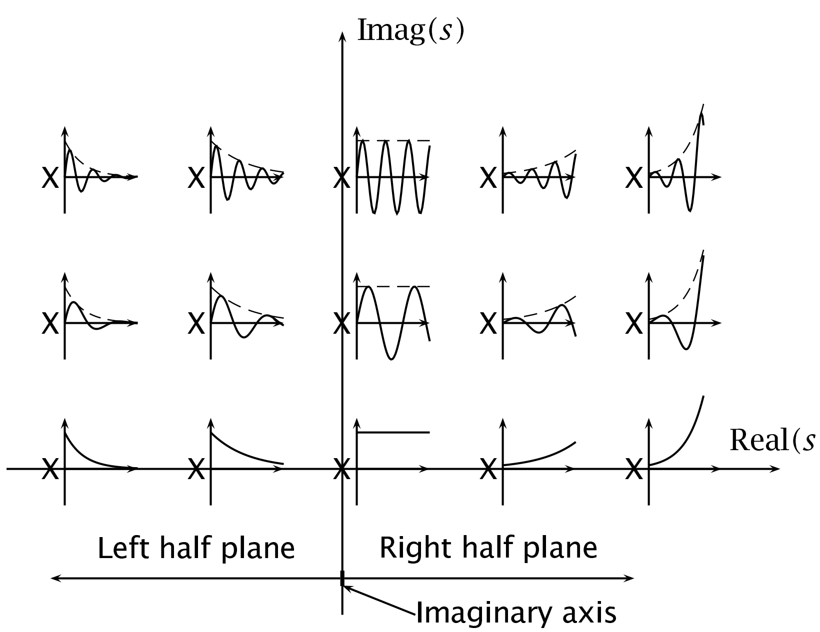
IB Paper 6, 2009: Solutions

SECTION A

1. *Pole locations, stability, closed loop transfer function, step response*

   (a) For an asymptotically stable system the real part of the poles must lie in the left hand part of the s-plane. If they are a complex conjugate pole pair they give rise to an oscillatory response: the cosine of the angle of the poles to the real axis is the damping factor. The larger the negative real part the faster the response decays. The imaginary part gives the frequency of oscillation. If poles are on the imaginary axis they are marginally stable, and if repeated or in the right-hand half plane unstable. An appropriate sketch is: [6]



Pole locations and corresponding transient responses

   (b) The CLTF is

$$\frac{G(s)}{1+K(s)G(s)}.$$

   Applying this formula to $G(s) = \frac{4}{s+3}$ and $K(s) = \frac{1}{s-1}$ gives the desired result. [3]

   (c) Open loop poles are at $s - 3$, $s = +1$. Hence it is unstable. The closed loop has (two) poles at $s = -1$ and hence it is stable. [3]

   (d) From the question the CLTF is

$$\frac{4(s-1)}{(s+1)^2}$$

With a unit step input then the transform of the output will be

$$\bar{y}(s) = \frac{4(s-1)}{s(s+1)^2}$$

Applying partial fractions:

