Engineering Tripos Part IB 2010

Paper 1 – Mechanics

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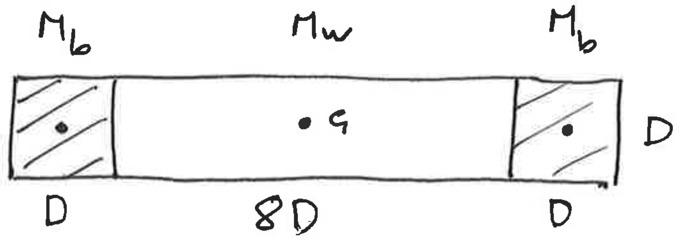
Solutions: Section A

G.McSHANE

Q1

a)



Data Book, rectangular lamina: $I_G = \frac{M}{12}(a^2+b^2)$

Wood: $I_G = \frac{M_w}{12}\left(64D^2 + D^2\right) = \frac{65}{12} M_w D^2$

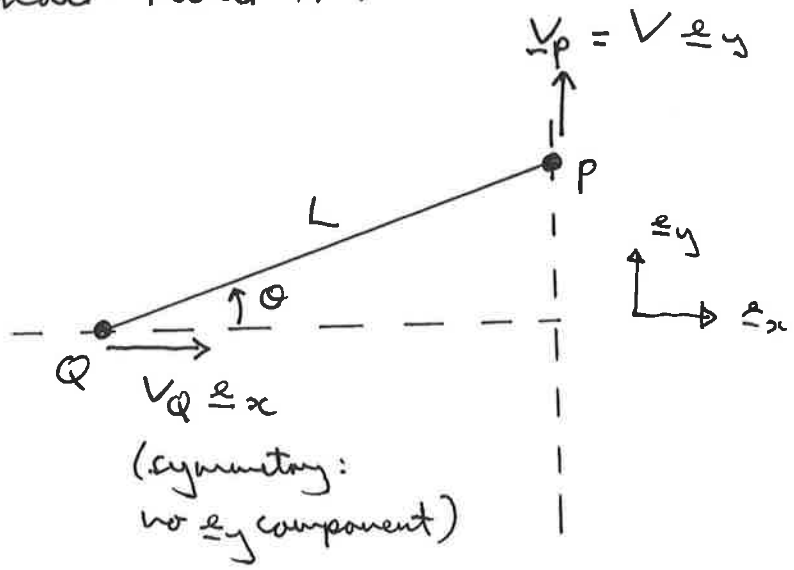
Brass (each): $I_G = \frac{M_b}{12}\left(2D^2\right) + \underbrace{M_b\left(\frac{9D}{2}\right)^2}_{\text{// axes term}}$

$$= \frac{245}{12} M_b D^2$$

Total (wood + 2×brass):

$$\boxed{I_G = \frac{65}{12} M_w D^2 + \frac{245}{6} M_b D^2}$$

b)(i) Consider ruler A:



Method 1 : vectors

$$\underline{r}_p = L\sin\theta\, \underline{e}_y$$

$$\dot{\underline{r}}_p = \dot{\theta} L\cos\theta\, \underline{e}_y = V\underline{e}_y$$

$$\therefore \boxed{\dot{\theta} = \frac{V}{L\cos\theta}}$$

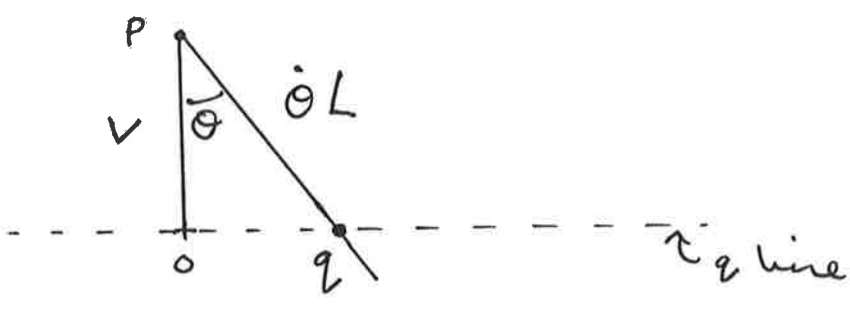
$$\ddot{\underline{r}}_p = \left(\ddot{\theta} L\cos\theta - \dot{\theta}^2 L\sin\theta\right)\underline{e}_y = 0$$

$$\therefore \boxed{\ddot{\theta} = \dot{\theta}^2\tan\theta = \frac{V^2\tan\theta}{L^2\cos^2\theta}}$$

