ENGINEERING TRIPOS PART IB

Friday 6 June 2008 9 to 11

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer not more than ***two*** *questions from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

*Answer not more than **two** questions from this section.*

1 (a) Sketch the region R in the first quadrant of the (x,y)-plane which lies between the curves $y=x$, $y=2x$, $xy=1$ and $xy=2$. [3]

(b) By using a change of variables or otherwise, evaluate the area of the region R. [10]

(c) Consider the vector field

$$\mathbf{V}=y\,\mathbf{i}-x\,\mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions of the x- and y- axes. Using Stokes' theorem, evaluate the integral

$$I=\oint_C \mathbf{V}\cdot d\mathbf{r}$$

where C is the curve that encloses R in an anticlockwise direction. [7]

2 (a) The cooling fin shown in Fig. 1 of width $2d$ extends from $x = 0$ to $x = \infty$. The temperature at the root of the fin $x = 0$ is $T = T^* \cos(\pi y/(2d))$, and the temperature on the faces $y = \pm d$ is $T = 0$. A steady state temperature field is established, governed by $\nabla^2 T = 0$. Find $T(x,y)$ using the method of separation of variables. [6]

(b) The heat flux vector is defined by $\mathbf{q} = -\lambda \nabla T$, where λ is the thermal conductivity, which may be considered as uniform.

(i) Find the value of the line integrals $\oint_C \mathbf{q} \cdot d\mathbf{l}$ and $\int_O^Q \mathbf{q} \cdot d\mathbf{l}$ where C is the curve $OPQRO$ shown in Fig. 1, and Q is the point $(d, -d)$. [4]

(ii) Find $\nabla \times \mathbf{q}$ and $\nabla \times (T\mathbf{q})$. [3]

(iii) Show that $\nabla \cdot \mathbf{q} = 0$ and $\nabla \cdot (T\mathbf{q}) < 0$. [3]

(iv) Use the coordinate-free definition of divergence to rewrite $\nabla \cdot \mathbf{q}$ as a surface integral and hence provide a physical interpretation of $\nabla \cdot \mathbf{q} = 0$. [4]

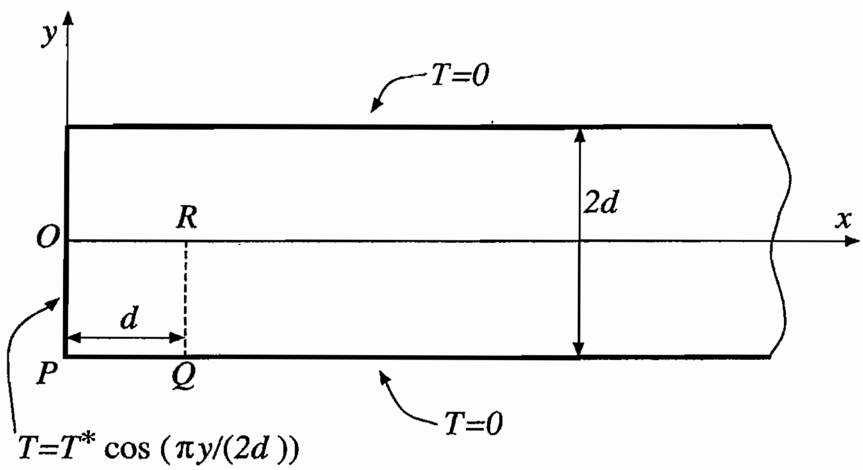

Fig. 1

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3 Consider the vector field

$$\mathbf{B} = (2x + yz)\,\mathbf{i} + xz\,\mathbf{j} + f(x,y,z)\,\mathbf{k}$$

where **i**, **j**, **k** are unit vectors in the x, y and z directions respectively.

(a) Describe the conditions on $f(x,y,z)$ such that $\mathbf{B}$ is a conservative field, and find a possible scalar potential $\phi(x,y,z)$ of $\mathbf{B}$. [6]

(b) Describe the conditions on $f(x,y,z)$ such that $\mathbf{B}$ is a solenoidal field. [5]

Let the surface S be the unit square in the (x,y)-plane $(z = 0)$ with corners $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$, and let $f(x,y,z) = xy$.

(c) Compute the flux of $\mathbf{B}$ through S. [4]

(d) For $f(x,y,z) = xy$ and V being the unit cube with corners $(0,0,0)$, $(0,0,1)$ ... $(1,1,1)$, compute the volume integral

$$\int_V \mathbf{B}\, dV.$$

[5]

SECTION B

*Answer not more than **two** questions from this section.*

4 The two random variables X_1 and X_2 represent the number of dots on each die after rolling a pair of dice. The random variables Y_1 and Y_2 are defined as:

$$Y_1 = X_1 + X_2 \quad \text{and} \quad Y_2 = X_1 - X_2.$$

(a) What is the covariance between Y_1 and Y_2? [3]

(b) Are Y_1 and Y_2 independent? Explain your answer. [3]

(c) What is the mean and variance of $Z = Y_1 - Y_2$? [4]

(d) What is the probability of observing $Y_2 = 0$ at least twice in 25 rolls of the two dice? [5]

(e) 25 rolls are performed and $Y_2 = 0$ occurs only once. Does this outcome constitute statistically significant evidence that the dice may not be "fair"? Explain your answer. [5]

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5 Consider the matrix A given by:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & -2 & x \end{bmatrix},$$

where x is a variable.

(a) Compute the value of $|A^3|$, i.e. the determinant of the third power of A. [4]

(b) Assuming $x = 1$, find a unit vector, $\mathbf{v}$ which solves $A\mathbf{v} = \mathbf{0}$, where $\mathbf{0}$ is a vector of zeros. [4]

(c) Assume x is a zero mean, unit variance Normal random variable, i.e. $x \sim N(0,1)$. What is the distribution of the determinant of A ? [4]

(d) Assume that $\mathbf{v} = a\,\mathbf{z}$, where a is a Normal random variable with mean 2 and variance 2, and

$$\mathbf{z} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

Find a value of x such that $A\mathbf{v} = \lambda\mathbf{v}$ has a solution for λ with non-vanishing probability. [8]

6 Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

(a) Calculate the inverse of A. [7]

(b) For a matrix with positive eigenvalues, the *power* method for finding the largest eigenvalue/eigenvector pair of a matrix iterates the following steps:

$$\mathbf{x} := A\mathbf{v}_k, \qquad \mathbf{v}_{k+1} := \mathbf{x}/|\mathbf{x}|,$$

starting from an initial arbitrary vector $\mathbf{v}_0$. Here $|\mathbf{x}|$ is the Euclidean norm of $\mathbf{x}$. Explain how the method works, and what requirements must be satisfied by the initial vector $\mathbf{v}_0$. [5]

(c) For a non-singular matrix, modify the power method to find the eigenvalue with the smallest absolute magnitude. [4]

(d) What happens when the power method is applied to a matrix with negative eigenvalues? [4]

END OF PAPER