$$\frac{4(s-1)}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

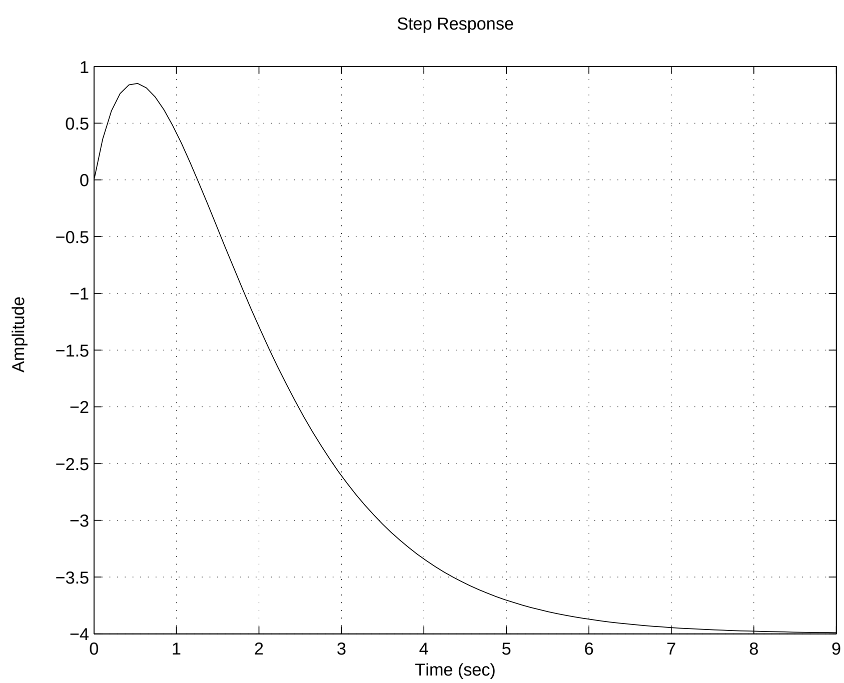
By cover up rule $A = -4, C = 8$. Multiply by LHS denominator and compare coefs of s^2 implies $A + B = 0$, and hence $B = 4$.

Hence finding the inverse Laplace transform (from data book)

$$\begin{aligned} y(t) &= -4 + 4e^{-t} + 8te^{-t} \\ y(0) &= 0 \\ y'(t) &= -4e^{-t} + 8e^{-t} - 8te^{-t} \\ y'(0) &= 4 \end{aligned}$$

As $t \to \infty$ then $y(t) \to 4$. For turning points set $y'(t) = 0$ which yields a maximum at $t = 0.5$. Also to find when the curve crosses the axis i.e.$y(t) = 0$ when $e^t = 1 + 2t$. This is at $t = 0$ and at approx $t = 1.25$ (by iteration).

A sketch of $y(t)$ showing the above features follows: [8]



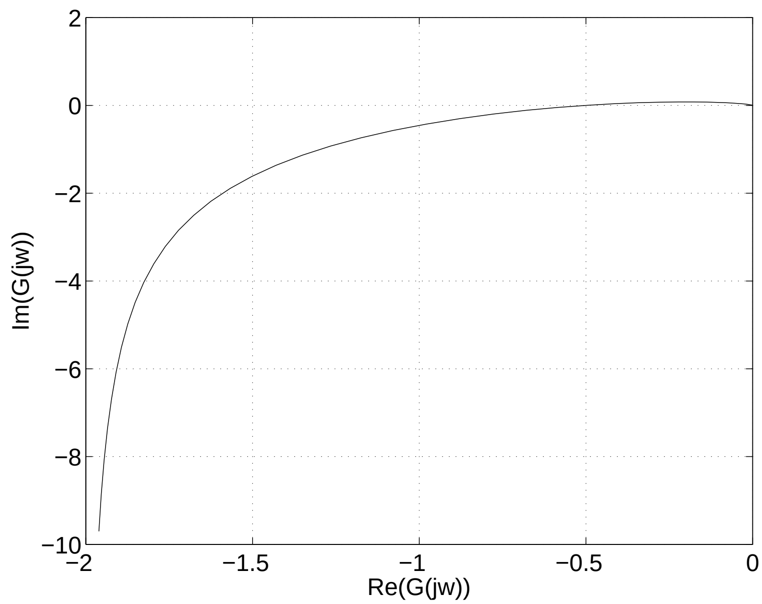
2. *Nyquist Diagrams*

(a) The Nyquist diagram shows the real and imaginary parts of the steady state frequency response for a range of frequencies for the open loop system. To construct, need a sinusoidal input to the open loop system, when at steady state, measure the magnitude and phase and convert to the real and imaginary parts. This should be done for a range of frequencies (often with logarithmic spacing) over the range of interest. [3]

(b)(i) To construct the Nyquist diagram set $s = j\omega$ and measure $G(j\omega)$.

