ENGINEERING TRIPOS PART IB

2009

Paper 7

MATHEMATICAL METHODS

Solutions

SECTION A

1 (a) Consider the vector field $\mathbf{u} = \mathbf{i} - \mathbf{j}\ \sin 2x$. Find the equations of the field lines of $\mathbf{u}$ and sketch the field lines. Make sure you include the field direction in your sketch. [5]

Solution:

Field lines must satisfy

$$\frac{dx}{dy} = \frac{u_x}{u_y} = \frac{-\sin 2x}{1}$$

so

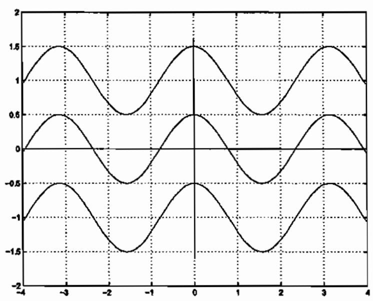
$$dy = -\sin 2x dx$$

integrating we get

$$y = \int dy = -\int \sin 2x dx = \frac{1}{2}\cos 2x + K$$

where K is a constant.

The field lines are therefore cosines for all constants K.



(cont.

(b) Consider two vector fields $\mathbf{v}$ and $\mathbf{w}$ where

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}$$

v_x is a function of x only, v_y is a function y only, and v_z is a function of z only,

$$\mathbf{w} = e^x\,\mathbf{i} + e^y\,\mathbf{j} + e^z\,\mathbf{k}$$

and

$$\nabla(\mathbf{v}\cdot\mathbf{w}) = 0$$

Solve for $\mathbf{v}$. [7]

Solution:

$$\mathbf{v}\cdot\mathbf{w} = v_x e^x + v_y e^y + v_z e^z$$

$$\nabla(\mathbf{v}\cdot\mathbf{w}) = (\frac{\partial v_x}{\partial x}e^x + v_x e^x)\mathbf{i} + (\frac{\partial v_y}{\partial y}e^y + v_y e^y)\mathbf{j} + (\frac{\partial v_z}{\partial z}e^z + v_z e^z)\mathbf{k}$$

solving

$$\frac{\partial v_x}{\partial x} + v_x = 0$$

we get

$$\frac{dv_x}{dx} = -dx$$

$$\int \frac{dv_x}{dx} = -\int dx$$

$$\log v_x = -x + c$$

so $v_x = Ke^{-x}$ where K is a constant. Similarly for v_y and v_z. So the solution is:

$$\mathbf{v} = K_1 e^{-x}\mathbf{i} + K_2 e^{-y}\mathbf{j} + K_3 e^{-z}\mathbf{k}$$

Some common partially correct solutions set all K_i equal, or omit the constants completely. The trivial solution $\mathbf{v} = 0$ also gets partial marks.

(TURN OVER for continuation of Question 1

(c) Let

$$\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{j} + u_z\,\mathbf{k}$$

be an irrotational field,

$$\mathbf{w} = y^2\,\mathbf{i} + z^2\,\mathbf{j} + x^2\,\mathbf{k}$$

and

$$\mathbf{v} = a\,\mathbf{i} + a\,\mathbf{j} + a\,\mathbf{k}$$

be a constant field. Evaluate

$$\nabla \cdot (\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))$$

in terms of a, u_x, u_y, u_z, and x, y and z, simplifying as much as possible. [8]

Solution:

Some definitions from DataBook and properties above used:

$$\nabla \cdot (\mathbf{u} \times (\mathbf{v} \times \mathbf{w})) = (\mathbf{u} \times \mathbf{w}) \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times (\mathbf{u} \times \mathbf{w})$$

by irrotationality of $\mathbf{u}$:

$$\nabla \cdot (\mathbf{u} \times (\mathbf{v} \times \mathbf{w})) = -\mathbf{u} \cdot \nabla \times (\mathbf{u} \times \mathbf{w})$$

using $\nabla \cdot \mathbf{w} = 0$ (solenoidal) and $\nabla \cdot \mathbf{v} = 0$ (constant/solenoidal) we have

$$\nabla \cdot (\mathbf{u} \times (\mathbf{v} \times \mathbf{w})) = -\mathbf{u} \cdot (\mathbf{v}\nabla \cdot \mathbf{w} - \mathbf{w}\nabla \cdot \mathbf{v} + (\mathbf{w} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{w}) = \mathbf{u} \cdot (\mathbf{v} \cdot \nabla)\mathbf{w}$$

$$= 2a(u_x y + u_y z + u_z x)$$

Note importantly that $\mathbf{v} \cdot \nabla \neq \nabla \cdot \mathbf{v}$ since ∇ is an operator, not a vector. This confusion can lead to the incorrect answer 0.

