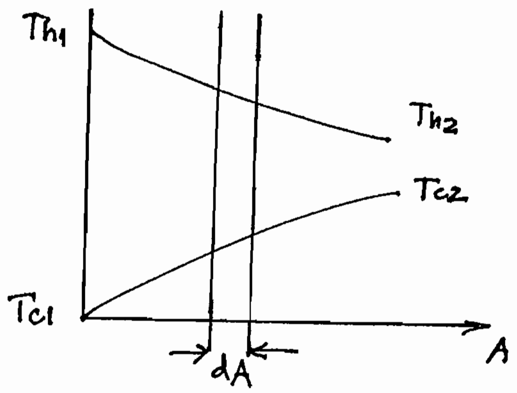
ENGINEERING TRIPOS PART IB, 2009

section A

QUESTION 1



WE CONSIDER THE HEAT TRANSFER BETWEEN HOT AND COLD FLOWS FOR AN ELEMENT dA ALONG THE HEAT EXCHANGER. THE ENERGY BALANCE FOR THE HOT AND COLD STREAMS READS:

$$dQ = \dot{m}_c c_c \, dT_c = -\dot{m}_h c_h \, dT_h \qquad (1)$$

FROM WHICH WE CAN WRITE:

$$dT_c = \frac{dQ}{\dot{m}_c c_c} \qquad dT_h = -\frac{dQ}{\dot{m}_h c_h} \qquad (2)$$

NOW WE NEED TO CONNECT THE ENERGY BALANCE TO HEAT TRANSFER. (THIS IS OFTEN THE BIT THAT IS NEGLECTED IN THE EXAM):

$$dQ = U\left(T_h - T_c\right) dA \qquad (3)$$

BUT FROM (2), WE HAVE:

$$dQ\left[\frac{1}{\dot{m}_c c_c} + \frac{1}{\dot{m}_h c_h}\right] = -d\left(T_h - T_c\right) \qquad (4)$$

COMBINING (3) AND (4) WE HAVE:

$$dQ = U(T_h - T_c)\,dA = -\underbrace{\left[\frac{1}{\dot m_c c_c} + \frac{1}{\dot m_h c_h}\right]}_{1/C}{}^{-1} d(T_h - T_c) \qquad (5)$$

INTEGRATING:

$$-\frac{UA}{C} = \ln(T_h - T_c)\Big|_1^2 = \left[\ln(T_{h2} - T_{c2}) - \ln(T_{h1} - T_{c1})\right] \qquad (6)$$

NOW WE NEED TO ELIMINATE $\underline{\underline{C}}$, AND CONNECT TO THE OVERALL HEAT TRANSFER Q.

FROM THE ENERGY BALANCE (1), WE HAVE, BY INTEGRATION:

$$Q = \dot m_c c_c (T_{c2} - T_{c1}) = \dot m_h c_h (T_{h1} - T_{h2}) \qquad (7)$$

$$\underbrace{\left(\frac{1}{\dot m_c c_c} + \frac{1}{\dot m_h c_h}\right)}_{1/C} Q = T_{c2} - T_{c1} + T_{h1} - T_{h2}$$

$$Q = C\left[(T_{h1} - T_{c1}) - (T_{h2} - T_{c2})\right] \qquad (8)$$

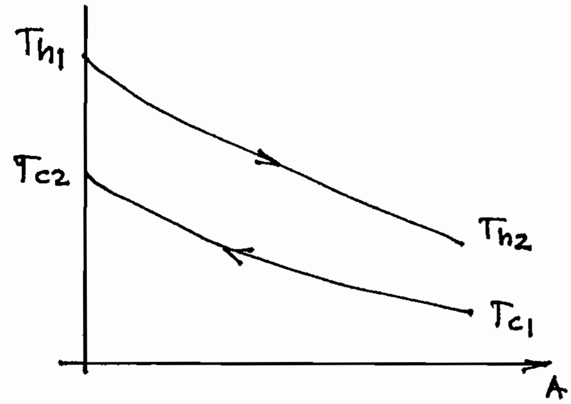
FROM (6) : $C = -UA\left[\ln(T_{h2} - T_{c2}) - \ln(T_{h1} - T_{c1})\right]^{-1}$

SO THAT, FROM (8)

$$Q = UA\,\frac{\left[(T_{h2} - T_{c2}) - (T_{h1} - T_{c1})\right]}{\ln\left(\frac{T_{h2} - T_{c2}}{T_{h1} - T_{c1}}\right)}$$

QED.

(b)



IN THE CASE OF COUNTERFLOW HEAT EXCHANGERS, THE SAME ANALYSIS AS IN (a) APPLIES, AS NO ASSUMPTION ABOUT THE DIRECTION OF THE FLOW WAS MADE.

(c) THE COUNTERFLOW HEAT EXCHANGER OFFERS A SMALLER MEAN LOG TEMPERATURE DIFFERENCE, AND THUS LOWER IRREVERSIBILITY.

