ENGINEERING TRIPOS PART IB

June 2008

Paper 6 - Worked solutions

INFORMATION ENGINEERING

STATIONERY REQUIREMENTS

SPECIAL REQUIREMENTS

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: 4

SECTION A

Answer not more than ***two*** *questions from this section.*

1 (a) Define the term frequency response for an asymptotically stable system having transfer function $G(s)$, and explain how it can be used to calculate the steady state response of such a system to inputs of the form $x(t) = \sin(\omega t)$, where ω rad.s^{-1} is a fixed frequency. [6]

Solution: Frequency response gives the amplitude and magnitude response of a linear stable system at a particular frequency, say ω, once all transients have died away. Can be obtained as:

$$H(s)|_{s=j\omega}$$

Response to $x(t)$ is

$$y(t) = |H(j\omega)|\sin(\omega t + \arg(H(j\omega)))$$

(b) A linear system has the following transfer function:

$$G(s) = \frac{100s}{(s+1)(s+10)}$$

Determine:

(i) The response of the system to an input signal $x(t) = 0.1H(t)$, after any initial transients have died away.

solution: System is stable as poles lie to left of imaginary axis (at -1 and -10). Hence steady state response is $0.1G(0) = 0$.

(ii) The response of the system to an input signal $x(t) = 0.3\cos(0.3t)$, again after any initial transients have died away.

solution: Frequency response is:

$$G(j0.3) = \frac{100j0.3}{(j0.3+1)(j0.3+10)} = 2.87\exp(j1.25)$$

Hence response is:

$$y(t) = 2.87 \times 0.3\cos(0.3t + 1.25) = 0.86\cos(0.3t + 1.25)$$

[6]

(c) Sketch, on the semi-log paper provided, the Bode diagram for $G(s)$. Determine from this the approximate range of frequencies where the input signal is passed unattenuated (assume that a 3dB tolerance is allowable before the signal is considered attenuated). [8]

Solution:

Bode diagram is obtained as:

Corner frequencies at $\omega = 1$ and $\omega = 10$ rad s^{-1}. Gain of s term on numerator is $0dB$ at $\omega = 1$. Hence first asymptote has gain of $100 \times 1/((1)(10)) = 10$, i.e. 20dB at $\omega = 1$. Finally, find frequencies where gain$= 1$. Since slope either side of range $1 < \omega < 10$ is approx. $\pm$20dB/decade, we need to move 1 decade below and above this range to find range of unattenuated frequencies, i.e.

$$0.1 < \omega < 100$$

is the unattentuated range.

2 (a) A linear system has impulse response $g(t)$. Under what condition is the system asymptotically stable? [6]

Solution:

We require $\int_{t=0}^{\infty} |g(t)| dt < \infty$

(b) A linear system has following impulse response:

$$g(t) = \exp(-t) \sum_{k=0}^{\infty} \delta(t - k\tau)$$

where τ is a positive constant. Determine whether the system is asymptotically stable. [6]

Solution:

Observe that $g(t) = 0$ always, so $|g(t)| = g(t)$. Hence, using summation of

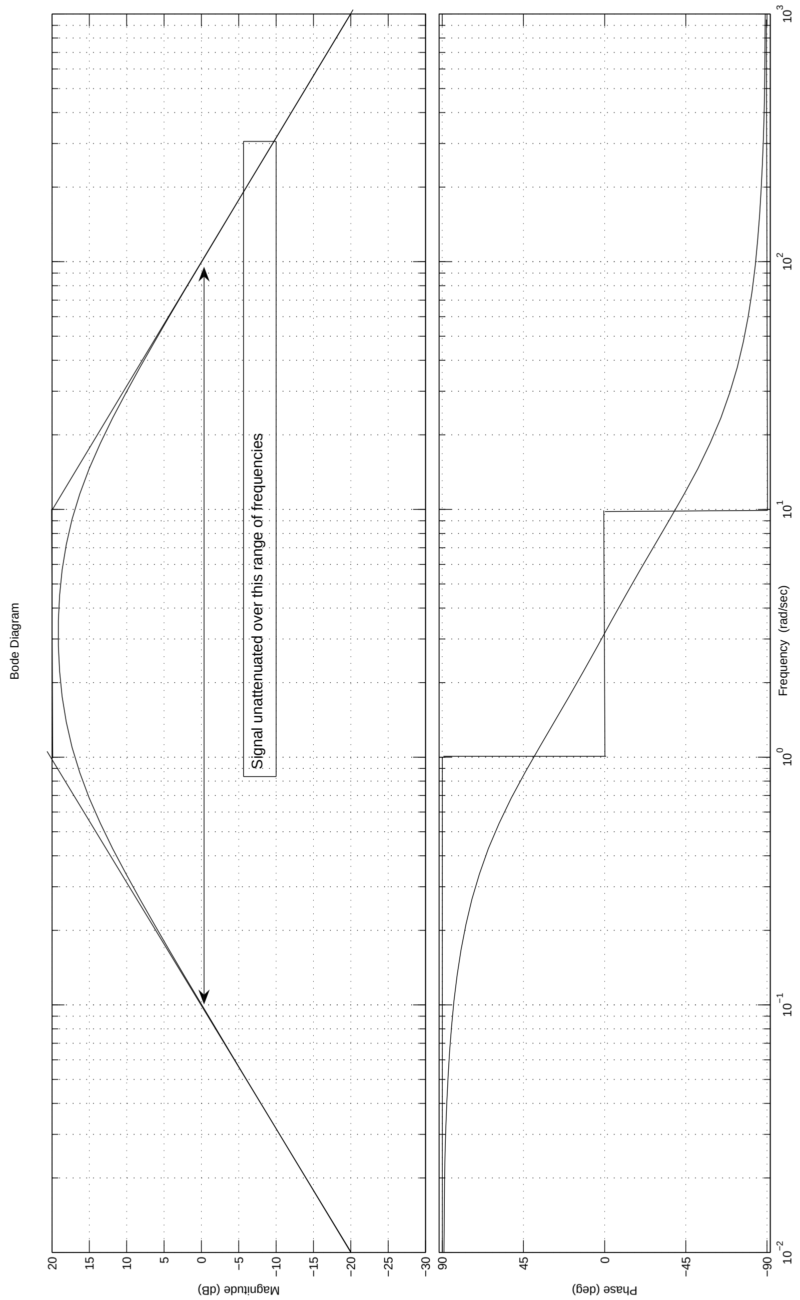

Fig. 1

geometric series:

$$
\begin{aligned}
\int_{t=0}^{\infty} |g(t)| dt &= \int_{t=0}^{\infty} \exp(-t) \sum_{k=0}^{\infty} \delta(t - k\tau) dt \\
&= \sum_{k=0}^{\infty} \exp(-k\tau) \\
&= \frac{1}{1 - \exp(-\tau))} \\
&< \infty
\end{aligned}
$$

(cont.

Therefore it is asymptotically stable.

(c) Show that the transfer function of the system is

$$G(s) = \frac{1}{1 - \exp(-(s+1)\tau)}$$

Consider a sine-wave input signal $x(t) = \sin(\omega t)$, where ω rad s^{-1} is a fixed frequency. Determine the smallest non-zero frequency ω for which the steady state output signal is in phase with the input signal, when $x(t)$ is input to the system $G(s)$. What is the gain of the system at that frequency? [8]

Solution: Transfer function is Laplace transform of impulse response:

$$\begin{aligned} G(s) &= \int_0^\infty \exp(-t) \sum_{k=0}^{\infty} \delta(t - k\tau) \exp(-st) dt \\ &= \sum_{k=0}^{\infty} \exp(-k\tau) \exp(-sk\tau) \\ &= \frac{1}{1 - \exp(-\tau(s+1))} \end{aligned}$$

Steady state output given by the frequency response, $G(j\omega)$. The output is in phase when $\arg G(j\omega) = 2n\pi$. This is obtained when

$$\arg(1 + \exp(-\tau(j\omega + 1)) = 2n\pi$$

i.e.

$$\Im \exp(-\tau(j\omega + 1)) = 0$$

[since $\Re 1 - \exp(-\tau(j\omega + 1))$ is never negative]

This occurs whenever

$$\sin(\tau\omega) = 0$$

i.e.

$$\tau\omega = n\pi$$

Hence smallest non-zero frequency is

$$\omega = \pi/\tau$$

The gain at this frequency, i.e. $\omega = \pi/\tau$ is:

$$|G(j\omega)| = \left| \frac{1}{1 - \exp(-\tau)\cos(\pi)} \right| = \frac{1}{1 + \exp(-\tau)}$$

3 A linear system has transfer function

$$G(s) = \frac{10}{s(s^2 + 2\sqrt{2}s + 4)}$$

The system is to be controlled by negative feedback with gain k_p, as shown in Figure 2.