2 Let S be the spherical surface defined by $x^2+y^2+z^2=a^2$.

(a) Calculate the total flux of the vector field

$$\mathbf{f}=r^2\,\mathbf{e}_r$$

where $\mathbf{e}_r$ is a unit radial vector, outwards through S. [6]

Solution: The flux is

$$\int\int_S r^2\,\mathbf{e}_r\cdot d\mathbf{A}.$$

The easiest way to solve this is to notice that the field is always perpendicular to the surface of the sphere. The surface area of a sphere of radius a is $4\pi a^2$ and given that the field has magnitude a^2 at the radius $r=a$, the total flux is simply the product of the two:

$$a^2(4\pi a^2)=4\pi a^4$$

This can also be obtained by Gauss's Theorem:

$$\int\int_S \mathbf{f}\cdot d\mathbf{A}=\int\int\int_V \nabla\cdot\mathbf{f}\,dV$$

$$\int\int\int_V \frac{1}{r^2}\frac{\partial}{\partial r}r^4\,dV=\int\int\int_V 4r\,dV=\int_0^a 4(4\pi r^3)dr=4\pi a^4.$$

(TURN OVER for continuation of Question 2

(b) Using Gauss' Theorem, evaluate the total flux of $\mathbf{f}$ outwards through the volume shown in Fig. 1, which is bounded by the surface S and the three planes $x=0$, $y=0$, $z=0$. [5]

Solution:

The easiest way to find this is to note that the answer is $1/8$ of the answer to part (a), since the surface of the region shown is an octant of the whole sphere S, and the radial field $\mathbf{f}$ is parallel to the three planes thus there is no flux through those planes. Therefore, the solution is:

$$\frac{\pi}{2}a^4$$

More laborious solutions involve integrals across each of the surfaces of S.

(c) Let $\mathbf{f}$ and $\mathbf{g}$ be two spherically symmetrical vector fields, $\mathbf{f}=f(r)\,\mathbf{e}_r$ and $\mathbf{g}=g(r)\,\mathbf{e}_r$. Calculate the flux of $\mathbf{f}\times\mathbf{g}$ outwards through S in terms of $f(r)$ and $g(r)$. [4]

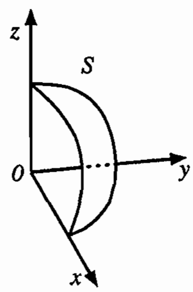
Solution:

Since $\mathbf{f}\times\mathbf{g}=0$, we have a flux of 0.

(d) Now assume $\mathbf{f}=x^2\,\mathbf{i}+y^2\,\mathbf{j}+z^2\,\mathbf{k}$. What is the total net flux of $\mathbf{f}$ outwards through S? [5]

Solution:

By symmetry of x^2 etc, for every octant of the sphere S there is one that cancels the flux, so the total flux is 0. Note that the field $\mathbf{f}$ here is *not* the same as the one in part (a), since it does not point radially outwards everywhere.



3 Consider the wave equation describing the motion of a string with transverse displacement $y(x,t)$:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad (1)$$

where c is the wave constant.

(a) Use separation of variables to find solutions for $y(x,t)$. [6]

Solution:

Separation of variables:

$$y(x,t) = X(x)F(t)$$

$$XF'' = c^2 X'' F$$

therefore

$$\frac{F''}{F} = c^2 \frac{X''}{X} = \text{constant}$$

Say constant $= \omega^2$ (positive), then solutions of the kind:

$$X(x) = Ae^{\omega x/c} + Be^{-\omega x/c}$$

$$F(t) = Ce^{\omega t} + De^{-\omega t}$$

exponential growth or decay, not interesting solutions generally for a string. Say constant $= -\omega^2$ (negative), then solutions of the kind:

$$X(x) = A\cos(\omega x/c) + B\sin(\omega x/c)$$

$$F(t) = C\cos(\omega t) + D\sin(\omega t)$$

which are sinusoidal waves travelling along the string left or right at speed c.

(TURN OVER for continuation of Question 3

(b) Given the boundary conditions $y = 0$ at $x = 0$ and at $x = L$, use the answer to part (a) to find solutions for $y(x,t)$. What is the physical significance of the form of the solutions? [6]

Solution:

Consider the sinusoidal form (exponential will not match bounary conditions here):

$$y(x,t) = (A\cos(\omega x/c) + B\sin(\omega x/c))(C\cos(\omega t) + D\sin(\omega t))$$

matching bounary conditions for all t we find that $A = 0$ and that

$$B\sin\frac{\omega L}{c} = 0.$$

An uninteresting solution is $B = 0$ which implies $y = 0$. More interestingly,

$$\frac{\omega L}{c} = \pi, 2\pi, 3\pi, \ldots$$

so $\omega = \frac{cn\pi}{L}$ for $n = 1, 2, 3, \ldots$ so

$$y(x,t) = B\sin\frac{n\pi x}{L}(C\cos(cn\pi t/L) + D\sin(cn\pi t/L))$$

These are vibration modes of a string (standing waves) where the shape is sinusoidal and $\frac{cn\pi}{L}$ is the natural frequency. The integer n controls how many nodes there are in the wave.