S. HOCHGREB, 18/06/09

QUESTION 2

(a) THE RATIONAL EFFICIENCY IS DEFINED AS:

$$\eta_r = \frac{\text{ACTUAL WORK}}{\text{MAXIMUM POSSIBLE WORK}} \qquad (1)$$

THE CYCLE EFFICIENCY IS DEFINED AS:

$$\eta_{cy} = \frac{\text{ACTUAL WORK}}{\text{HEAT INPUT}} \qquad (2)$$

THE CARNOT EFFICIENCY IS DEFINED AS:

$$\eta_c = \frac{\text{MAXIMUM POSSIBLE WORK}}{\text{HEAT INPUT}} \qquad (3)$$

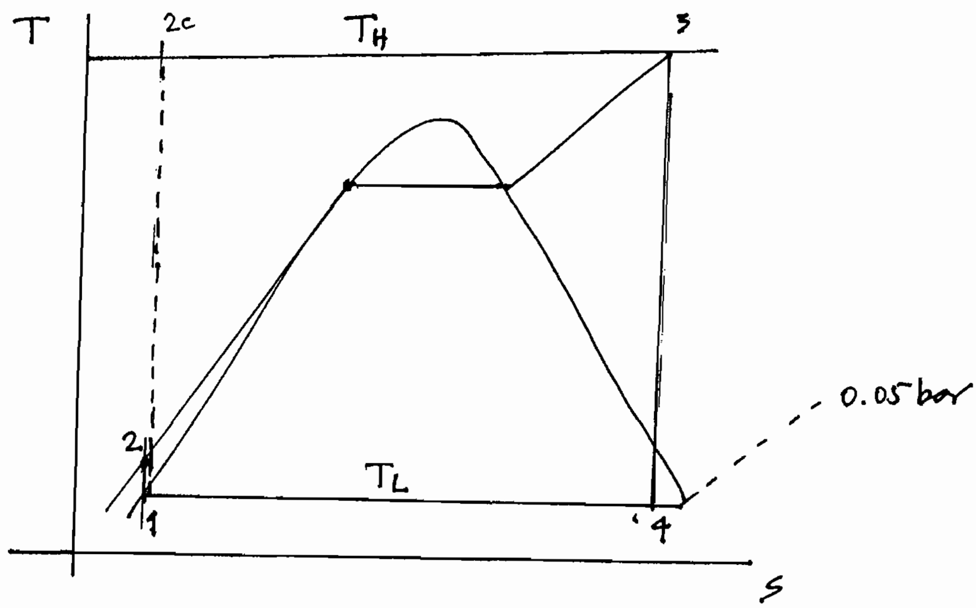
CLEARLY,

$$\eta_{cy} = \frac{\text{ACTUAL WORK}}{\text{HEAT INPUT}} = \underbrace{\frac{\text{ACTUAL WORK}}{\text{MAX. POSS. WORK}}}_{\eta_r} \cdot \underbrace{\frac{\text{MAX. POSS. WORK}}{\text{HEAT INPUT}}}_{\eta_c}$$

QED.

QUESTION 2 / (b)

(b)



RANKINE : 1-2-3-4

CARNOT : 1-2c-3-4

IN ORDER TO CALCULATE RANKINE CYCLE EFFICIENCIES, WE NEED TO WORK OUT THE WORK OUTPUT AND HEAT INPUT. WE ASSUME $W_{12} \approx 0$.

PROPERTIES (FROM STEAM CHARTS IN DATABOOK):

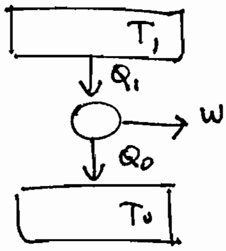
$h_3 = 3560$ kJ/kg
$s_3 = 7.6$ kJ/kg/K $= s_4$
$h_4 \simeq 2320$ kJ/kg K $\longrightarrow T \simeq 33°C$
$h_1 \simeq 138$ kJ/kg K (FROM TABLES)

$w = h_3 - h_4 = (3560 - 2320)$ kJ/kg $= 1240$ kJ/kg

$q = h_3 - h_1 = (3560 - 138)$ kJ/kg $\simeq 3422$ kJ/kg

(a) (CONT).

ALTERNATIVELY, ONE CAN USE EXERGY FLOWS :



$$\eta_{cy} = \frac{W}{Q_1} \qquad \eta_c = 1 - \frac{T_0}{T_1} \qquad \eta_r = \frac{W}{Q_1\left(1 - \frac{T_0}{T_1}\right)}$$

$$\therefore \quad \eta_{cy} = \frac{W}{Q_1} = \underbrace{\frac{W}{Q_1\left(1 - T_0/T_1\right)}}_{\eta_r} \cdot \underbrace{\left(1 - \frac{T_0}{T_1}\right)}_{\eta_c}$$

<u>NOTE</u> : STARTING FROM AN EXPRESSION FOR EXERGY SUCH AS $\left\{ \begin{array}{l} H - T_0 S \\ E - T_0 S \end{array} \right.$ DOES NOT HELP IN THIS CONTEXT, AS THE SYSTEM IS GENERIC, SO THAT THE EXERGY <u>FLOWS</u> RATHER THAT EXERGY STATE SHOULD BE CONSIDERED.