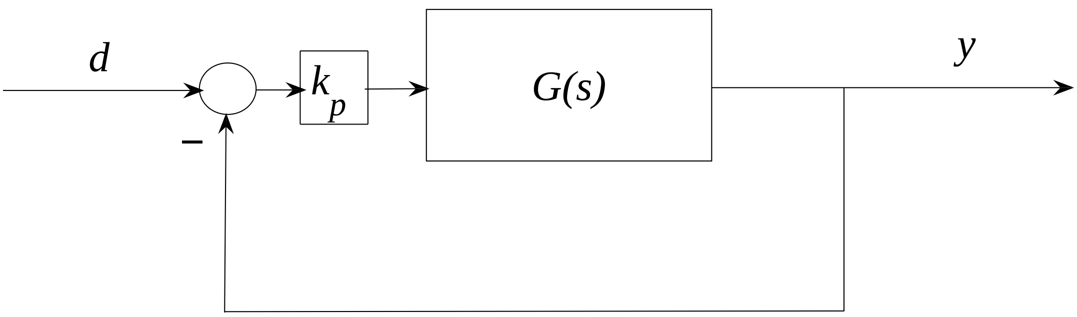

Fig. 2

(a) Determine the closed-loop transfer function of the system. [6]

solution:

$$\frac{k_p G(s)}{1 + k_p G(s)} = \frac{10k_p}{s(s^2 + 2\sqrt{2}s + 4) + 10}$$

(b) A part of the Nyquist diagram for the system is shown in Fig. 3. Determine the gain and phase margins for the system, with $k_p = 1$. How would you expect the system to respond to changes in the desired signal d? [10]

Solution:

Gain margin is measured as $a \approx 1/0.88 = 1.14$, see annotated figure. Phase margin is $\psi \approx 5$degrees, again see annotated figure.

These margins are very small, so we expect an oscillatory response to the changes in desired r.

(c) When $k_p = 0.5$, determine the range of frequencies for which

$$\left| \frac{1}{1 + k_p G(j\omega)} \right| \geq 1$$

Explain the implications of this for system behaviour. [6]

Solution:

When $k_p = 0.5$ the Nyquist diagram has the same phase but half the magnitude. In order to find the frequency where

$$\left| \frac{1}{1 + k_p G(j\omega)} \right| = 1$$

we note that $1 + k_p G(j\omega)$ is the complex vector joining $k_p G(j\omega)$ to the point $(-1, 0)$. On our graph, which corresponded to $k_p = 1$, this will therefore correspond to the vector joining $G(j\omega)$ and $(-2, 0)$. We look for the frequency where this vector has magnitude equal to 2, i.e. the frequency 0.94 rad s$^-$1, see annotated diagram.

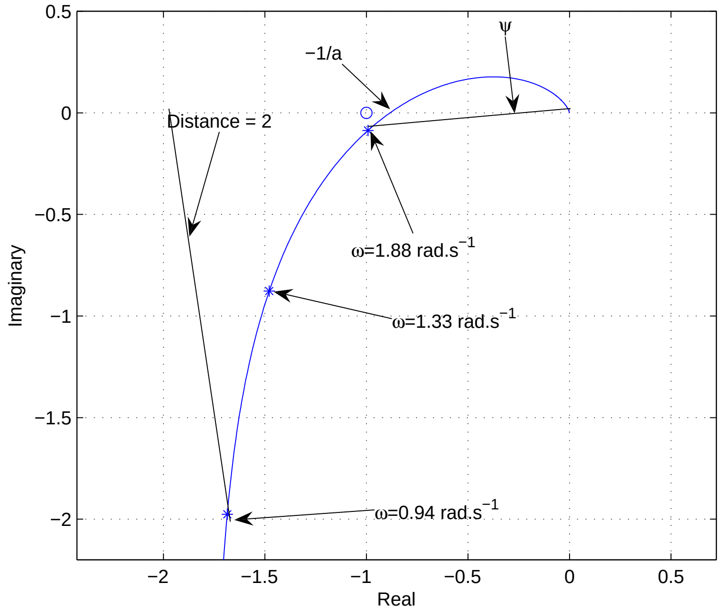
ψ
−1/a
Distance = 2
ω=1.88 rad.s⁻¹
ω=1.33 rad.s⁻¹
ω=0.94 rad.s⁻¹
Imaginary
Real

Fig. 3

SECTION B

*Answer not more than **two** questions from this section.*

4 (a) A signal $f(t)$ is as shown in Figure 4.

