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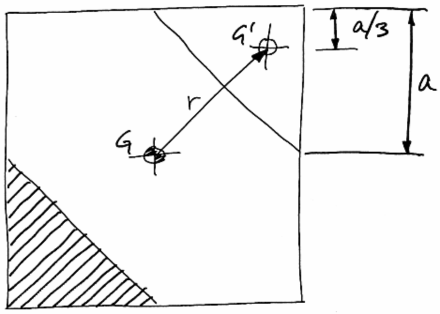
Engineering Tripos Part IB

2009

Paper 1 Mechanics

Worked Solutions

1a)



$$r = \sqrt{\left(\frac{2a}{3}\right)^2 + \left(\frac{2a}{3}\right)^2} = \frac{a\sqrt{8}}{3}$$

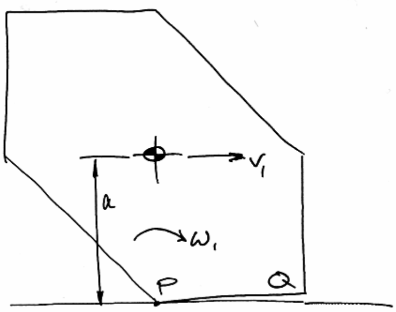
MASS OF SQUARE SECTION WOULD BE $\frac{4m}{3}$

MASS OF EACH TRIANGLE $= \frac{m}{6}$

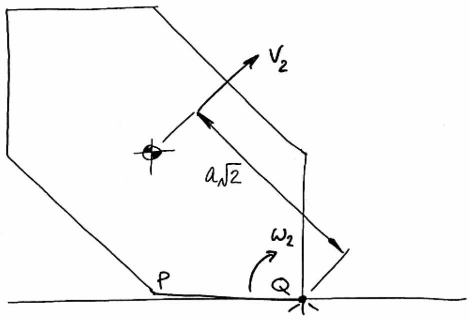
FROM DATABOOK (PAGE 18):

$$J_G = \underbrace{\frac{4m}{3}}_{\text{MASS OF SQUARE}} \underbrace{\frac{2a^2}{3}}_{J_G \text{ OF SQUARE (BY ⊥ AXIS)}} - \underbrace{2}_{\text{2 TRIANGLES}} \underbrace{\frac{m}{6}}_{\text{MASS OF TRIANGLES}} \left[\underbrace{\frac{a^2}{9}}_{J_{G'} \text{ OF TRIANGLE (BY ⊥ AXIS)}} + \underbrace{\left(\frac{a\sqrt{8}}{3}\right)^2}_{\text{// AXIS}} \right] = ma^2\left[\frac{8}{9} - \frac{1}{3}\left(\frac{1}{9} + \frac{8}{9}\right)\right] = \frac{5ma^2}{9}$$

b) BEFORE IMPACT AT Q (t_1)



AFTER IMPACT AT Q (t_2)



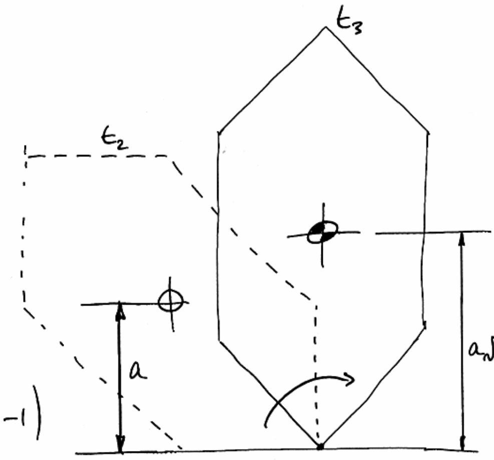
BY CONSERVATION OF MOMENT OF MOMENTUM ABOUT Q:

$$J_G\omega_1 + V_1 m a = J_G\omega_2 + V_2\, m\, a\sqrt{2}$$

$$\therefore J_G\omega_1 + a^2\omega_1 m = J_G\omega_2 + 2a^2\omega_2 m$$

$$\therefore \frac{14}{9}\omega_1 = \frac{23}{9}\omega_2$$

$$\therefore \omega_2 = \frac{14}{23}\omega_1$$



c) BY CONSERVATION OF ENERGY:

KE AT t_2 = GAIN IN PE FROM t_2 TO t_3

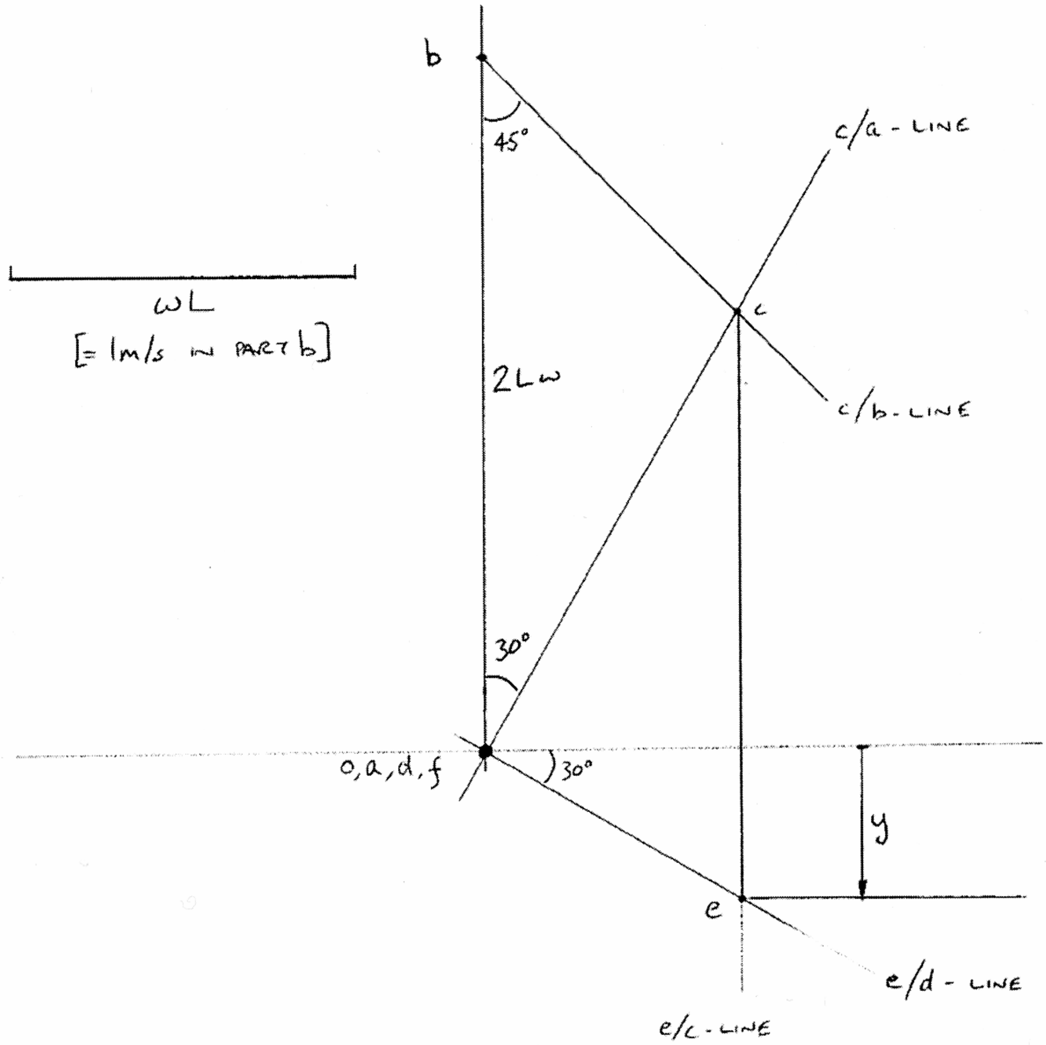
$$\therefore \frac{1}{2}J_G\omega_2^2 + \frac{1}{2}mV_2^2 = mg\left(a\sqrt{2} - a\right)$$