WE CAN OBTAIN THE CYCLE EFFICIENCY AS:

$$\eta_{cy} = \frac{w}{q} = \frac{1240}{3422} = \underline{\underline{0.36}}$$

THE <u>CARNOT EFFICIENCY</u> REQUIRES THE KNOWLEDGE OF THE MAXIMUM $(T_H = 540°C = 813K)$ AND MINIMUM TEMPERATURES $(T_L = 33°C = 306K)$:

$$\eta_c = \frac{w_{MAX}}{q} = 1 - T_L/T_H = 1 - 306/813 = 0.62$$

THE <u>RATIONAL EFFICIENCY</u> CAN THEREFORE BE OBTAINED AS:

$$\eta_r = \frac{w}{w_{MAX}} = \frac{\eta_{cy}}{\eta_c} = \frac{0.36}{0.62} = 0.58$$

<u>NOTE</u>: IN THIS QUESTION, ONE CAN SOLVE FOR THE RANKINE CYCLE EASILY. THE ISSUE IS RECOGNIZING THE ASSOCIATED IDEALIZED CARNOT CYCLE, DEFINED BY THE SAME HEAT INPUT AT THE PEAK TEMPERATURE, AND DISCHARGE AT THE LOWEST TEMPERATURE.

(c) $\Delta E_D = T_0 S_{GEN} = W_{IDEAL} - W_{ACTUAL} = W_{LOST}$

THE EXERGY OR IRREVERSIBILITY LOSS IS CALCULATED BY COMPARING THE <u>ACTUAL</u> WORK TO WHAT AN IDEAL CARNOT CYCLE COULD HAVE PRODUCED WITH THE SAME INPUT HEAT:

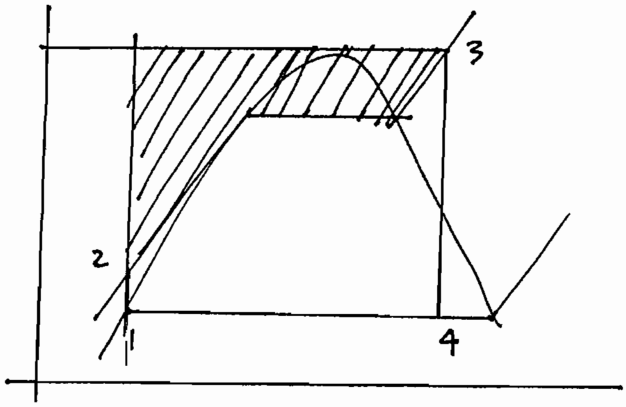
$$w_{MAX} = \eta_c q = (3422\ kJ/kg)(0.62) =$$

THUS, THE LOST WORK OR IRREVERSIBILITY IS:

$$T_0 S_{GEN} =$$

(c) (CONT.)

GRAPHICALLY, ONE CAN REPRESENT THE LOSS EXERGY AS THE AREA BETWEEN THE CARNOT CYCLE AND THE RANKINE CYCLE.



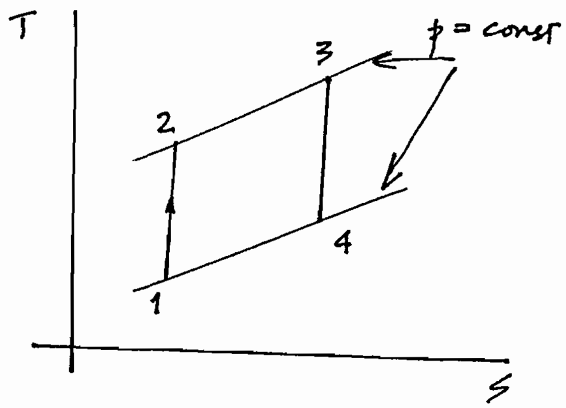
NOTICE THAT EVEN THOUGH THIS IS AN <u>IDEAL</u> RANKINE CYCLE, IRREVERSIBILITIES ARISE IF THE HOT SOURCE IS AT T_3 AND THE COLD SINK IS AT T_4, DUE TO EXERGY LOSSES DURING HEAT TRANSFER.

<u>NOTE</u> : IT IS <u>NOT</u> USEFUL TO TACKLE THIS QUESTION USING $\Delta E = T_0 \Delta S$, USING $\Delta s = s_3 - s_1$, AS THIS SIMPLY REPRESENTS THE EXERGY FLUX LEAVING THE SYSTEM, NOT THE <u>LOST</u> EXERGY.

S. HOCHGREB 18/06/09

QUESTION 3

(a)



$p_1 = p_4$
$p_2 = p_3$

$r_p = \frac{p_2}{p_1} = \frac{p_3}{p_4}$

THE EFFICIENCY OF THE CYCLE IS GIVEN BY:

$$\eta = \frac{w}{q} = \frac{q_{23} - q_{14}}{q_{23}} = 1 - \frac{q_{14}}{q_{23}} = 1 - \frac{c_p (T_4 - T_1)}{c_p (T_3 - T_2)}$$

$$= 1 - \frac{T_1 \; (T_4/T_1 - 1)}{T_2 \, (T_3/T_2 - 1)} \qquad (1)$$

1-2 and 3-4 ARE ISENTROPIC PROCESSES, AND 2-3 AND 1-4 ARE p = CST:

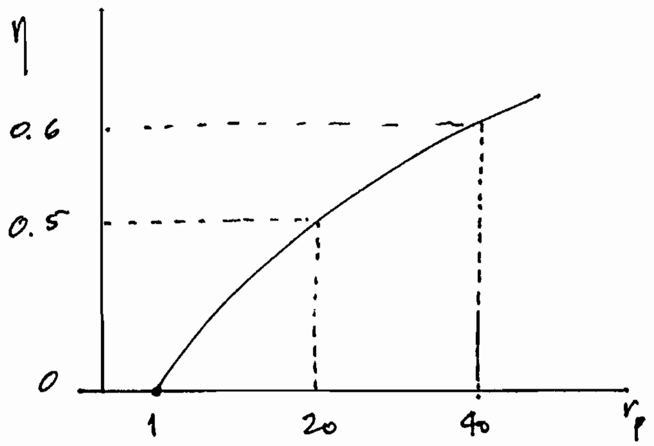
$$\frac{T_4}{T_1} = \frac{T_4}{T_3} \cdot \frac{T_3}{T_2} \cdot \frac{T_2}{T_1} = r_p^{-\frac{(\gamma-1)}{\gamma}} \frac{T_3}{T_2} \; r_p^{\frac{(\gamma-1)}{\gamma}} = \frac{T_3}{T_2}$$

FROM (1): $\eta = 1 - r_p^{-\frac{(\gamma-1)}{\gamma}} \frac{(T_3/T_2 - 1)}{(T_3/T_2 - 1)}$

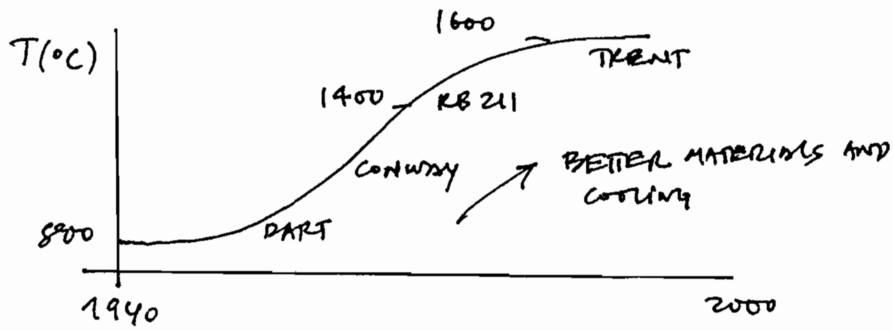
$$\underline{\eta = 1 - r_p^{-\frac{(\gamma-1)}{\gamma}}}$$

QED.

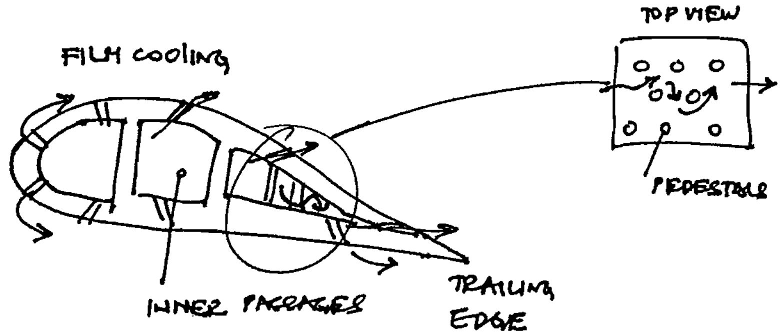
(b) JOULE CYCLE EFFICIENCY



THE PRACTICAL EFFICIENCY LIMIT IS SET BY THE ABILITY TO KEEP MATERIALS WORKING AT THE HIGH TEMPERATURES CREATED BY HIGH PRESSURE RATIOS. MATERIAL PROPERTIES (ESPECIALLY CREEP) REQUIRE THAT TURBINE BLADES MUST BE COOLED IF THE TURBINE INLET TEMPERATURES ARE HIGHER THAN 1200°C. THIS CAN USE A LARGE (10 - 15%) FRACTION OF THE COMPRESSOR AIR, WITH A CORRESPONDING DECREASE IN POWER AND EFFICIENCY.



(c) TURBINE BLADES ARE COOLED BY AIR INJECTION THROUGH INNER PASSAGES. THE AIR CIRCULATES THROUGH THE BLADE AND EXITS THROUGH HOLES LEAVING A COOLING FILM THAT PROTECTS THE BLADE.



(d) <u>REAL CYCLES</u>:

- FUEL MASS FLOW ADDED IN COMBUSTOR (~4%)
- PRESSURE DROP ACROSS COMBUSTOR (2-4%)
- TEMPERATURE DEPENDENCE OF GAS PROPERTIES (γ, c_p)
- CHANGE IN GAS COMPOSITION IN COMBUSTOR
- VISCOUS EFFECTS (BOUNDARY LAYERS, SHOCK DISSIPATION) IN BOTH TURBINE AND COMPRESSOR
- MIXING OF COOLING AIR AND HOT AIR (see (c))

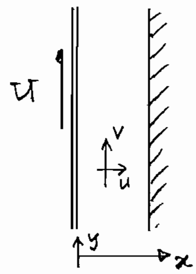
<u>NOTES</u>:

(1) "IRREVERSIBILITIES" IS A GENERAL TERM, AND THE ANSWER SHOULD BE MORE SPECIFIC THAN THAT ("VISCOUS", OR "MIXING")

(2) ADDITION OF HEAT EXCHANGERS OR BOTTOMING RANKINE CYCLE ARE ALSO NOT GOOD ANSWERS, AS THEY CHANGE THE QUESTION ABOUT THE "PURE" JOULE CYCLE.

S. HOCHGREB, 10/06/09

2009, 1B Paper 4, Q4.