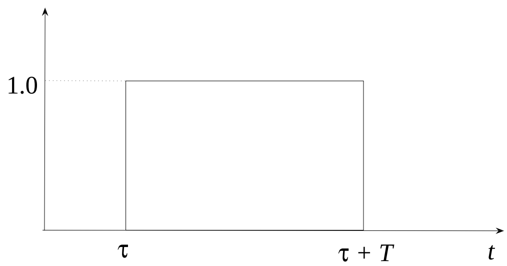

Fig. 4

Show by direct integration that its Fourier Transform is

$$F(\omega) = Te^{-j\omega(\tau+T/2)}\mathrm{sinc}(\omega T/2)$$

[5]

Solution:

$$\begin{aligned}
F(\omega) &= \int_{\tau}^{\tau+T} 1.\exp(-j\omega t)dt \\
&= -\frac{1}{j\omega}(\exp(-j\omega(\tau+T) - \exp(-j\omega(\tau)) \\
&= -\frac{1}{j\omega}\exp(-j\omega(\tau+T/2))(\exp(-j\omega(T/2)) - \exp(-j\omega(-T/2)) \\
&= -\frac{1}{j\omega}\exp(-j\omega(\tau+T/2))(-2j\sin(\omega(T/2))) \\
&= T\exp(-j\omega(\tau+T/2))\mathrm{sinc}(\omega T/2)
\end{aligned}$$

as required.

(b) It is proposed to sample $f(t)$ at regular times $t = ...0, T/M, 2T/M, ...$, etc., where M is an integer constant. Now, take $\tau = 0$. Show that the Discrete Time Fourier Transform (DTFT) $F_s(\omega)$ of the sampled waveform is:

$$F_s(\omega) = \exp(-j\omega T/2)\frac{\sin((M+1)\omega T/(2M))}{\sin(\omega T/(2M))}$$ [8]

Solution:

Sampled values of $f(t)$ are equal to 1 for times $t = 0, T/M, 2T/M...T$. Hence DTFT is:

$$\begin{aligned}
F_s(\omega) &= \sum_{n=0}^{M} 1.\exp(j\omega nT/M) \\
&= \frac{1-\exp(-j\omega T/M)^{M+1}}{1-\exp(-j\omega T/M)} \\
&= \frac{\exp(-j\omega T(M+1)/(2M))}{\exp(-j\omega T/(2M))}\frac{2j\sin(\omega T(M+1)/(2M))}{2j\sin(\omega T/(2M))} \\
&= \exp(-j\omega T/2)\frac{\sin(\omega T(M+1)/(2M))}{\sin(\omega T/(2M))}
\end{aligned}$$

as required, using the summation formula for a finite term geometric series, see maths databook.

(c) Hence, using sampling theory or otherwise, explain why the following identity holds:

$$\sum_{k=-\infty}^{+\infty}(-1)^{kM}\mathrm{sinc}(\omega T/2+\pi kM) = \frac{\sin((M+1)\omega T/(2M))}{M\sin(\omega T/(2M))}$$ [7]

Solution: Sampling theory states that the spectrum of a signal sampled at times nT/M will be:

$$F_s(\omega) = \frac{M}{T}\sum_{m=-\infty}^{+\infty} F(\omega - m\omega_0)$$

where $\omega_0 = 2\pi M/T$ rad.s^{-1} is the sampling frequency. Hence, since we know $F_s(\omega)$ directly from part (b), and we know $F(\omega)$ from part (a), we can equate the two representations (with $\tau = 0$):

$$F_s(\omega) = \frac{M}{T}\sum_{m=-\infty}^{+\infty} Te^{-j(\omega+2m\pi M/T)(T/2)}\mathrm{sinc}((\omega+2m\pi M/T)T/2)$$

$$= \exp(-j\omega T/2)\frac{\sin(\omega T(M+1)/(2M))}{\sin(\omega T/(2M))}$$

But $e^{-j(\omega+2m\pi M/T)(T/2)} = e^{-j(\omega T/2)}(-1)^{mM}$, so

$$\sum_{m=-\infty}^{+\infty}(-1)^{mM}\operatorname{sinc}((\omega+2m\pi M/T)T/2) = \frac{1}{M}\frac{\sin(\omega T(M+1)/(2M))}{\sin(\omega T/(2M))}$$

as required.

Examiner's Comment: this last part was only solved by a handful of candidates in the examination.