$$\therefore \frac{1}{2}\left(\frac{5ma^2}{9}\right)\left(\frac{14}{23}\omega_1\right)^2 + ma^2\left(\frac{14}{23}\omega_1\right)^2 = mga\left(\sqrt{2}-1\right)$$

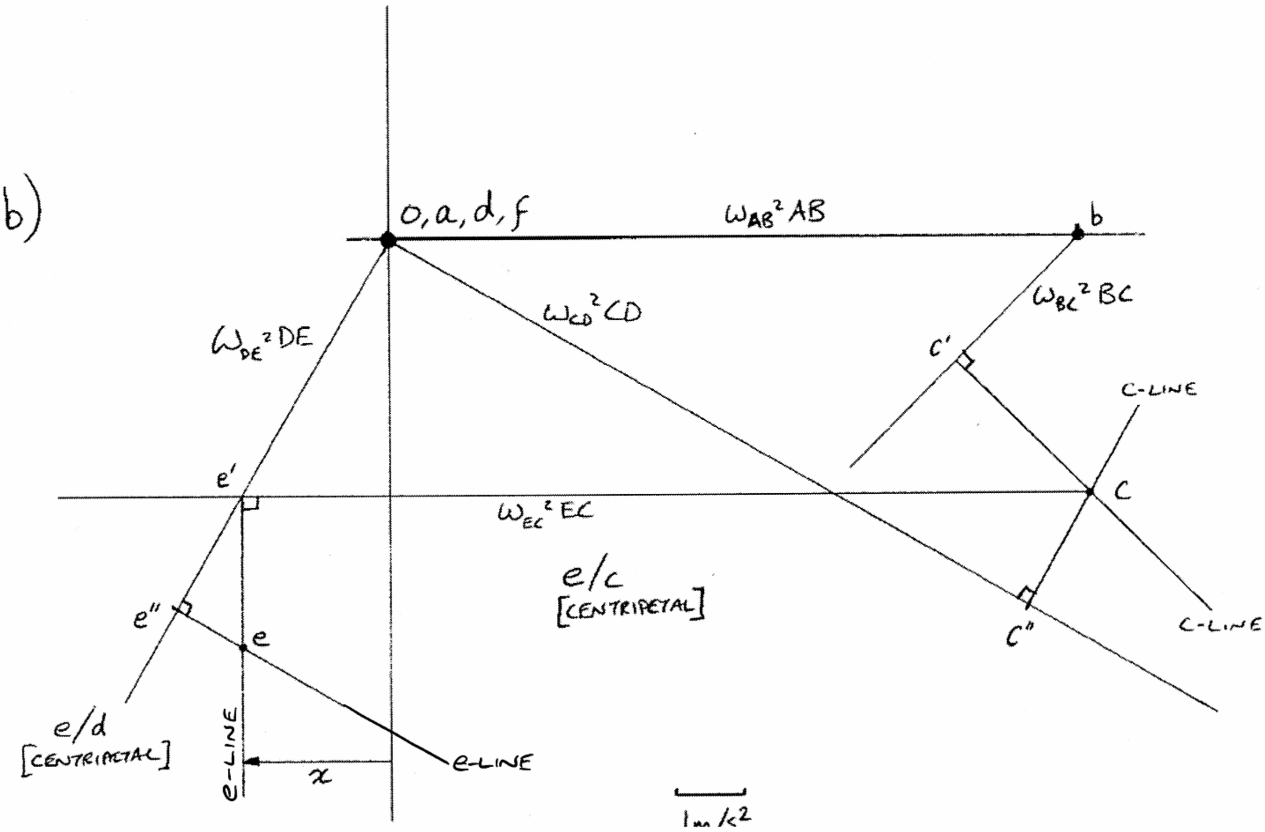
$$\therefore \frac{23}{18}\cancel{m}a^2\left(\frac{14}{23}\omega_1\right)^2 = \cancel{m}ga\left(\sqrt{2}-1\right)$$