(a) i) For incompressible flow, the continuity equation $\nabla \cdot \underset{\sim}{u} = 0$ must be satisfied :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 .$$

The streamlines are straight so $u = 0$ everywhere. Therefore $\partial u/\partial x = 0$ everywhere and, by continuity, $\partial v/\partial y = 0$. Therefore the vertical velocity, v, is a function only of x.

An equivalent way to say this is that if the streamlines are straight then conservation of mass requires that v does not vary in y.

ii) There are two ways to justify that the pressure is uniform in the film:

<u>method 1</u>:

The streamlines are straight, so there can be no pressure gradient across them (otherwise they would be curved), so $\partial p/\partial x = 0$.

The streamlines are straight and there is no acceleration in the vertical direction so the imposed shear must balance with gravity and, if there is any, the pressure gradient in the y-direction: $\partial p/\partial y$. The imposed shear and the gravity do not vary in the y-direction so the pressure gradient, $\partial p/\partial y$, cannot vary in the y-direction. The pressure at the bottom of the film is the same as the pressure at the top of the film (both are exposed to atmospheric pressure) so $\Delta P/\Delta y = 0$ across the whole film and therefore, because $\partial p/\partial y$ cannot vary in the y-direction, $\partial p/\partial y$ must be zero throughout the film.

In summary, $\partial p/\partial x = 0$ and $\partial p/\partial y = 0$ so the pressure, p, is uniform everywhere in the film.

method 2:

Start from the Navier-Stokes equation (the momentum equation):

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = -\frac{\nabla p}{\rho} + \frac{\mu}{\rho} \nabla^2 \underset{\sim}{u} - g \underset{\sim}{e_y} \qquad \text{where } \underset{\sim}{u} = u \underset{\sim}{e_x} + v \underset{\sim}{e_y}.$$

The flow is steady so $\partial \underset{\sim}{u} / \partial t = 0$.

Let us split this into the component in the x-direction and the component in the y-direction:

x-direction:
$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

The streamlines are parallel so $u = 0$ and, by inspection, $\partial p / \partial x = 0$.

y-direction:
$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - g.$$

We know already that $u = 0$ and $\partial v / \partial y = 0$ and therefore that $\partial^2 v / \partial y^2 = 0$

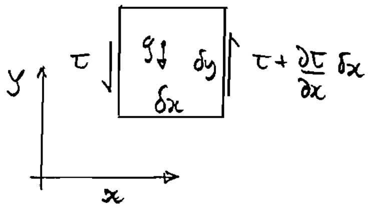
$$\Rightarrow \quad 0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \frac{\partial^2 v}{\partial x^2} - g \qquad (1)$$

We know that g and v (and therefore $\partial^2 v / \partial x^2$) do not vary in the y-direction. Therefore $\partial p / \partial y$ cannot vary in the y-direction. The top and bottom of the film are both exposed to atmospheric pressure so therefore $\partial p / \partial y = 0$ throughout the film, by the same argument as in method 1.

(b) If method 2 has been used for part (a) then the governing equation for the vertical velocity can be written down from equation (1) above:

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \frac{\partial^2 v}{\partial x^2} - g \quad \text{with } \frac{\partial p}{\partial y} = 0 \quad \Rightarrow \quad \mu \frac{\partial^2 v}{\partial x^2} = \rho g.$$

If method 1 has been used for part (a) then the governing equation can either be derived from the Navier-Stokes equation as in method 2 of part (a) or can be derived from a force balance around a small control volume:



Sum of shear forces acting in the vertical direction on the fluid = $\frac{\partial \tau}{\partial x}\delta x \delta y$

gravitational force = $-\rho g \, \delta x \, \delta y$

There is no acceleration so the forces must sum to zero:

$$\frac{\partial \tau}{\partial x}\delta x \delta y - \rho g \, \delta x \, \delta y = 0$$

$$\Rightarrow \quad \frac{\partial \tau}{\partial x} = \rho g$$

Now, $\tau = \mu \frac{\partial v}{\partial x}$ (this is the definition of viscosity)

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) = \rho g$$

And $\underline{\underline{\text{if}}}$ μ does not vary in x:

$$\Rightarrow \quad \mu \frac{\partial^2 v}{\partial x^2} = \rho g$$

And we know already that v is a function only of x so the partial derivative $\partial^2 v / \partial x^2$ collapses to an ordinary derivative $d^2 v / dx^2$.

(c) It is easiest to use indefinite integrals to solve this problem:

$$\frac{d^2 v}{dx^2} = \frac{\rho g}{\mu}$$

$$\Rightarrow \int d\left(\frac{dv}{dx}\right) = \int \frac{\rho g}{\mu} dx$$

$$\Rightarrow \frac{dv}{dx} = \frac{\rho g}{\mu} x + C_1$$

$$\Rightarrow \int dv = \int \left(\frac{\rho g}{\mu} x + C_1\right) dx$$

$$\Rightarrow \quad v = \frac{\rho g}{\mu}\frac{x^2}{2} + C_1 x + C_2 .$$

(It is perfectly acceptable to write this down without working)

At $x = 0$, $v = U$ so $\quad C_2 = U$.

