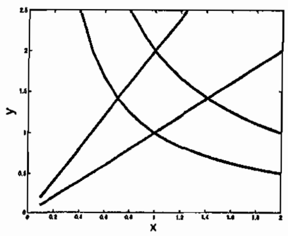
SECTION A

1 (a) Figure as follows, with enclosed region shaded:



(b) The area of region R is evaluated using a double integral,

$$I_1 = \int\int_R dx\,dy$$

. The best way to do this is via a change of variables: $u = xy$ and $v = y/x$ which makes both limits of integration 1 to 2. The remaining step is to compute the determinant of the Jacobian of the transformation

$$du = y\,dx + x\,dy$$

$$dv = -\frac{y}{x^2}\,dx + \frac{1}{x}\,dy$$

The determinant is $2\frac{y}{x} = 2v$, so

$$I_1 = \int_1^2 \int_1^2 \frac{1}{2v}\,du\,dv = \frac{1}{2}\ln 2$$

(c) Stokes' theorem (in Databook) states that

$$I = \oint_C \mathbf{V} \cdot d\mathbf{r} = \int_R (\nabla \times \mathbf{V}) \cdot d\mathbf{A}$$

The curl of $\mathbf{V}$ is

$$\nabla \times \mathbf{V} = -2\mathbf{k}$$

So the integral of interest is

$$I = -\int\int_R 2\,dx\,dy = -2I_1 = -\ln 2$$

2 (a) We use separation of variables $T(x,y) = f(x)g(y)$. Laplaces equation results in

$$f''g + g''f = 0$$

hence

$$\frac{f''}{f} = c = -\frac{g''}{g}$$

where c is a constant. Solving for $f(x) = b\exp(-ax)$ and $g(y) = e\cos(hy)$, we get $c = a^2 = h^2$, using the boundary conditions and this separation to variables we obtain

$$T(x,y) = T^* \cos(\pi y/(2d)) \exp(-\pi x/(2d)).$$

(b) (i)

$$\oint_C \mathbf{q} \cdot d\mathbf{l} = \oint_C -\lambda \nabla T \cdot d\mathbf{l} = -\lambda \oint_C \nabla T \cdot d\mathbf{l} = 0$$

$$\int_O^Q \mathbf{q} \cdot d\mathbf{l} = -\lambda \int_O^Q \nabla T \cdot d\mathbf{l} = -\lambda [T_Q - T_O] = \lambda T^* [1 - 0] = \lambda T^*$$

(ii)

$$\nabla \times \mathbf{q} = \nabla \times (-\lambda \nabla T) = -\lambda \nabla \times \nabla T = 0$$

$$\nabla \times (T\mathbf{q}) = T \nabla \times \mathbf{q} + \nabla T \times \mathbf{q} = 0 + \nabla T \times (-\lambda \nabla T) = 0$$

(iii)

$$\nabla \cdot \mathbf{q} = \nabla \cdot (-\lambda \nabla T) = -\lambda \nabla (\nabla T) = 0$$

$$\nabla \cdot (T\mathbf{q}) = T \nabla \cdot \mathbf{q} + \mathbf{q} \cdot \nabla T = -\lambda (\nabla T)^2 < 0$$

(iv)

$$\nabla \cdot q = \lim_{\delta V \to 0} \frac{\oint_S \mathbf{q} \cdot d\mathbf{s}}{\delta V}$$

where **s** encloses a small volume δV. So $\nabla \cdot \mathbf{q} = 0$ implies $\oint_S \mathbf{q} \cdot d\mathbf{s} = 0$. Steady state temperature distribution requires no net flux if energy into volume δV, i.e. $\oint_S \mathbf{q} \cdot d\mathbf{s} = 0$.

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3 (a) Several ways of defining conservative vector fields, but the one that can be easily satisfied by choosing f is the condition that the curl is zero.

$$\nabla \times \mathbf{B} = 0.$$

For this to hold,

$$\mathbf{i}(\frac{\partial f}{\partial y} - \frac{\partial xz}{\partial z}) - \mathbf{j}(\frac{\partial f}{\partial x} - \frac{\partial(2x+yz)}{\partial z}) + \mathbf{k}(\frac{\partial xz}{\partial x} - \frac{\partial(2x+yz)}{\partial y}) = 0$$

which results in:

$$\frac{\partial f}{\partial y} = x$$

$$\frac{\partial f}{\partial x} = y$$

and $z = z$! For example $f(x,y,z) = xy$ would work. A more general solution is $f(x,y,z) = xy + h(z)$ for an arbitrary function $h(z)$.
The potential field ϕ needs to satisfy $-\nabla\phi = \mathbf{B}$. By matching the partial derivatives we get for the above

$$\phi(x,y,z) = -x^2 - xyz + c$$

where c is a constant.

(b) For $\mathbf{B}$ to be solenoidal, it needs to satisfy $\nabla \cdot \mathbf{B} = 0$. Therefore

$$\frac{\partial(2x+yz)}{\partial x} + \frac{\partial xz}{\partial y} + \frac{\partial f}{\partial z} = 0$$

so

$$\frac{\partial f}{\partial z} = -2$$

so $f(x,y,z) = -2z + c$ for any constant c.