$$\therefore \omega_1 = 2.93 \text{ RAD/S}$$

2a)


b
45°
c/a - LINE
ωL
[= 1m/s IN PART b]
c
2Lω
c/b - LINE
30°
o, a, d, f
30°
y
e
e/d - LINE
e/c - LINE

b)


o, a, d, f
$\omega_{AB}^2 AB$
b
$\omega_{BC}^2 BC$
$\omega_{CD}^2 CD$
$\omega_{DE}^2 DE$
c'
C-LINE
e'
c
$\omega_{EC}^2 EC$
e/c
[CENTRIPETAL]
e''
e
c''
C-LINE
e/d
[CENTRIPETAL]
e-LINE
x
e-LINE
1m/s²

2a) FROM VELOCITY DIAGRAM (TRIANGLE abc):

RESOLVING VERTICALLY: $bc\cos 45 + dc\cos 30 = 2\omega L$

$$\therefore \quad \frac{bc}{\sqrt{2}} + \frac{dc\sqrt{3}}{2} = 2\omega L \quad \text{——— ①}$$

RESOLVING HORIZONTALLY: $bc\sin 45 = dc\sin 30$

$$\frac{bc}{\sqrt{2}} = \frac{dc}{2} \quad \text{——— ②}$$

SUB ② INTO ①:

$$\frac{dc}{2} + \frac{dc\sqrt{3}}{2} = 2\omega L$$

$$\therefore \quad dc = \frac{4\omega L}{1+\sqrt{3}} \quad \text{——— ③}$$

FROM VELOCITY DIAGRAM (TRIANGLE cde):

LET y BE VERTICAL COMPONENT OF de

$$\tan 30 = \frac{y}{dc\sin 30} \qquad \therefore \quad y = \frac{dc}{2\sqrt{3}}$$

SUB IN FROM ③:

$$y = \frac{2\omega L}{3+\sqrt{3}}$$

b) FROM VELOCITY DIAGRAM: $\omega_{CD} = \frac{cd}{CD} = \frac{cd}{L} \; [= \omega_{CE} = \omega_{DE}]$

SUB IN FROM ③:

$$= \frac{4\omega}{1+\sqrt{3}} = 1.464\omega = 7.321 \text{ RAD/S}$$

$$\omega_{BC} = \frac{bc}{BC} = \frac{bc}{2L}$$

SUB IN FROM ②:

$$= \frac{dc\sqrt{2}}{4L}$$

SUB IN FROM ③:

$$= \frac{\omega\sqrt{2}}{1+\sqrt{3}} = 0.518\omega = 2.588 \text{ RAD/S}$$

ON ACCELERATION DIAGRAM: $ab = AB\,\omega_{AB}^2 = 0.4\,(5^2) = 10\,m/s^2$

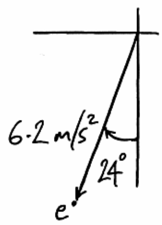
CENTRIPETAL COMPONENT OF $c/b = BC\,\omega_{BC}^2 = 0.4\,(2.588^2) = 2.679\,m/s^2$

CENTRIPETAL COMPONENT OF $c/d = CD\,\omega_{CD}^2 = 0.2\,(7.321^2) = 10.719\,m/s^2$