(cont.

(c) Using the substitutions $\alpha = x - ct$ and $\beta = x + ct$, but ignoring the boundary conditions in part (b), express the wave equation (1) as a function of α and β. Hence find a general solution for y as a function of α and β, and give an interpretation for the form of this solution. [8]

Solution:

$$\alpha = x - ct$$

$$\beta = x + ct$$

Use the chain rule to rewrite the general wave equation in α and β.

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial \alpha} + \frac{\partial y}{\partial \beta}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial \alpha^2} + 2\frac{\partial^2 y}{\partial \alpha \partial \beta} + \frac{\partial^2 y}{\partial \beta^2}$$

Similarly,

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial \alpha^2} - 2c^2\frac{\partial^2 y}{\partial \alpha \partial \beta} + c^2\frac{\partial^2 y}{\partial \beta^2}.$$

Substituting into the wave equation (1), some terms cancel and we find,

$$-2c^2\frac{\partial^2 y}{\partial \alpha \partial \beta} = 2c^2\frac{\partial^2 y}{\partial \alpha \partial \beta}$$

therefore

$$\frac{\partial^2 y}{\partial \alpha \partial \beta} = 0.$$

Integrating twice, we find that the general solution for this PDE is of the form:

$$y(\alpha, \beta) = f(\alpha) + g(\beta) = f(x - ct) + g(x + ct)$$

for general functions f and g. This is the superposition of arbitrary leftward and rightward travelling waves, travelling at speed c.

SECTION B

4 (a) A bag contains 10 coins, of which 9 are conventional coins, but one is fake with 2 heads.

(i) Probability of heads is $9/10 \times 1/2 + 1/10 \times 1 = 11/20$. [4]

(ii) Probability of not picking the fake: $9/10 \times 8/9 = 4/5$, so probability of getting the fake is $1/5$. Probability of two heads $4/5 \times 1/4 + 1/5 \times 1/2 = 3/10$. [4]

(iii) Apply Bayes rule:

$$p(F|H) = \frac{p(H|F)p(F)}{p(H)} = \frac{p(H|F)p(F)}{p(H|F)p(F) + p(H|\overline{F})p(\overline{F})} = \frac{1/10}{1/10 + 9/10 \times 1/2} = 2/11.$$

[4]

(b) A discrete uniform random variable X takes on values $-2, -1, 0, 1, 2$.

(i) mean 0, standard deviation $\sqrt{2}$, entropy $\log(5)$. [4]

(ii) Y has higher entropy (Z only has three possible values). [4]

5 (a) Unique solution only if $\det(A) \neq 0$. We have $\det(A) = 2a - 13$, so no unique solution if $a = 13/2$. [5]

(b) Since $\det(A) = 0$, we have (at least) one zero eigenvalue. Solve $A\mathbf{x} = \mathbf{0}$ (eg Gaussian elimination) to get $\mathbf{v} \pm [1, -2, 4]/\sqrt{21}$. [4]

(c) Use Gaussian elimination to find $c = 1/6$. The complete solution is $[11, 2, 0]/24 + \alpha\mathbf{v}$ (eigenvector from above). [5]

(d) Find roots of $\det(A - (\alpha - 2)I) = 0$; We know that the roots of $\det(A - \beta I) = 0$ are, $\beta = (19 + \sqrt{(129)})/4$, $(19 - \sqrt{(129)})/4$ and 0 (from part (a)), so substituting $\alpha = 2 + \beta$, we get $\alpha = (27 + \sqrt{(129)})/4$, $(27 - \sqrt{(129)})/4$ and 2. [3]

(e) $A^3 = (VLV')^3 = VL^3V'$, so eigenvalues of A^3 are the eigenvalues of A cubed. Thus, $\lambda = 0$, $((19 + \sqrt{(129)})/4)^3$ and $((19 - \sqrt{(129)})/4)^3$. [3]

(TURN OVER

6 (a) The region is a unit disc centered on the origin. From normalisation, we get $\alpha = 1/\pi$. [4]

(b) Not independent, eg $x_1 = 1$ implies $x_2 = 0$. [4]

(c) A is a rotation matrix. Because of rotation symmetry, the distribution on $\mathbf{z}$ is the same as the distribution on $\mathbf{x}$. [4]

(d) The conditional distribution $p(x_1|x_2)$ lies on a horizontal slice through the unit disc. The length of the slice is $2\sqrt{1-x_2^2}$, so the conditional probability comes out to be $p(x_1|x_2) = (2\sqrt{1-x_2^2})^{-1}$, $-\sqrt{1-x_2^2} < x_1 < \sqrt{1-x_2^2}$. [4]

(e) Use that y is the radius. The mass of points at a certain radius is proportional to the radius, so $p(y) \propto y$. Normalisation then gives $p(y) = 2y, 0 \leq y < 1$. [4]

END OF PAPER