$$G((j\omega) = \frac{1}{j\omega(1 - \omega^2 + 2j\omega)}$$

As $\omega \to \infty$, the $|\cdot| \to 0$ and phase to $\frac{-3\pi}{2}$.
As $\omega \to 0$, $|\cdot| \to \infty$ and phase to $\frac{-\pi}{2}$.
In fact as can be seen from a Taylor expansion, as $G(j\omega \to 0) \to -2$ (which is the required asymptote). For $\omega = 1, G(j\omega) = -0.5$. Hence the Nyquist plot can be drawn: [5]



(b)(ii) The magnitude of the closed-loop frequency response is

$$\frac{|K.G(j\omega)|}{|1 + K.G(j\omega)|}$$

From the Nyquist plot, for $K = 1$ the numerator is the distance from the origin to the point on $G(j\omega)$ curve at a particular frequency. The denominator is the distance from (-1,0) to the point on the $G(j\omega)$ curve. [2]

(b)(iii) Gain margin is simply the reciprocal of the intersection with the negative real axis. Since $\omega = 1, G(j\omega) = -0.5$, the gain margin is equal to 2. [2]

(b)(iv) The $j\omega$ term on the denominator gives a phase of $-90°$. If the phase margin is $60°$ then

$$\begin{aligned} \tan^{-1}\frac{2\omega}{1-\omega^2} &= 30° \\ \frac{2\omega}{1-\omega^2} &= \frac{1}{\sqrt{3}} \\ \omega &= -\sqrt{3} \pm 2 \\ \omega &= 2-\sqrt{3} \\ &= 0.268 \end{aligned}$$

The magnitude is

$$\frac{K}{\omega(1+\omega^2)}$$

and at the phase margin the magnitude is equal to 1. Equating gives the magnitude as 3.482. Hence the corresponding value of $K = \frac{1}{3.482} = 0.287$. The closed-loop frequency response is simply given by the ratio of two unit lengths if the phase margin is $60°$, and is hence equal to one.

If the phase margin were smaller than $60°$, this would imply the magnitude of the CL frequency response and the value of ω_c both increase, which implies that K increases.

[8]

3. *Bode Plots*

(a) For an asymptotically stable open-loop system: the phase margin (if it exists) is the increase in (negative) phase before the system becomes unstable at constant gain; the gain margin (if it exists) is the factor that the gain must increase at constant phase before the system becomes unstable (measured when phase angle = $-180°$). [3]

(b) For the GM, find when the phase plot is at $-180°$, and find the difference between 0dB and the plot. This is equal to 3dB (at 25 rad/s). For the PM, find when the gain plot is at 0dB, and find the difference between the plot and $-180°$. This is equal to $10°$ (at 22 rad/s). [4]

(c) First draw the plot for the compensator. Note that the numerator has a breakpoint frequency of 10 rad/s (20 dB/decade increase in magnitude at this point), and the denominator has a breakpoint at 40 rad/s. The overall compensator has a 0.5 term on the numerator which reduces the gain by 6dB at low frequencies (the high frequency gain is increased by 6dB). The effect on phase will be over a wide range of frequencies (approx 1 rad/s to 400 rad/s). Need to compute some sample points of the phase advance due to the compensator in order to yield an accurate plot.

For instance at $\omega = 10$ phase advance is $tan^{-1}1 - tan^{-1}0.25 = 31°$. At $\omega = 20$ phase advance is $tan^{-1}2 - tan^{-1}0.5 = 37°$. etc. Combining these yields a Bode plot for the original (solid line), compensator alone (dash-dot line) and compensated loop (dashed line): see Fig. 1.

Measuring from the plot yields a phase margin of $44°$ (at 22rad/s) and a gain margin of 11dB (at 45 rad/s). [9]

(d) The main difference in response is that the compensated loop is less oscillatory (the uncompensated loop has a small PM and is close to instability). To find the steady-state error for a unit step input need to use final value theorem. This means that final value of $y(t)$ can be found as the zero frequency value of the closed-loop frequency response:

$$\frac{K(0)G(0)}{1 + K(0)G(0)} = \frac{12K(0)}{1 + 12K(0)}$$

The steady state error is $1 -$ final-value or

$$\frac{1}{1 + 12K(0)}$$

Hence for $K = 1$, steady-state error $= \frac{1}{1+12} = \frac{1}{13}$.
For compensator $K(0) = 0.5$, steady state error $= \frac{1}{1+6} = \frac{1}{7}$. [4]