At $x = d$, $v = 0$ so $\quad 0 = \frac{\rho g}{\mu}\frac{d^2}{2} + C_1 d + U$

$$\Rightarrow C_1 = -\frac{\rho g}{\mu}\frac{d}{2} - \frac{U}{d}$$

$$\Rightarrow \quad v = \frac{\rho g}{2\mu}\left(x^2 - xd\right) + U\left(1 - \frac{x}{d}\right)$$

$$\Rightarrow \quad v = \left(\frac{U}{d} - \frac{\rho g x}{2\mu}\right)(d - x) \qquad (2)$$

checks: when $x = d$, $v = 0$ ✓

when $x = 0$, $v = U$ ✓

when $g = 0$, $v = \frac{U}{d}(d - x)$ $\Rightarrow$ v is a linear function of x, which is Couette flow. This must be the limit when $g \to 0$ ✓

(d) $$Q = \int_0^d v(x)\, dx$$, where $v(x)$ is given in equation (2) above. and Q is defined as positive upwards (as is v)

$$\Rightarrow \quad Q = \int_0^d \left(\frac{\rho g}{2\mu} x^2 - \frac{\rho g d x}{2\mu} - \frac{U}{d} x + U\right) dx$$

$$= \left[\frac{\rho g}{2\mu}\frac{x^3}{3} - \frac{\rho g d}{2\mu}\frac{x^2}{2} - \frac{U}{d}\frac{x^2}{2} + Ux\right]_0^d$$

$$= \frac{\rho g}{6\mu} d^3 - \frac{\rho g}{4\mu} d^3 - \frac{Ud}{2} + Ud$$

$$= \frac{Ud}{2} - \frac{\rho g d^3}{12\mu}$$

checks:

The first term arises from the motion induced by the moving plate. As U increases, it increases and causes an increase in Q ✓

The second term arises from the motion induced by gravity. As g, ρ or d increase, this term increases in magnitude, as expected

$Q = 0$ when $$\frac{Ud}{2} = \frac{\rho g d^3}{12\mu}$$

$$\Rightarrow \quad U = \frac{\rho g d^2}{6\mu}$$

Matthew Juniper

5th June 2009

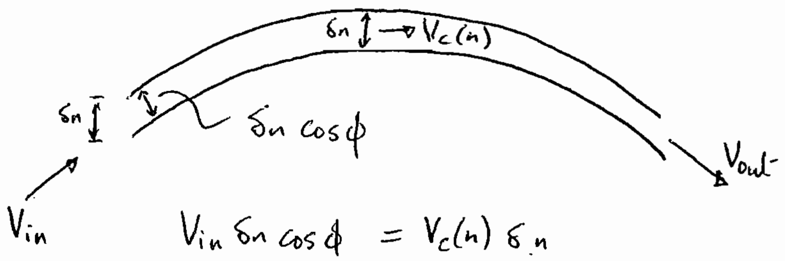
(This level of explanation would not be expected in an exam script but is included here to help students understand the question)

5) a) (i) $\partial p/\partial n$ is the force (per unit volume) perpendicular to streamline. V^2/R is the centripetal acceleration of the fluid; hence $\rho V^2/R$ is the rate of change of momentum (per unit vol.) $\perp$ s/line.

(ii) Stagnation pressure is constant throughout flow.

b) (i) Applying continuity between a plane far upstream and a plane going through the vanes, vol. flow rate $= V_\infty h$.

(ii) Continuity for a streamtube, edges separated by δn:



$$V_{in}\,\delta n \cos\phi = V_c(n)\,\delta n$$

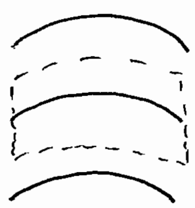
$$V_{in} = \frac{V_c(n)}{\cos\phi}, \quad \underline{\underline{V_{out} = V_{in}}}$$

(iii) Normal eqn: $\dfrac{\partial p}{\partial n} = \dfrac{\rho V_c^2}{R}$; const. stag'n pressure: $p + \frac{1}{2}\rho V_c^2 = \text{const.}$

$$\therefore \; -\rho V_c \frac{\partial V_c}{\partial n} = \frac{\rho V_c^2}{R} \quad \text{with sol'n} \quad \underline{\underline{V_c(n) = V_c(0)\,e^{-n/R}}}$$

(iv) Small h/R: $V_c(n) = V_c(0)\left[1 - \frac{n}{R} + O\left(\frac{n}{R}\right)^2\right]$, so total vol'c flow rate is approx. $V_c(0)h$; hence (cf (i)) $\underline{\underline{V_c(0) = V_\infty}}\left[1 + O(h/R)^2\right]$

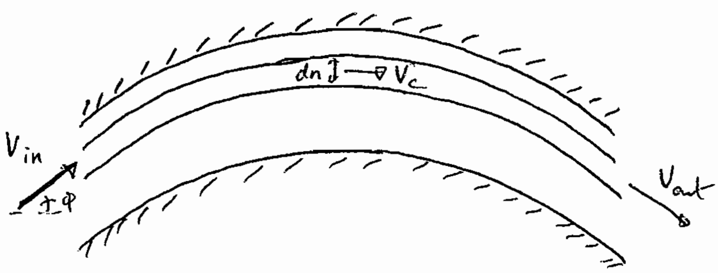
(v) Consider one of two possible control volumes:



- Pressure forces on sides cancel
- Vane force acts on fluid inside

- No force acting on fluid inside
- Vane force from sum of pressure integrals on curved surfaces
- Slightly easier to formulate integral.