5 (a) Explain the meaning of channel capacity. [2]

(b) The binary erasure channel is a channel for which the transmitted bits are either received correctly with probability $1-\varepsilon$ or erased with probability ε, i.e. there are no bit-flips . The capacity of this channel is

$$C = 1-\varepsilon \quad \text{bits/channel use}$$

where ε is the erasure probability. Sketch the capacity of the channel as a function of ε and shade the region of rates that are not achievable. [2]

(c) Assume now that the erasure probability ε depends on the signal-to-noise ratio (SNR) of a given modulation. What sort of relationship would you expect between ε and SNR? Justify your answer. [3]

(d) If $\varepsilon = 1/\text{SNR}$, sketch the capacity of the channel as a function of SNR, shading again the region of rates that are not achievable. [3]

(e) A speech signal of bandwidth $B = 3$ kHz is to be transmitted over the above channel, but needs to be digitised first.

(i) How many samples per second do we need from this signal so that we can reconstruct it perfectly? [2]

(ii) Show that the quantisation noise variance of a uniform quantiser is $\Delta^2/12$, where Δ is the quantisation step. [4]

(iii) Assume that we use a 10-bit uniform quantiser, that one channel use of the above binary erasure channel corresponds to $10\ \mu$s, and that a rate $r = 1/2$ code is used to correct the errors that occur in the channel. Is the corresponding rate achievable at $\text{SNR} = 10$ dB? [4]

(a) *Explain the meaning of channel capacity.*

The channel capacity is the largest data rate that can be transmitted over a given communications channel with arbitrarily low error probability.

(b) *The binary erasure channel is a channel for which the transmitted bits are either received correctly with probability $1-\varepsilon$ or erased with probability ε, i.e., there are* *no bit-flips. The capacity of this channel is*

$$C = 1 - \varepsilon \quad \textit{bits/channel use} \tag{1}$$

where ε is the erasure probability. Sketch the capacity of the channel as a function of ε and shade the region of rates that are not achievable.

The capacity of the above channel is shown in Figure 5. Achievability regions are shown in the figure. Achievable rates are rates R such that $R < C$, i.e., the region underneath the curve. Conversely, rates that are not achievable are the rates $R > C$, i.e., the region above the curve.

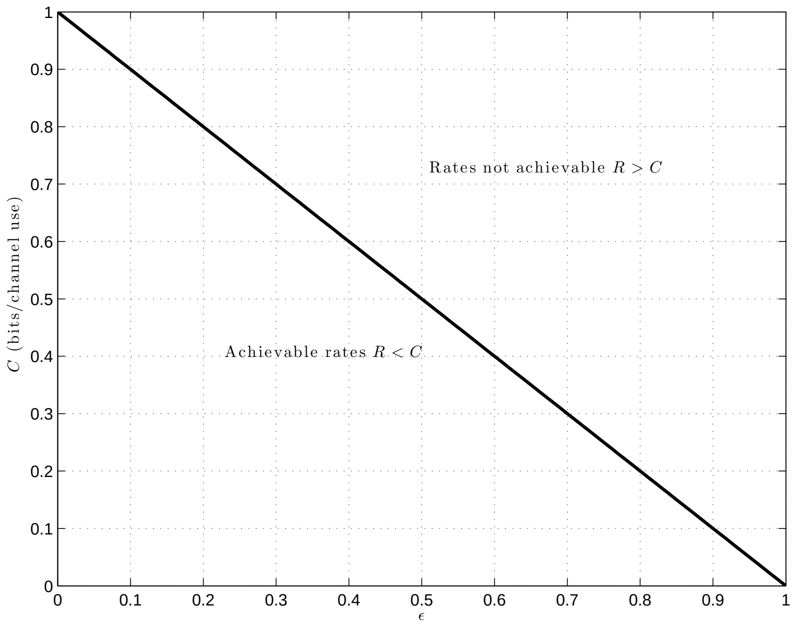

Fig. 5

(c) *Assume now that the erasure probability ε depends on the signal-to-noise ratio (SNR) of a given modulation. What sort of relationship would you expect between ε and SNR? Justify your answer.*

We would expect a decreasing relationship, i.e., ε gets smaller as SNR increases. The SNR represents the quality of a given communications link, and therefore, the larger the SNR the less errors (erasures in this case) we will have.

(d) *If $\varepsilon = \frac{1}{\mathrm{SNR}}$, sketch the capacity of the channel as a function of SNR, shading again the region of rates that are not achievable.*

The corresponding plot is shown in Figure 6.