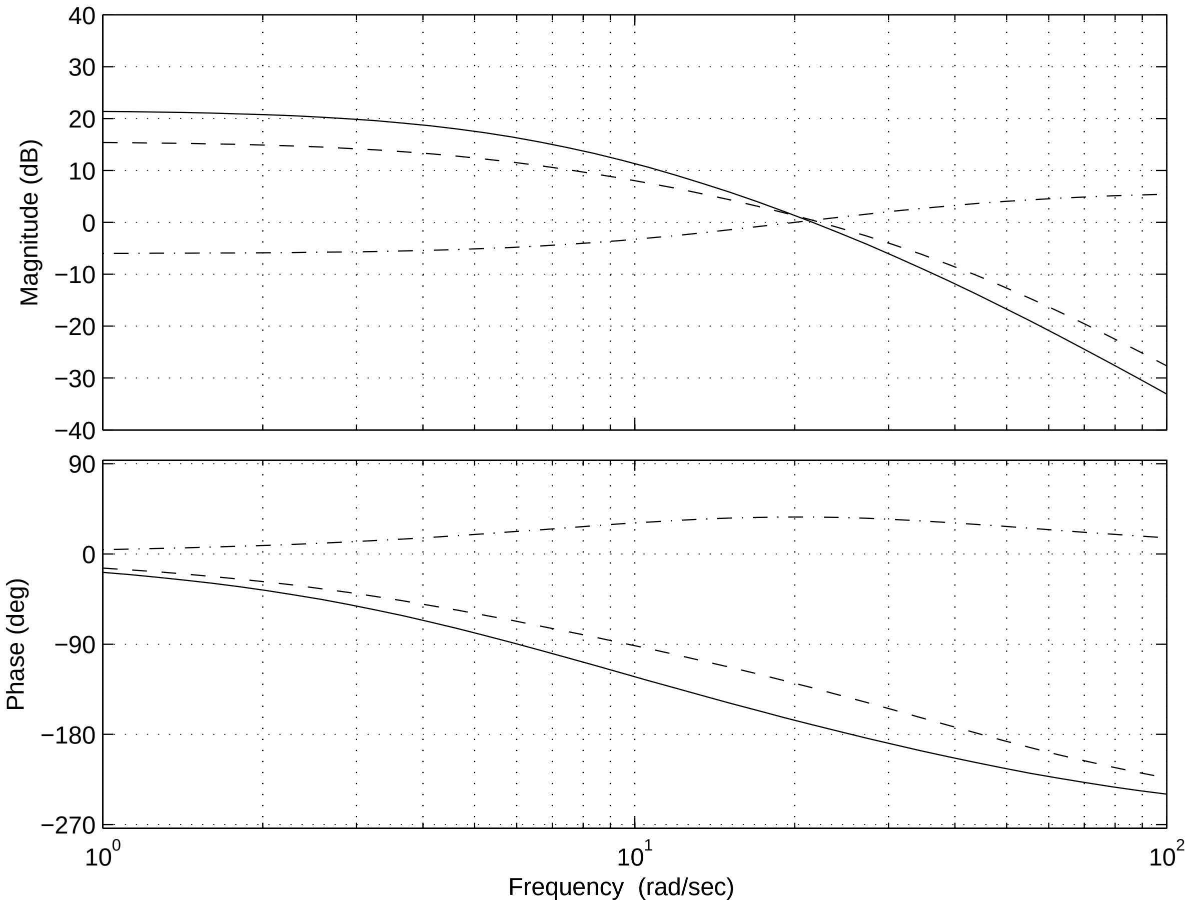


Figure 1: Bode plots for the original loop, compensator and compensated loop, for use with phase margin and gain margin calculations