[to write in terms of time : $L\sin\theta = Vt$]

Method 2 : diagrams

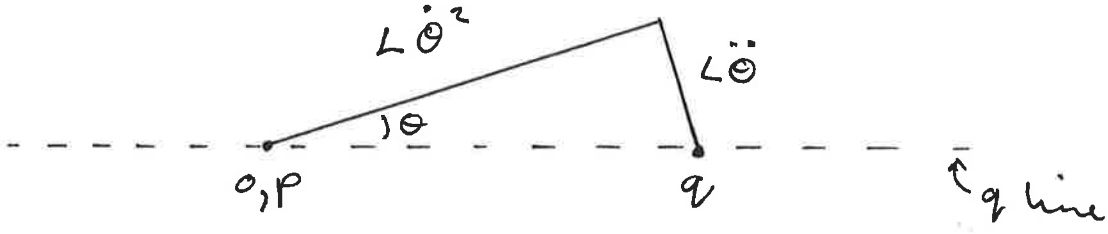
velocity:



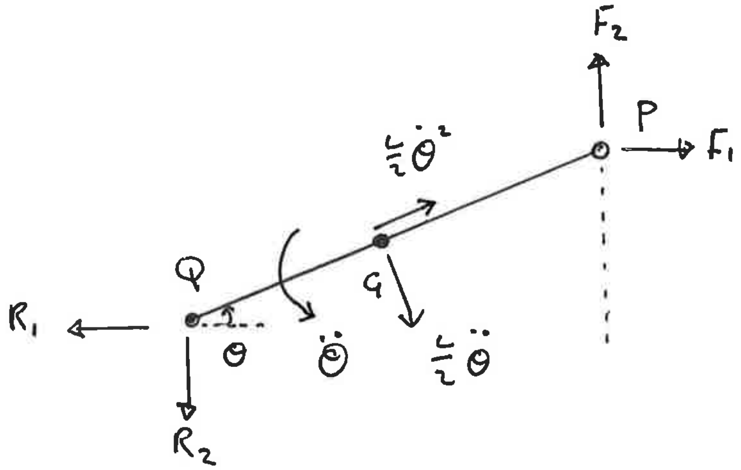
$$\therefore \boxed{\dot{\theta} = \frac{V}{L\cos\theta}}$$

acceleration :

$$\therefore \; L\ddot{\theta}\cos\theta = L\dot{\theta}^2\sin\theta \quad \Rightarrow \quad \boxed{\ddot{\theta} = \frac{V^2\tan\theta}{L^2\cos^2\theta}}$$



(ii) Consider the acceleration of G :



Symmetry $\Rightarrow \boxed{R_1 = 0}$ $\therefore$ reaction acts in y-direction only

Let : $M = M_w + 2M_b$ (total mass)

Moments about Q :

$$R_2\, L\cos\theta = I_G\ddot{\theta} + M\left(\frac{L}{2}\right)^2\ddot{\theta}$$

$$\boxed{R_2 = \left(I_G + \frac{ML^2}{4}\right)\frac{V^2\tan\theta}{L^3\cos^3\theta}}$$

Comments

Question 1 was the most popular question in Section A, attempted by nearly every candidate. The kinematics is similar to the ladder sliding down a wall in the examples paper, as picked up by some candidates. Although part (a) (a standard calculation of moments of inertia requiring the parallel axes theorem), was answered well, part (b) (analysing the kinematics and inertia forces) caused problems for most. Common errors in this question:

(i) Neglecting the width of the beam (*D*) – i.e. treating it as a rod – was common in part (a). The formula for the moment of inertia of a rod was clearly familiar, and most used this in conjunction with the perpendicular axes theorem to find the moment of inertia of each lamina.

(ii) The kinematics in part (b) caused difficulties. The best solutions involved a sketch of velocity and acceleration diagrams. Most attempted to first find the angular acceleration using D'Alembert forces, and then integrate it (often incorrectly) to get the angular velocity. This resulted in answers expressed in terms of the tip forces acting on the ruler rather than the prescribed velocity. Most also incorrectly assumed that only vertical forces exist at the tips of the ruler.