5)b)(v) cont'd



- $\rightarrow$ force component is zero by symmetry
- $\updownarrow$ force component is $\displaystyle\int_{n=-h/2}^{n=h/2} (V_{out}\sin\phi + V_{in}\sin\phi)\, d\dot{m}$

with $d\dot{m} = \rho V_c(n)\,dn$ and $V_{out} = V_{in} = \dfrac{V_c(n)}{\cos\phi}$

$$\text{Force} = 2\rho\tan\phi\, V_c^2(0)\int_{-h/2}^{h/2}\left[1 - \frac{2n}{R} + O\left(\frac{n}{R}\right)^2 \cancel{+O\left(\frac{n}{R}\right)}\right] dn$$

$$= 2\rho h\tan\phi\, V_c^2(0)\left[1 + O\left(h/R\right)^2\right]$$

$$= 2\rho h\tan\phi\, V_\infty^2\left[1 + O\left(h/R\right)^2\right]$$

N.B. Acceptable without $O(\;)$ terms, but beware spurious term in $(h/R)^2$ if $(1-n/R)^2$ is integrated exactly.

c) • Velocity decreases from inlet to ℄, then increases again, so wall b-ls subject to adverse pressure gradient in first region; may cause separation upstream, not downstream, of ℄.

~~• Skin friction will make $\rightarrow$ force component non-zero.~~

~~• (Subtle!) B-L thickening / separation will reduce angle of incoming flow, thereby lowering $\updownarrow$ force component.~~
~~N.B. Due to asymmetry in real inlet flow; as this is outside scope of question, this point is not required.~~

2009 Thermofluids Paper 4 Q6

(a) Neglect viscous effects in the nozzle. Exit losses will be negligible because the water exits into air, rather than more water. Therefore the mechanical energy per unit volume at the nozzle exit equals that at the top of the fountain. In other words, all the K.E. of the fluid at the nozzle is converted to G.P.E. by the time it reaches the top of the fountain.

Bernoulli can be used because mechanical energy is conserved:

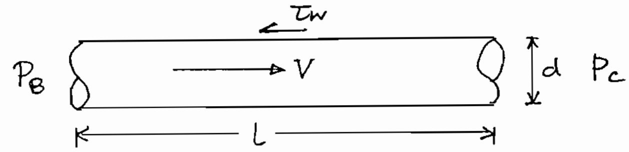
$$P_0 = P_{atm} + \frac{1}{2}\rho V_{exit}^2 = P_{atm} + \rho g \Delta h$$

$\Delta h = 10\text{m}$

But the gauge stagnation pressure is $P_0 - P_{atm} = \rho g \Delta h$

$$\rho g \Delta h = 1000 \times 9.81 \times 10 = 9.81 \times 10^4 \text{ Pa}$$

(b)



Consider the forces on all the fluid within the pipe.

- due to pressure: $(p_B - p_C)\frac{\pi d^2}{4}$ to the right
- due to wall friction: $\tau_w \pi d L$ to the left

By definition of c_f, $\tau_w = \frac{1}{2}\rho V^2 c_f$

And $Q = AV = \frac{\pi d^2}{4} V \Rightarrow V = \frac{4Q}{\pi d^2}$

There is no acceleration so the forces must sum to zero. By substitution:

$$(p_B - p_C)\frac{\pi d^2}{4} = \tau_w \pi d L = \frac{1}{2}\rho V^2 c_f \pi d L = \frac{1}{2}\rho\left(\frac{4Q}{\pi d^2}\right)^2 c_f \pi d L$$

$$\Rightarrow (p_B - p_C) = \frac{32}{\pi^2}\rho Q^2 c_f \frac{1}{d^5}$$

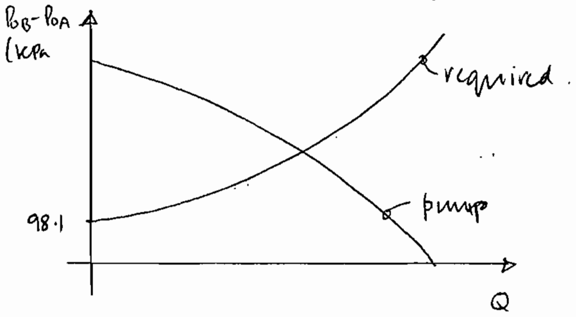
Because $V_B = V_C$, $p_{0B} - p_{0C} = p_B - p_C = \frac{32}{\pi^2}\rho Q^2 c_f \frac{1}{d^5}$

(c) $P_{0B} - P_{0C} = \frac{32}{\pi^2} \times 1000 \times 0.005 \times \frac{5}{(0.1)^5} Q^2 = 8.105694 \times 10^6 Q^2$

$P_{0C} = \rho g h = 98100$ Pascals

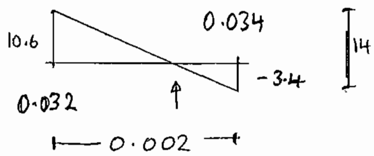
$\Rightarrow P_{0B} = 8.105694 \times 10^6 Q^2 + 98100$ Pascals

$P_{0A} = 0$ (atmospheric pressure, in gauge pressure)



Q $m^3 s^{-1}$	(kPa) $P_{0B} - P_{0A}$ required	(kPa) $P_{0B} - P_{0A}$ pump	difference
0.032	106.4	117	10.6
0.034	107.4	104	-3.4

linear interpolation:



$$Q = \frac{10.6}{14} \times 0.034 + \frac{3.4}{14} \times 0.032 = 0.0335 \ m^3 s^{-1}$$