SECTION B

4. *Fourier transforms*

(a) Need to show that

$$\mathcal{F}\{x(t) * y(t)\} = \mathcal{F}\{x(t)\}\,\mathcal{F}\{y(t)\}$$

where $*$ and $\mathcal{F}\{.\}$ denote the convolution and Fourier transform operators, respectively.

$$\begin{aligned}
\mathcal{F}[x(t) * y(t)] &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \right) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) y(t-\tau) e^{-j\omega t} dt \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} y(t-\tau) e^{-j\omega t} dt \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} y(x) e^{-j\omega(x+\tau)} dx \right) d\tau \\
&= Y(\omega) \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \\
&= X(\omega) Y(\omega)
\end{aligned}$$

where the first step follows from the definitions of convolution and Fourier transform, the second stem follows from a change of integration order and the fourth step follows from the change of variables $x = t - \tau$. The other steps are straightforward. [4]

(b)(i) The signals $x_i(t)$ are sinusoids multiplied by a rectangular pulse $p(t)$. Therefore, by applying the convolution property in the frequency domain, the overall Fourier transform is the convolution of the Fourier transform of the sinusoid and the Fourier transform of the rectangular pulse. Writing

$$\sin \omega_i t = \frac{1}{2j}(e^{j\omega_i t} - e^{-j\omega_i t})$$

$$X_i(\omega) = \frac{1}{2j}(P(\omega - \omega_i) - P(\omega + \omega_i))$$

where $P(\omega)$ is the rectangular pulse Fourier transform. Since the pulse is centered at the origin, we obtain that [5]

$$P(\omega) = T\text{sinc}\left(\frac{\omega T}{2}\right)$$

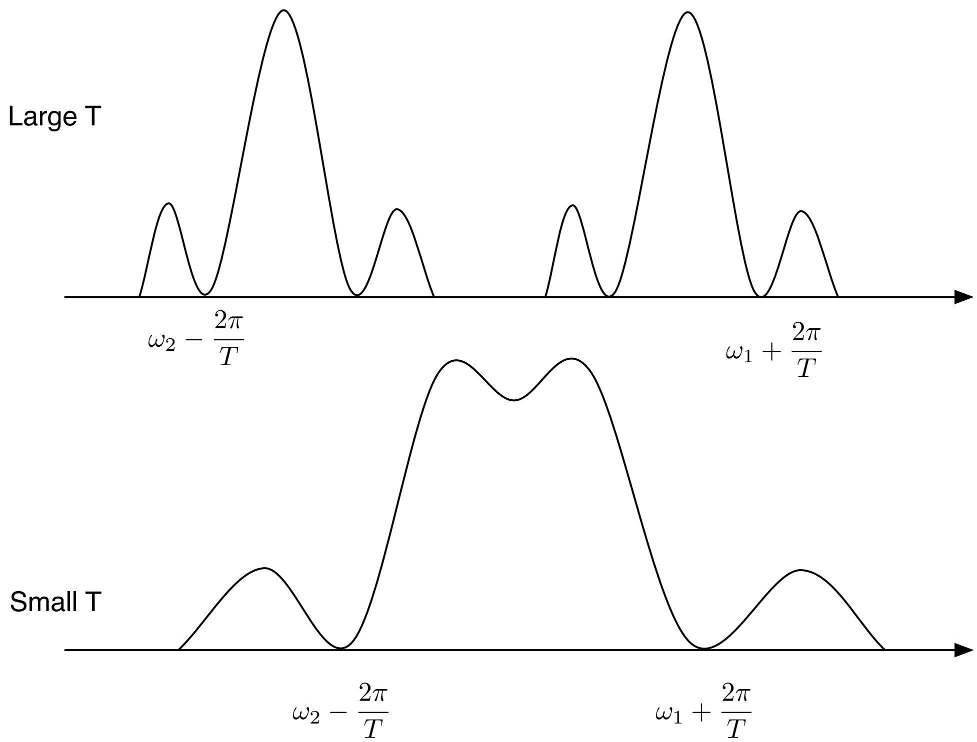


Figure 2: Sketch of $|Y(\omega|$

(b)(ii) The Fourier transform of $y(t)$ consists of 2 shifted sinc functions. The sinc functions have zeros at $\omega_i \pm \frac{n2\pi}{T}$. Hence, for large T (more observation, more resolution), the resulting transform will resemble 2 delta functions, while for small T (poor observation, poor resolution) we might not be able to distinguish the 2 peaks.