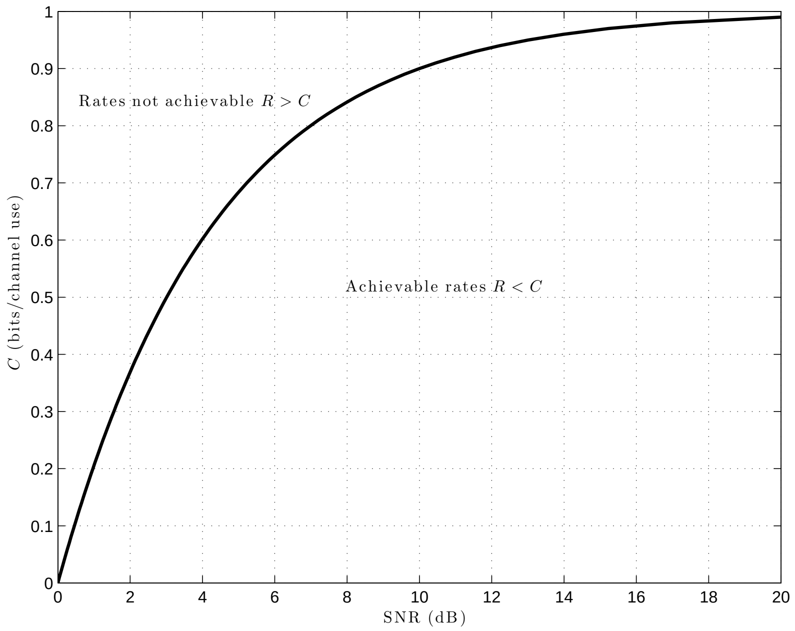

Fig. 6

(e) *A speech signal of bandwidth $B = 3$ kHz is to be transmitted over the above channel, but needs to be digitised first.*

(i) *How many samples per second do we need from this signal so that we can reconstruct it perfectly?*

According to Nyquist's theorem we need $f_s = 2B = 6000$ samples/second.

(ii) *Show that the quantisation noise variance of a uniform quantiser is $\frac{\Delta^2}{12}$, where Δ is the quantisation step.*

Following the notes, we model the quantisation error as a uniformly distributed random variable on the interval $\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$. In order for the probability density function of the quantisation error to integrate to 1, the value of the pdf is $\frac{1}{\Delta}$. Then, the quantisation noise power is

$$N_Q = E[E^2] = \int x^2 \frac{1}{\Delta} dx = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} x^2 \frac{1}{\Delta} dx = \frac{1}{\Delta} x^3 3 \Big|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{\Delta^2}{12}. \quad (2)$$

(iii) *If we use a* 10*-bit uniform quantiser, and one channel use of the above binary erasure channel corresponds to* $10\mu s$*, is the corresponding rate achievable at* SNR $=10$ *dB if we use a rate* $r=\frac{1}{2}$ *code to correct the errors that occur in the channel?*

The resulting data rate is $R = 10 \times 2B \times 1/r = 120$ kbits/second.

The capacity at SNR $=10$ dB is

$$c = 1 - \frac{1}{10^{0.1\times 10}} = 1 - 0.1 = 0.9 \text{ bits/channel use} \tag{3}$$

If a channel use corresponds to 10 μs, then the capacity in bits/second is $C = \frac{0.9}{10\times 10^{-6}} = 90$ kbit/second. Therefore the rate $R < C$ is not achievable.

6 (a) Describe the principal analogue and digital amplitude modulation techniques and outline the main differences between them. [6]

(b) Consider a signal with $B = 5$ kHz. Derive, sketch and compare the spectra obtained with double sideband (DSB) analogue modulation and with binary amplitude shift-keying (ASK), using the association $0 \rightarrow -A$ and $1 \rightarrow +A$. Assume sampling at the Nyquist rate, a 5-bit quantiser and a rectangular pulse. For the purposes of sketching, assume that the carrier frequency is much larger than B . [8]

(c) For the same set-up as in part (b), how many users can be accommodated in an FDMA cellular system with total bandwidth 20 MHz, with DSB and ASK modulations? For ASK modulated users, assume that frequencies beyond the second zero do not cause interference to other users. How would the result change if a triangular pulse were used instead of rectangular? [6]

(a) *Describe analogue and digital amplitude modulation techniques and outline the main differences. Sketch the corresponding spectra.*

All amplitude modulation techniques (both analogue and digital) modulate the amplitude of a carrier signal of frequency f_c Hz, $x_c(t) = a(t)\cos(2\pi f_c t)$ with the information signal $x(t)$ in different ways.