(d) Δp_0 has units $N m^{-2} = kg\, m^{-1} s^{-2}$
ρ " " $kg\, m^{-3}$
N " " s^{-1}
D " " m
Q " " $m^3 s^{-1}$

5 quantities, 3 dimensions
$\Rightarrow$ 2 dimensionless parameters.
(we want one to describe Δp_0 and another to describe Q)

By inspection, $\frac{\Delta p_0}{\rho D^2 N^2}$ and $\frac{Q}{N D^3}$ are dimensionless

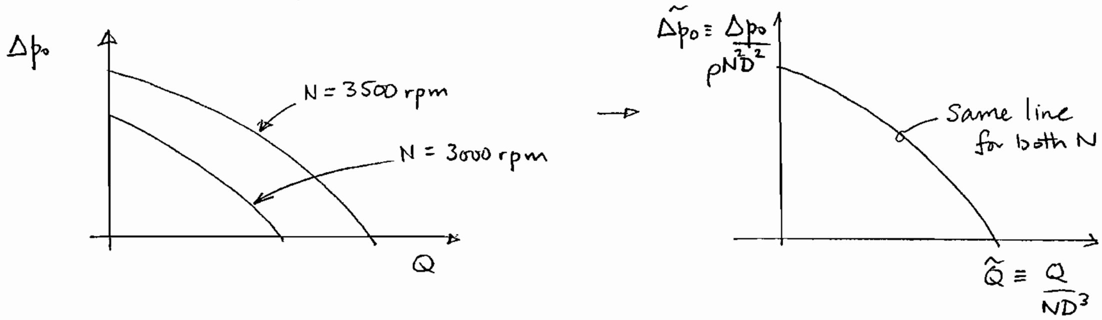
Therefore, for pumps that are geometrically similar to this one, we expect to find:

$$\frac{\Delta p_0}{\rho D^2 N^2} = f\left(\frac{Q}{N D^3}\right)$$

(e) The pump speed, N, will be reduced. The size of the pump, D, will remain the same. The fountain height will remain the same (note that this will require a new nozzle, but this does not matter for this question) so the relation for $p_{oB} - p_{oA}$ remains:

$$\Delta p_o = p_{oB} - p_{oA} = 8.105694 \times 10^6 Q^2 + 98100 \text{ Pascals}.$$

We are given the pump characteristics at $N = 3500$ rpm. The pump characteristics at $N = 3000$ rpm will be different, but both would collapse to the same line when plotted as $\tilde{\Delta p_o} = f(\tilde{Q})$ where $\tilde{\Delta p_o} = \dfrac{\Delta p_o}{\rho N^2 D^2}$ and $\tilde{Q} = \dfrac{Q}{ND^3}$:



We need to find a way to simulate the graph on the right from the data at $N = 3500$ rpm on the graph on the left; or, equivalently, we need to find the functional form of the $N = 3000$ rpm line from that of the $N = 3500$ rpm line on the graph on the left.

We do this by converting the relationship between Q and Δp_o at 3000 rpm to an equivalent relationship at 3500 rpm and then using the graph as before:

$$Q_{3000} = \tilde{Q} N_{3000} D^3 = Q_{3500} \frac{N_{3000}}{N_{3500}} \frac{D^3}{D^3} = \frac{6}{7} Q_{3500}$$

(n.b. this is <u>not</u> the new flowrate; it is simply the conversion from the $N = 3000$ line to the $N = 3500$ line on the graph on the left)

similarly,

$$\Delta p_{o\,3000} = \tilde{\Delta p_o} \rho N_{3000}^2 D^2 = \Delta p_{o\,3500} \frac{N_{3000}^2}{N_{3500}^2} = \left(\frac{6}{7}\right)^2 \Delta p_{o\,3500}$$

The fountain remains at the same height so the relationship between stagnation pressure and flowrate remains:

$$\Delta p_{o_{3000}} = 8.105694 \times 10^6 \, Q_{3000}^2 + 98100 \text{ Pascals}$$

Substitute in our new expressions for $\Delta p_{o_{3000}}$ and Q_{3000}:

$$\Rightarrow \left(\frac{6}{7}\right)^2 \Delta p_{3500} = 8.105694 \times 10^6 \left(\frac{6}{7}\right)^2 Q_{3500}^2 + 98100 \text{ Pascals}$$

$$\Rightarrow \quad \Delta p_{3500} = 8.105694 \times 10^6 \, Q_{3500}^2 + 133525 \text{ Pascals}$$

Now we can find the point of intersection of this line with the pump characteristic data that we are given in the question, following the same procedure as before.

Q_{3500} ($m^3 s^{-1}$)	Δp_{3500} required (kPa)	Δp_{3500} of pump (kPa)	Difference (kPa)
0.02	136.7	166	
0.025	138.6	149	
0.03	140.8	127	
0.0275	139.7	139	← close enough not to need linear interpolation

The new flowrate is $Q_{3000} = \frac{6}{7} Q_{3500} = \frac{6}{7} \times 0.0275$

$$= 0.0236 \ m^3 s^{-1}$$

(This level of explanation would not be expected in an exam script but is included here to help students to understand the question)

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June 2009