As above, large T implies better resolution, as the zeros of the sinc are at $\frac{n}{T}$. Hence the larger T, the better estimation of ω_i. On the other hand, for small T, the main lobe of the sincs will be wide, and it will be difficult to distinguish the two tones if they are close to each other. [7]

(b)(iii) The triangular pulse has a steeper decay (although the zeros are in the same positions). Therefore, it will increase the resolution in the frequency domain. [4]

5. *Sampling/Quantisation*

(a)

$$x_s(t) = x(t) \sum_n \delta(t - nT_s) = \sum_n x(nT_s)\delta(t - nT_s)$$

where T_s is the sampling period. Since the Fourier transform of a train of delta functions is a train of delta functions,

$$\sum_n \delta(t - nT_s) \longleftrightarrow \frac{1}{T_s} \sum_m \delta\left(f - \frac{m}{T_s}\right)$$

the Fourier transform of the sampled signal $x_s(t)$ is given by

$$\begin{aligned} X_s(f) = \mathcal{F}\left[x(t) \sum_n \delta(t - nT_s)\right] &= \mathcal{F}[x(t)] * \mathcal{F}\left[\sum_n \delta(t - nT_s)\right] \\ = X(f) * \frac{1}{T_s} \sum_m \delta\left(f - \frac{m}{T_s}\right) &= \frac{1}{T_s} \sum_m X\left(f - \frac{m}{T_s}\right). \end{aligned}$$

Hence the Fourier transform of the sampled signal consists of the the Fourier transform of the original signal translated at multiples of the sampling frequency. Hence, if the the sampling frequency is larger or equal than $2B$ we can recover the original signal with an ideal filter. Otherwise, the multiple copies will overlap, causing aliasing, and perfect reconstruction is not possible. [5]

(b) The quantisation noise is $e(t) \triangleq x(t) - x_Q(t) \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$, where $x_Q(t)$ is the quantised signal. We model $e(t)$ as a uniformly distributed random variable between $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$ The noise power can therefore be calculated as

$$N_Q = \mathbb{E}[e^2] = \int x^2 \frac{1}{\Delta} dx = \frac{1}{\Delta} \frac{x^3}{3}\bigg|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12}$$

and its corresponding RMS is $\frac{\Delta}{\sqrt{12}}$. [5]

(c) For sinusoidal waves, the SNR depends on the number of bits, n, as

$$\text{SNR} = 1.76 + 6.02n\text{dB}$$

Hence $n = \left\lceil \frac{\text{SNR}-1.76}{6.02} \right\rceil = 5$. [4]

(d) The minimum data rate is $R = 2Bn = 10B$. [2]

(e) The quantiser gives a finer quantisation when the input signal has a small magnitude. Therefore, the quantiser is advantageous for signals that take on small values more often than large values. Signals that have an approximately Gaussian pdf would be suited for this type of quantisers. [4]

6. *Modulation/Multiple Access*

(a) Modulation is the process of shaping one or multiple parameters of a carrier wave according to a given information signal $x(t)$. Modulation is used in practice to translate the baseband information signal to a frequency range where the attenuation introduced by the channel is low. In wireless communications, modulation is further used to keep the size of the antennas small. [2]

(b) From lecture notes, the bandwidth of AM and DSB-SC is $2B$, while that of SSB-SC is B. Using Carson's approximated rule, we obtain that for FM, the bandwidth is $2(B+\Delta f)$, where Δf is the frequency deviation. Clearly, the bandwidth consumption of FM is larger than those of amplitude modulations. As for BPSK, the bandwidth depends on the rate of the digitised signal. In particular, if we sample at the Nyquist rate and quantise with an n-bit uniform quantiser, we obtain a rate of $R = \frac{1}{T} = 2Bn$ bit/s. Then, assuming a rectangular pulse, we know the zeros happen at multiples of $1/T$, where T is the signaling period. Hence, the first zero (or main lobe) bandwidth for BPSK is $2 \times 2Bn = 4Bn$, which is again larger than that of analogue amplitude methods. The second zero bandwidth for BPSK is $2 \times 4B = 8Bn$. [5]