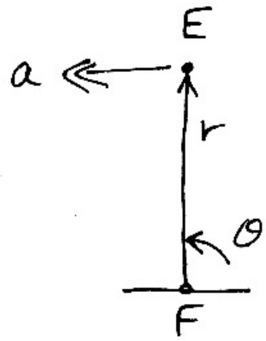
CENTRIPETAL COMPONENT OF $e/d = DE\,\omega_{DE}^2 = \frac{0.2}{\sqrt{3}}(7.321^2) = 6.189\,m/s^2$

CENTRIPETAL COMPONENT OF $e/c = CE\,\omega_{CE}^2 = \frac{2(0.2)}{\sqrt{3}}(7.321^2) = 12.378\,m/s^2$

MEASURING FROM ACCELERATION DIAGRAM:



c) FROM DATABOOK (PAGE 1):



ACCELERATION, $a = x$ (WHERE x IS DEFINED ON THE ACCELERATION DIAGRAM AND IS MEASURED AS $2.1\ m/s^2$).

$$x = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$\Rightarrow \ddot{\theta} = \frac{x - 2\dot{r}\dot{\theta}}{r}$$

$r = L = 0.2\ m$

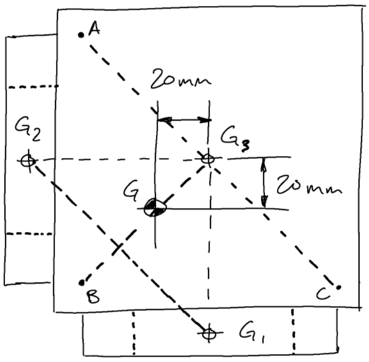
FROM PART a: $\dot{r} = -y = \frac{-2L\omega}{3+\sqrt{3}} = -0.423\ m/s$

$\dot{\theta} = \omega_{EF} = \frac{ef \cos 30}{EF} = -3.7\ RAD/s$ (CLOCKWISE)

$$\therefore \ddot{\theta} = \frac{2.1 - 2(-0.423)(-3.7)}{0.2}$$

$$= \underline{-5.151\ RAD/s^2}$$ (CLOCKWISE)

3 a)



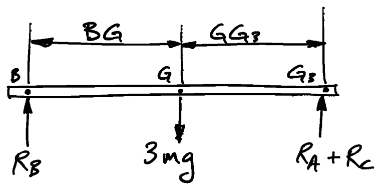
G_1 AND G_2 ARE THE CENTRES OF MASS FOR THE ROTORS

G_3 IS THE CENTRE OF MASS FOR THE CUBE

G IS THE OVERALL CENTRE OF MASS FOR THE MACHINE

AG_3C FORM A STRAIGHT LINE
BGG_3 FORM A STRAIGHT LINE

b) LOOKING PERPENDICULAR TO BGG_3:



$BG = GG_3 \quad [= \sqrt{800}\text{ mm}]$

AND $AG_3 = G_3C \quad [= \sqrt{3200}\text{ mm}]$

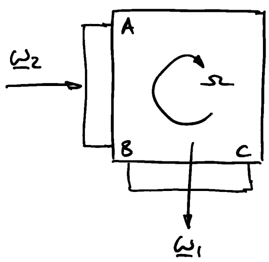
$\therefore R_A = R_C \qquad \therefore R_B = 2R_A$

$\therefore R_A : R_B : R_C = 1 : 2 : 1$

c) FROM DATABOOK (PAGE 19):

$$J_{ROTOR} = m\left[a^2 + \frac{t^2}{4}\right] = m\left[0.03^2 + \frac{0.02^2}{4}\right] = 6\times10^{-4}\ \text{kgm}^2$$

d)



TORQUE PROVIDED BY REACTIONS, $T = J\Omega \times \omega$

$\therefore$ ω_1 CAUSES INCREASED REACTION AT B, DECREASE AT A (NO CHANGE AT C).

AND ω_2 CAUSES INCREASED REACTION AT C, DECREASE AT B (NO CHANGE AT A).

