe, imply $p(x)$ and $v(r)$. The terms in the equation can only balance if they are all constants (idea of "separation of variables"). Thus:

$$\underline{\underline{\frac{\partial p}{\partial x}=constant}}$$

$$\frac{1}{r}\frac{\partial(r\tau)}{\partial r}=\frac{\partial p}{\partial x}-\rho g$$

$fnc(r)$ $fnc(x)$ $constant$

{Most candidates stated parallel streamlines and did not analyse stated equation!}

(d) $\dfrac{d(r\tau)}{dr}=r\left(\dfrac{dp}{dx}-\rho g\right)$ with $\dfrac{dp}{dx}-\rho g=constant$ and $\tau=\mu\dfrac{dv}{dr}$

Integrating: $\left[r\mu\dfrac{dv}{dr}\right]_0^r=\left(\dfrac{dp}{dx}-\rho g\right)\left[\tfrac{1}{2}r^2\right]_0^r$ and, by symmetry, $\left.\dfrac{dv}{dr}\right|_{r=0}=0$

Hence: $r\mu\dfrac{dv}{dr}=\left(\dfrac{dp}{dx}-\rho g\right)\tfrac{1}{2}r^2$ Thus: $\dfrac{dv}{dr}=\dfrac{1}{2\mu}\left(\dfrac{dp}{dx}-\rho g\right)r$

$\left[v\right]_r^{d/2}=\dfrac{1}{2\mu}\left(\dfrac{dp}{dx}-\rho g\right)\left[\tfrac{1}{2}r^2\right]_r^{d/2}$ where integration is between "r" and "$d/2$"

$v(d/2)=0$ $0-v(r)=\dfrac{1}{4\mu}\left(\dfrac{dp}{dx}-\rho g\right)\left(\dfrac{d^2}{4}-r^2\right)$ rearrange: $\underline{\underline{v(r)=\dfrac{1}{16\mu}\left(\rho g-\dfrac{dp}{dx}\right)(d^2-4r^2)}}$

(e) Pressure at the base of the reservoir: $p_{atm} + \rho g H$

Pressure at bottom of vertical pipe: p_{atm}

Thus: $\dfrac{dp}{dx} = \dfrac{(p_{atm}) - (p_{atm} + \rho g H)}{h} = \underline{\underline{-\dfrac{\rho g H}{h}}}$

{Few candidates realised that hydrostatic gave pressure at base of reservoir!}

(f) Volume flow rate: $\dot{Q} = \displaystyle\int_0^{d/2} 2\pi r\, v(r)\, dr = \int_0^{d/2} 2\pi r \frac{1}{16\mu}\left(\rho g + \frac{\rho g H}{h}\right)\left(d^2 - 4r^2\right) dr$

$$\dot{Q} = \frac{\pi}{8\mu}\left(\rho g + \frac{\rho g H}{h}\right)\int_0^{d/2}\left(r d^2 - 4r^3\right) dr$$

$$\dot{Q} = \frac{\pi}{8\mu}\rho g\left(1 + \frac{H}{h}\right)\left[\tfrac{1}{2} r^2 d^2 - r^4\right]_0^{d/2}$$

$$\dot{Q} = \frac{\pi}{8\mu}\rho g\left(1 + \frac{H}{h}\right)\left[\tfrac{1}{8} d^2 d^2 - \tfrac{1}{16} d^4\right]$$

$$\underline{\underline{\dot{Q} = \frac{\pi}{128\mu}\rho g\left(1 + \frac{H}{h}\right) d^4}}$$

{Vast majority of students were unable to write the expression for volume flow rate and, consequently, were unable to complete this question.}

In some questions there are several methods that could have been used. These alternative methods are outlined in this crib for comparison and completeness. Clearly, only one method was required in the examination.

Dr J. P. Longley, April/July 2008