(c) Transmission of BPSK modulation over an additive white Gaussian noise (AWGN) can be modelled as

$$Y = X + Z$$

where $X \in \{-A, +A\}$ and Z is a zero-mean Gaussian random variable with variance σ^2. The optimal demodulator will decide $\widehat{X} = +A$ if $Y > 0$ and $\widehat{X} = -A$ otherwise, where $\widehat{X}$ denotes the estimated transmitted signal. We want to calculate the error probability $P_e = p(\widehat{X} \neq X)$. The error probability can be expressed as

$$\begin{aligned} P_e &= p(\widehat{X} \neq X) \\ &= p(\widehat{X} = +A|X = -A)p(X = -A) + p(\widehat{X} = -A|X = +A)p(X = +A) \\ &= \frac{1}{2}\big(p(\widehat{X} = +A|X = -A) + p(\widehat{X} = -A|X = +A)\big) \\ &= \frac{1}{2}\big(p(Y > 0|X = -A) + p(Y < 0|X = +A)\big) = p(Y < 0|X = +A) \end{aligned}$$

due to the symmetry of the problem. Conditioned on $X = +A$, Y is a Gaussian random variable, with mean $+A$ and variance σ^2, i.e., $Y \sim N(+A, \sigma^2)$. Given that $Y \sim N(+A, \sigma^2)$, the error probability can be expressed as

$$\begin{aligned} P_e = p(Y < 0|X = +A) &= \int_{-\infty}^{0} p_Y(y)dy = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-A)^2} dy \\ &= \int_{-\infty}^{-\frac{A}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = \Phi\left(-\frac{A}{\sigma}\right) \\ &= Q\left(\frac{A}{\sigma}\right) = Q\left(\sqrt{\frac{A^2}{\sigma^2}}\right) = Q\left(\sqrt{2\mathrm{SNR}}\right) \end{aligned}$$

where $\Phi(x)$ is the Gaussian cumulative distribution function, $Q(x) \triangleq 1 - \Phi(x)$ and $\text{SNR} \triangleq \frac{A^2}{2\sigma^2}$. [5]

(d) We find the minimum SNR by inverting the above error probability expression, i.e.,

$$\text{SNR} = \frac{1}{2}\left[Q^{-1}(P_e)\right]^2$$

We note that $5.10^{-6} = \frac{1}{2}.10^{-5}$. Since $Q(x) = 1 - \Phi(x)$ we get that

$$1 - \Phi(x) = \frac{1}{2}.10^{-5}$$
$$2(1 - \Phi(x)) = 10^{-5}$$

From the maths databook (last line page 27), we see that $x = 4.417$. Then, we obtain that

$$\text{SNR} = \frac{1}{2}\left[x\right]^2 = 9.7549$$

Hence $\text{SNR} \geq 9.8922 \approx 10$ dB for the error probability to be $P_e \leq 5.10^{-6}$. [4]

(e)

The data rate of each user is $R = \frac{1}{T} = 50$ kbit/s.The spectrum of a BPSK signal is given by

$$|S_{\text{BPSK}}(f)|^2 = \frac{1}{4}\left[|X(f - f_c)|^2 + |X(f + f_c)|^2\right]$$

where $X(f) = |X(f)|^2 = \frac{1}{T}|P(f)|^2$. Since we assume no interference is caused beyond the second sinc zero, we assume each user occupies a bandwidth of $B_u = \frac{4}{T} = 200$ KHz. Hence the total required bandwidth is $B = 4$ MHz. If a guard band of 10 kHz is used, we have that each user occupies a total band of 210 KHz. Hence, the new total bandwidth is $B = 4.2$ MHz. [4]