ENGINEERING TRIPOS PART IB

Friday 5 June 2009 9 to 11

Paper 7

MATHEMATICAL METHODS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

*Answer not more than **two** questions from this section.*

1 (a) Consider the vector field $\mathbf{u} = \mathbf{i} - \sin(2x)\,\mathbf{j}$. Find the equations of the field lines of $\mathbf{u}$ and illustrate these with a sketch. Make sure you include the field direction in your sketch. [5]

(b) Consider two vector fields $\mathbf{v}$ and $\mathbf{w}$ where

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}$$

v_x is a function of x only, v_y is a function y only, and v_z is a function of z only,

$$\mathbf{w} = e^x\,\mathbf{i} + e^y\,\mathbf{j} + e^z\,\mathbf{k}$$

and

$$\nabla(\mathbf{v} \cdot \mathbf{w}) = 0$$

Determine expressions for v_x, v_y and v_z. [7]

(c) Assume

$$\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{j} + u_z\,\mathbf{k}$$

is an irrotational field,

$$\mathbf{v} = a\,\mathbf{i} + a\,\mathbf{j} + a\,\mathbf{k}$$

is a constant field, and

$$\mathbf{w} = y^2\,\mathbf{i} + z^2\,\mathbf{j} + x^2\,\mathbf{k}$$

Evaluate

$$\nabla \cdot (\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))$$

in terms of a, u_x, u_y, u_z, and x, y and z, simplifying as much as possible. [8]

2 Let S be the spherical surface defined by $x^2+y^2+z^2=a^2$.

(a) Calculate the total flux of the vector field

$$\mathbf{f}=r^2\,\mathbf{e}_r$$

where $\mathbf{e}_r$ is a unit radial vector, outwards through S. [5]

(b) Using Gauss' Theorem, evaluate the total flux of $\mathbf{f}$ outwards through the volume shown in Fig. 1, which is bounded by the surface S and the three planes $x=0$, $y=0$, $z=0$. [5]

(c) Let $\mathbf{f}$ and $\mathbf{g}$ be two spherically symmetrical vector fields, $\mathbf{f}=f(r)\,\mathbf{e}_r$ and $\mathbf{g}=g(r)\,\mathbf{e}_r$. Calculate the flux of $\mathbf{f}\times\mathbf{g}$ outwards through S in terms of $f(r)$ and $g(r)$. [5]

(d) Now assume $\mathbf{f}=x^2\,\mathbf{i}+y^2\,\mathbf{j}+z^2\,\mathbf{k}$. What is the total net flux of $\mathbf{f}$ outwards through S? [5]

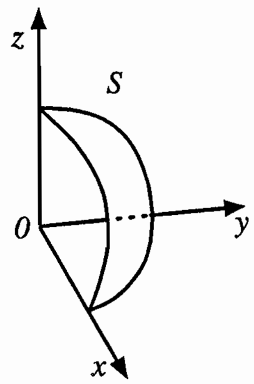

Fig. 1

3 Consider the wave equation describing the motion of a string with transverse displacement $y(x,t)$:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

where x is the distance along the string, t is time, and c is the wave constant.

(a) Use the method of separation of variables to find solutions for $y(x,t)$. [5]

(b) Given the boundary conditions $y = 0$ at $x = 0$ and at $x = L$, use the answer to part (a) to find solutions for $y(x,t)$. What is the physical significance of the form of the solutions? [6]

(c) Using the substitutions $\alpha = x - ct$ and $\beta = x + ct$, but ignoring the boundary conditions in part (b), express the wave equation as a function of α and β. Hence find a general solution for y as a function of α and β, and give an interpretation for the form of this solution. [9]

SECTION B

*Answer not more than **two** questions from this section.*

4 (a) A bag contains 10 coins, of which 9 are conventional coins, but one is fake with 2 heads.

(i) A coin is drawn at random and tossed. What is the probability of heads? [4]

(ii) Two coins are drawn at random and tossed. What is the probability of getting two heads? [4]

(iii) A third coin is drawn and tossed, and comes out heads. What is the probability that it was the fake coin? [4]

(b) A discrete uniform random variable X takes on values $-2,-1,0,1,2$.

(i) What is the mean, standard deviation and entropy of X? [4]

(ii) Define two random variables $Y = 6X$ and $Z = X^2$. Which has higher entropy? Explain your answer. [4]

(TURN OVER

5 Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & a & 3 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix}.$$

(a) Determine a such that $A\mathbf{x} = \mathbf{b}$ does not have a unique solution. [3]

(b) For the value of a found in part (a), determine a normalized eigenvector for A. [4]

(c) For the value of a found in part (a), find the value of c such that $A\mathbf{x} = \mathbf{b}$ has solutions, and give the complete solution. [5]

(d) For the value of a found in part (a), the characteristic equation for the matrix A has three distinct roots, two of which are $(19-\sqrt{129})/4$ and $(19+\sqrt{129})/4$. Find the eigenvalues of the matrix $A+2I$, where I is the identity matrix. [4]

(e) For the value of a found in part (a), find the eigenvalues of the matrix A^3. [4]

6 A continuous probability distribution is given by

$$p(x_1,x_2) = \begin{cases} \alpha & \text{if } x_1^2 + x_2^2 < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{where} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) Sketch the region where the probability density is non-zero and calculate α. [3]

(b) Explain whether or not x_1 and x_2 are independent. [3]

(c) Let $\mathbf{z} = A\mathbf{x}$, where

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Calculate the joint distribution $p(z_1, z_2)$. [4]

(d) Calculate the conditional distribution $p(x_1|x_2)$. [5]

(e) Let $y = \sqrt{x_1^2 + x_2^2}$. Calculate $p(y)$. [5]

END OF PAPER