(i) Analogue

AMor Amplitude Modulation is a the simplest amplitude modulation technique. The amplitude of the carrier $a(t) = a_0 + x(t)$, yielding

$$s_{\mathrm{AM}}(t) = [a_0 + x(t)]\cos(2\pi f_c t). \tag{4}$$

From the above expression, AM modulation transmits the carrier signal, and will therefore be power inefficient. The spectrum is shown in Figure 7. AM requires a transmission bandwidth of $B_{\mathrm{AM}} = 2B$ and power $P_{\mathrm{AM}} = \frac{a_0^2}{2} + \frac{P_x}{2} = P_c + 2P_{\mathrm{SB}}$ where $P_c = \frac{a_0^2}{2}$ is the carrier power and $P_{\mathrm{SB}} = \frac{P_x}{4}$ is the power required to transmit a single sideband.

DSB-SCor Double Side Band - Suppressed Carrier is effectively equal to AM by letting $a_0 = 0$, i.e., the carrier is not transmitted, which improves power efficiency. The resulting modulated signal is

$$s_{\mathrm{DSB-SC}}(t) = x(t)\cos(2\pi f_c t) \tag{5}$$

and its spectrum is shown in Figure 7. The bandwidth required to transmit a DSB-SC signal is $B_{\mathrm{DSB-SC}} = 2B$ and power $P_{\mathrm{DSB-SC}} = 2P_{\mathrm{SB}}$ being therefore more power efficient than standard AM.

SSB-SCor Single Side Band - Suppressed Carrier, transmits only a single sideband, since the other sideband can be obtained from the other (by symmetry), thus saving bandwidth. The corresponding spectrum is shown in Figure 7. The bandwidth required to transmit an SSB-SC signal is $B_{\mathrm{SSB-SC}} = B$ and power $P_{\mathrm{SSB-SC}} = P_{\mathrm{SB}}$ being therefore more bandwidth and power efficient than AM or DSB-SC.

(ii) Digital

ASKor Amplitude Shift Keying produces a digitally modulated signal, once the information signal $x(t)$ has been digitised (sampled and quantised). The resulting string of bits is then mapped onto the amplitude of a carrier signal as follows

$$s_{\mathrm{ASK}}(t) = \sum_k b_k p(t - kT)\, a \cos(2\pi f_c t) \tag{6}$$

where b_k are coefficients that the information bits (typically $0, 1$ or ± 1), and $p(t)$ is the pulse shape (assumed to be rectangular pulse) that represents bits. In the case of $b_k = \pm 1$ ASK is equal to BPSK (Binary Phase Shift Keying), since multiplying a carrier signal by ± 1 is equivalent to change the phase from 0 to π radians. The spectrum in digital modulations depends heavily on the pulse shape. In the case of a rectangular pulse (whose spectrum is a sinc function), the spectrum is shown in Figure 8.

(b) *Consider a signal with $B = 5$ kHz. Calculate, sketch and compare the spectra obtained with double sideband (DSB) analogue modulation and with binary amplitude shift-keying (ASK), using the association $0 \to -A$ and $1 \to +A$, assuming sampling at Nyquist rate, a 5-bit quantiser and a rectangular pulse. For sketching purposes, assume that the carrier frequency is much larger than B.*

As shown above, the DSB signal at positive frequencies occupies a bandwidth of 5 kHz. Instead the ASK signal, sampled Nyquist rate, and quantised with a 5-bit quantiser, yields a rate of $R = 5 \times 2B = 50$ kbit/s. This means that we are transmitting one bit every $T = 1/50{,}000$ seconds. The sinc function has zeros at $\pm\frac{k}{T}$, $k = 1, \ldots$. The approximate

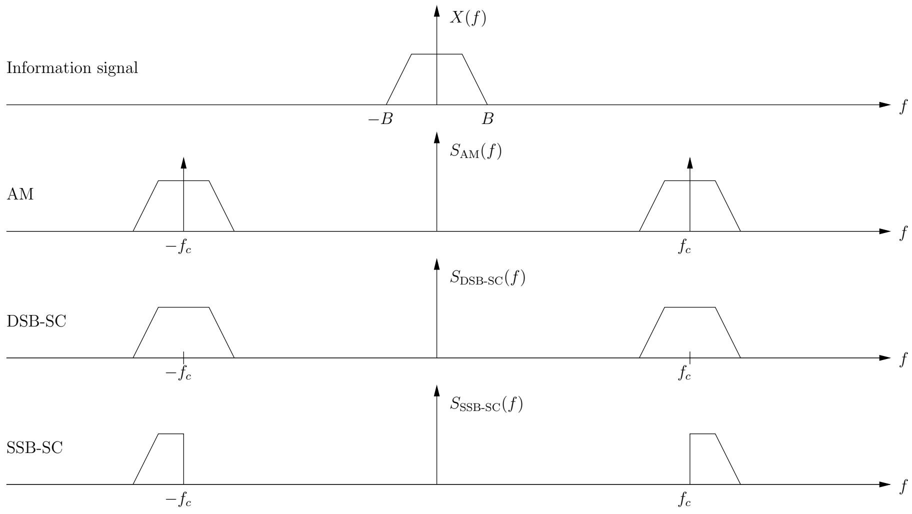

Fig. 7

plot is given in Figure 9. In this plot, only bandwidths are up to scale, not the amplitudes.