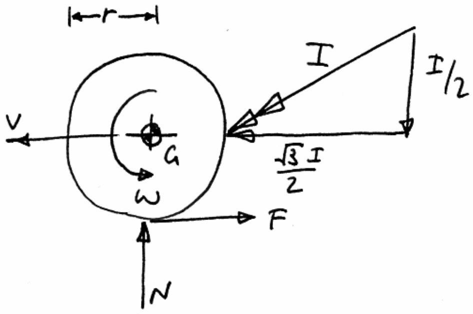
STATIC REACTIONS: $R_{A_0} = R_{C_0} = \dfrac{3mg}{4} \qquad R_{B_0} = \dfrac{3mg}{2}$

$$\therefore R_A = \frac{3mg}{4} - J\Omega\frac{\omega_1}{L}, \quad R_C = \frac{3mg}{4} + J\Omega\frac{\omega_2}{L}, \quad R_B = \frac{3mg}{2} + J\Omega\left(\frac{\omega_1 - \omega_2}{L}\right)$$

WHERE L IS THE DISTANCE AB OR BC (= 80 mm).

$$\therefore \text{FOR } R_B = 0, \quad \omega_2 = \omega_1 + \frac{3mgL}{2J\Omega} = 500 + \frac{3(0.6)g(0.08)}{2(6\times10^{-4})(0.5)} = 2854\ \text{RAD/s.}$$

4a) Consider impulsive loads:



Linear momentum ($\rightarrow$) $mv = \frac{\sqrt{3}}{2}I - F$

— " — — " — ($\uparrow$) $\frac{I}{2} - N = 0$

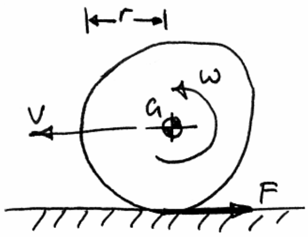
angular — " — (G:) $Fr - \frac{Ir}{2} = J\omega$

$v = r\omega \Rightarrow \frac{\sqrt{3}I}{2} - F = mr\omega$ $\qquad J = \frac{2}{5}mr^2 \Rightarrow F - \frac{I}{2} = \frac{2}{5}mr\omega$

hence: $F = \frac{2}{7}\left(\frac{\sqrt{3}}{2} + \frac{5}{4}\right)I$ and $N = \frac{I}{2}$

$$\mu \geqslant \frac{F}{N} = \frac{(2\sqrt{3} + 5)}{7}$$

b)



$F = \mu mg = -m\frac{dv}{dt}$ $\qquad J\frac{d\omega}{dt} = \mu mgr$

$\Rightarrow \frac{dv}{dt} = -\mu g$ $\qquad \frac{d(r\omega)}{dt} = \frac{5}{2}\mu g$

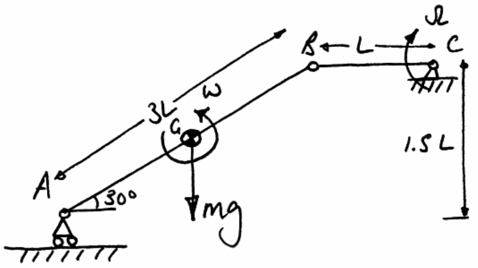
sliding velocity, $v_s = v - r\omega \Rightarrow \frac{dv_s}{dt} = -\mu g - \frac{5}{2}\mu g = -\frac{7}{2}\mu g$

since rate of change of v_s is constant use "$v^2 = u^2 + 2as$"

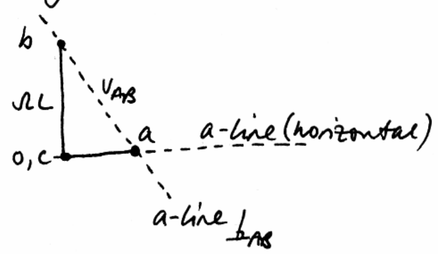
$$\Rightarrow 0 = (v_0 - r\omega_0)^2 - 7\mu g s$$

$$s = \frac{(v_0 - r\omega_0)^2}{7\mu g} \qquad \mu = \frac{2}{7} \Rightarrow s = \frac{(v_0 - r\omega_0)^2}{2g}$$