(c) The flux of $\mathbf{B}$ through the unit square is

$$I = \int_0^1 \int_0^1 \mathbf{B} \cdot d\mathbf{A} = \int_S x\, y\, dx\, dy = \int_0^1 \frac{1}{2} y\, dy = \frac{1}{4}$$

(d) The volume integral is

$$\mathbf{I} = \int_V \mathbf{B}\, dV.$$

(cont.

We can calculate each element separately,

$$I_x = \int_0^1 \int_0^1 \int_0^1 (2x + yz)\, dx\, dy\, dz = x^2 + \frac{1}{4}y^2z^2|_0^1 = \frac{5}{4}$$

$$I_y = \int_0^1 \int_0^1 \int_0^1 xz\, dx\, dy\, dz = \frac{1}{4}x^2z^2y|_0^1 = \frac{1}{4}$$

$$I_z = \int_0^1 \int_0^1 \int_0^1 xy\, dx\, dy\, dz = \frac{1}{4}x^2y^2z|_0^1 = \frac{1}{4}$$

So we have

$$\mathbf{I} = \frac{5}{4}\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{1}{4}\mathbf{k}$$

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SECTION B

4 (a) The covariance between Y_1 and Y_2 is

$$\frac{1}{36}\sum_{x_1=1}^{6}\sum_{x_2=1}^{6}(x_1+x_2-7)(x_1-x_2) \;=\; \frac{1}{36}\sum_{x_1=1}^{6}\sum_{x_2=1}^{6}x_1^2-x_2^2-7x_1+7x_2 \;=\; 0,$$

since X_1 and X_2 have identical distributinons.

(b) Y_1 and Y_2 are not idependent. For example, $Y_1 = 12$ implies $Y_2 = 0$.

(c) $Z = Y_1 - Y_2 = X_1 + X_2 - X_1 + X_2 = 2X_2$. We have $E[X_2] = \frac{1}{6}\sum_{x=1}^{6} x = 7/2$, and $V[X_2] = \frac{1}{6}\sum_{x=1}^{6}(x-7/2)^2 = 35/12$. Thus, $E[Z] = 2E[X_2] = 7$ and $V[Z] = 4V[X_2] = 35/3$.

(d) $p(Y_2 = 0) = 1/6$. The number of times you observe $Y_2 = 0$ in 25 rolls follows a Binomial $B(25, 1/6)$. The probability of oberving $Y_2 = 0$ at least twice, is $1 - p(\text{zero times}) - p(\text{once})$. which is $1-(5/6)^{25} - 25(1/6)(5/6)^{24} = 0.94$

(e) Under the null-hypothesis, that the dice are fair, this outcome, or something more extreme occours with probability $(5/6)^{25} + 25(1/6)(5/6)^{24} = 0.06$, so we would reject the null hypothesis at the $> 6\%$ level, but not at the 5% level.

5 (a) The determinant $|A|$ is $(2x+2) - 2(2x) = 2 - 2x$. So $|A^3| = |A|^3 = (2-2x)^3$.

(b) Solve $A\mathbf{v} = \mathbf{0}$, for $x = 1$. Eg, $v_1 + 2v_2 = 0 \Rightarrow v_2 = -v_1/2$ and $-2v_2 + v_3 = 0 \Rightarrow v_3 = 2v_2 = -v_1$. So, $\mathbf{v} = (v_1, -v_1/2, -v_1)^\top$. Normalize: $9/4v_1^2 = 1 \Rightarrow v_1 = 2/3$. Solution $\mathbf{v} = \pm(2/3, -1/3, -2/3)$. This is an eigenvector corresponding to $\lambda = 0$.

(c) $|A| \sim N(\mu = 2, \sigma^2 = 4)$, since adding an offset translates the mean, and multiplying by a factor, scales the variance by the square of this factor.

(d) $\mathbf{v}$ is a vector of random length in the direction given by $\mathbf{z}$. The equation $A\mathbf{v} = \lambda\mathbf{v}$ only has a solution if $\mathbf{v}$ is an eigenvector of A. So, we must ensure that $\mathbf{v}$ is in the

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direction of an eigenvector of A. In the previous question we saw that $\mathbf{z}$ is an eigenvector of A for $x = 1$, so one solution is $x = 1$.

6 (a) E.g. Gauss-elimination. Answer

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}.$$

(b) At every iteration, the component of $\mathbf{v}$ in the direction of the eigenvector corresponding to the largest eigenvector gets amplified the most. So, $\mathbf{v}$ turns to align with λ. The initial vector cannot be orthogonal to the largest eigenvector.

(c) Use the inverse of A instead of A, or $(sI - A)$ if $s > \lambda_{\text{max}}$.

(d) The power method finds the eigenvector with the largest absolute eigenvalue. If the eigenvalue is negative, then (at "convergence") the elements of $\mathbf{v}$ change sign at every iteration.

END OF PAPER