(c) *How many users can be accommodated in an FDMA cellular system with total bandwidth* 20 *MHz, with DSB and ASK modulations? For ASK modulated users, assume that frequencies beyond the second zero do not cause interference to other users. How would the result change if a triangular pulse is used instead?*

An FDMA system divides the spectrum in equal parts for all users. Therefore we can accommodate

$$N_{\text{DSB}} = \frac{20,000,000}{5,000} = 4000 \quad \text{DSB users.} \tag{7}$$

Since the second zero is at $\frac{2}{T}$, then the ASK bandwidth is $\frac{4}{T}$, and therefore we can accommodate

$$N_{\text{ASK}} = \frac{20,000,000}{\frac{4}{50,000}} = 100 \quad \text{ASK users.} \tag{8}$$

Nothing would change in terms of number of users with a triangular pulse, since the zeros are in exactly the same places.

END OF PAPER

Examiner: Simon Godsill

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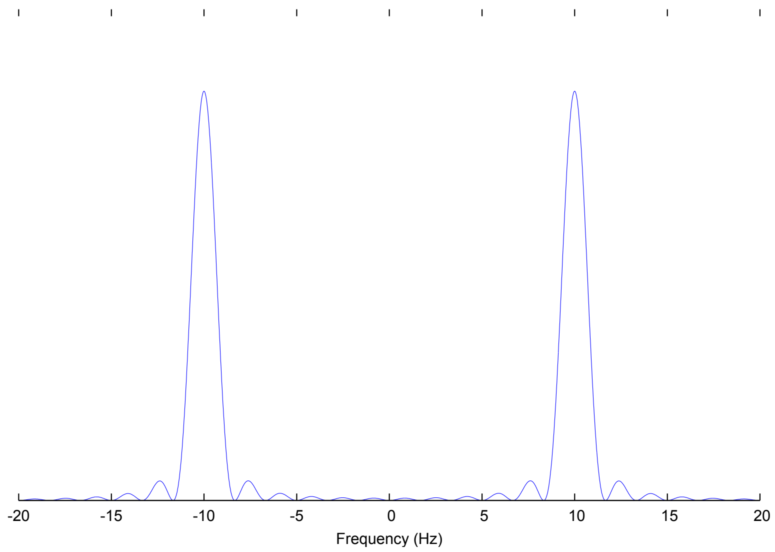

Fig. 8

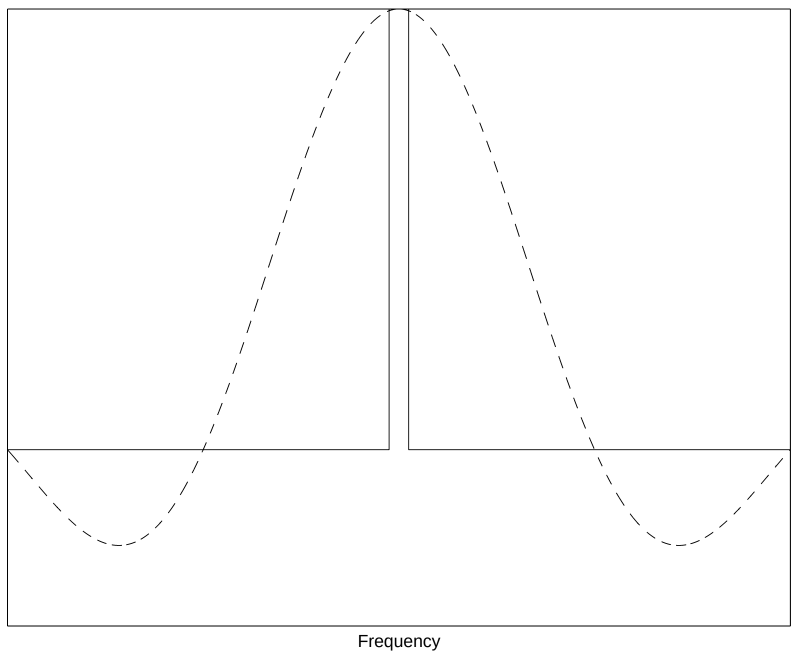

Fig. 9

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