5a) Space:

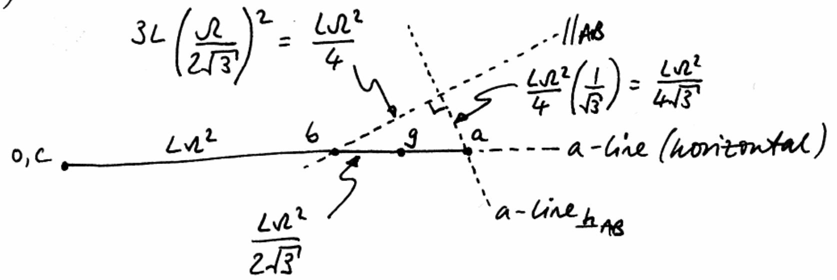


velocity:



$$V_{AB} = r_{AB}\omega_{AB} \Rightarrow \omega_{AB} = \frac{\sqrt{3}}{2}\left(\frac{L\Omega}{3L}\right) = \frac{\Omega}{2\sqrt{3}}$$

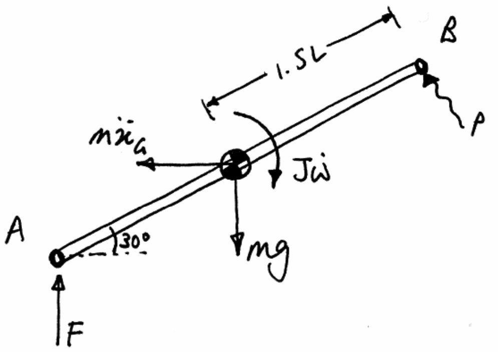
b) Acceleration:



$$\dot{\omega} = \frac{a_{a\perp AB}}{|AB|}$$

$$= \left(\frac{L\Omega^2}{4\sqrt{3}}\right)\Big/3L = \frac{\Omega^2}{12\sqrt{3}}$$

c) Unknown force P through pin @ B ∴ moments of d'Alembert forces: B+)



$$\frac{3\sqrt{3}}{2}FL + J\dot{\omega} - \frac{3\sqrt{3}}{4}mgL + \frac{3}{4}m\ddot{x}_G L = 0$$

$$J = \frac{m(3L)^2}{12} = \frac{3mL^2}{4}$$

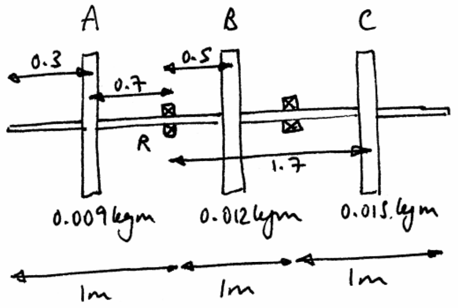
$$\ddot{x}_G = L\Omega^2 + \frac{1}{2}\left(\frac{L\Omega^2}{2\sqrt{3}}\right) = \left(1 + \frac{1}{4\sqrt{3}}\right)L\Omega^2$$

∴ for F=0 : $$\frac{3mL^2}{4}\left(\frac{\Omega^2}{12\sqrt{3}}\right) - \frac{3\sqrt{3}}{4}mgL + \frac{3}{4}\left(1+\frac{1}{4\sqrt{3}}\right)mL^2\Omega^2 = 0$$

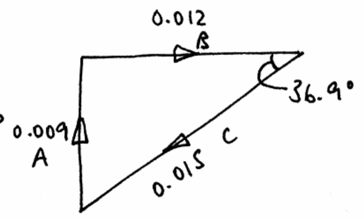
$$\Rightarrow \Omega = \sqrt{\frac{\left(\frac{3\sqrt{3}}{4}\right)\frac{g}{L}}{\left(\frac{\sqrt{3}}{48} + \frac{3}{4}\left(1+\frac{1}{4\sqrt{3}}\right)\right)}} = 1.21\sqrt{g/L}$$

6a) Static balance: centre of mass of system lies on axis of rotation: shaft will sit at equilibrium at any angular position.

Dynamic balance: bearing forces for steady rotation are all zero.

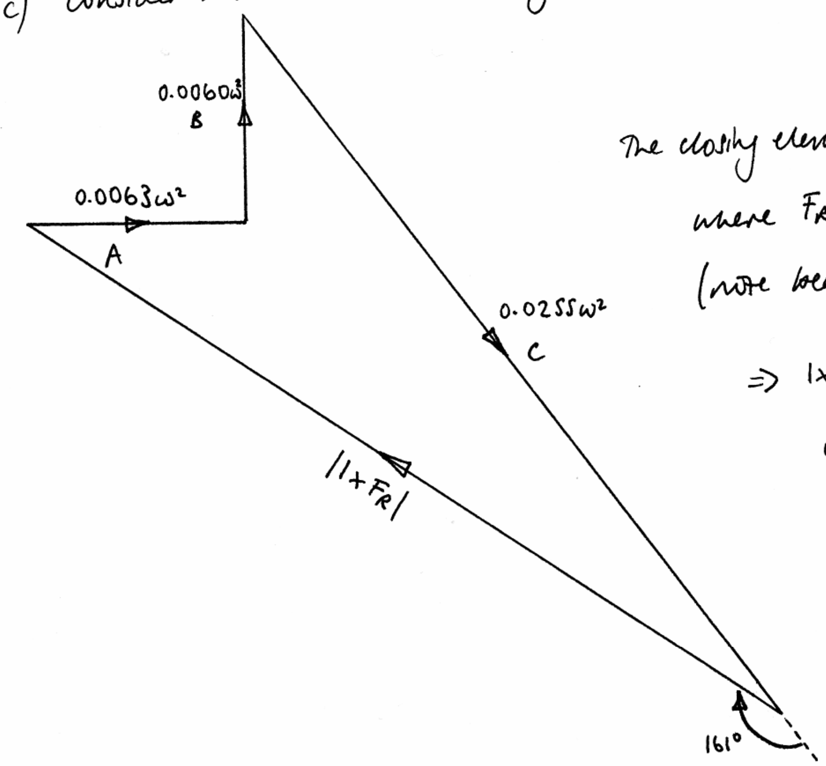


b) Static balance: by inspection 3:4:5 Δ ⇒



⇒ A & B 90° out of balance,
C at $(270 - 36.9) = 233°$

c) Consider moments about bearing "R". Contributions: (−) $0.009 \times 0.7\omega^2 = 0.0063\omega^2$
$0.012 \times 0.5\omega^2 = 0.0060\omega^2$
$0.015 \times 1.7\omega^2 = 0.0255\omega^2$



The closing element of the polygon $= |1 \times F_R|$
where F_R is the bearing load
(note bearings are 1m apart)

$\Rightarrow 1 \times F_R = 0.026\omega^2$ Nm

$\omega = 8000 \times \frac{2\pi}{60} = 838$ rad/s

$\Rightarrow F_R = 18.2$ kN

d) Close polygon by adding balance masses to rims of A & C (i.e. @ r = 0.5m)

$|1 \times F_R| = (m \times 0.5 \times 0.7)\omega^2 + (m \times 0.5 \times 1.7)\omega^2 \Rightarrow m = 0.012$ kg

Balance weights must have same mass and be 180° out of phase to preserve static balance. Diag indicates that mass at C should be added at $161° + 233° = 34°$ to that at A ⇒ mass at A = 214°.