(iii) The acceleration of the centre of gravity of the ruler was often incorrect, the most common mistake being to assume that the hinge Q is not accelerating.

(iv) Many candidates assumed the forces in the pin to act parallel to the short edge of the ruler. Very few could identify correctly the effect of symmetry on the direction of the forces at the pin, although this is a common idea in the structural mechanics course when dealing with pin jointed arches with symmetric external loads.

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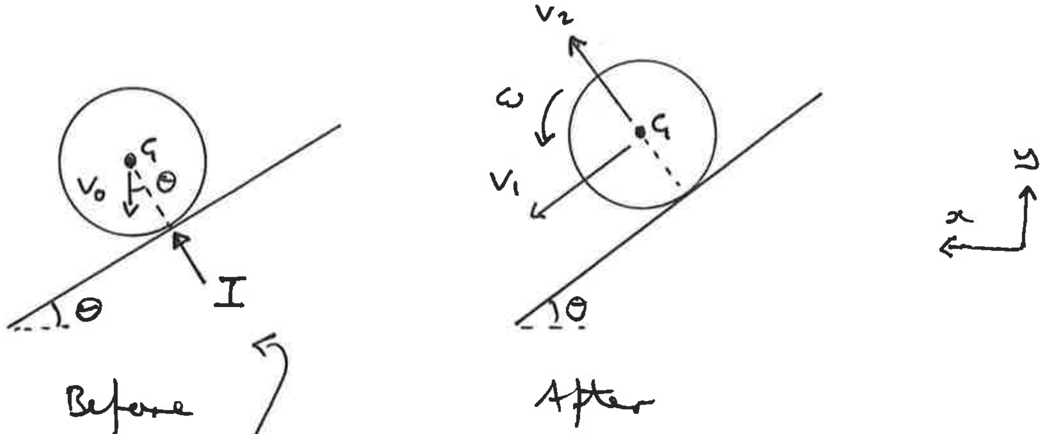
17/06/10

Q2

a) Smooth slope :



No friction : impulse acts normal

i) Moments of momentum about G :

$$\omega = 0$$

ii) Conservation of momentum along the plane :

$$M V_0 \sin\theta = M V_1$$

$$\therefore V_1 = V_0 \sin\theta$$

Perfectly elastic collision :

$$\frac{1}{2} M V_0^2 = \frac{1}{2} M (V_1^2 + V_2^2)$$

$$\therefore V_2 = V_0 \cos\theta$$

Conservation of momentum normal to plane :

$$I = M V_2 + M V_0 \cos\theta$$

$$\therefore \boxed{I = 2 M V_0 \cos\theta}$$

(iii) $V_x = V_1 \cos\theta + V_2 \sin\theta$

$= 2V_0 \sin\theta \cos\theta$

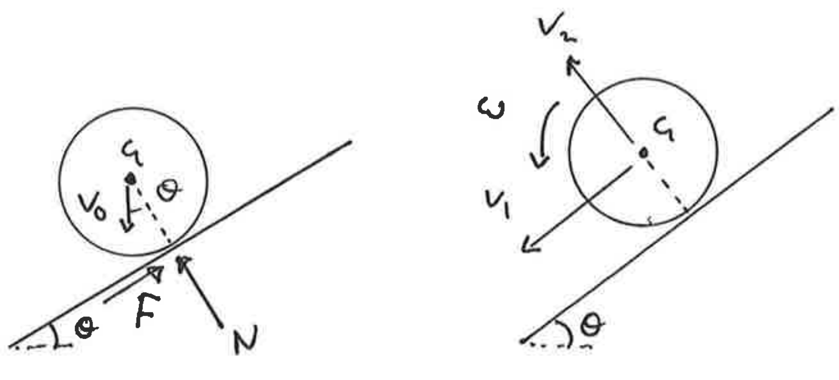
$V_y = V_2 \cos\theta - V_1 \sin\theta$

$= V_0 (\cos^2\theta - \sin^2\theta)$

For $\theta = \frac{\pi}{4}$:

$$\boxed{\begin{aligned} V_x &= V_0 \\ V_y &= 0 \end{aligned}}$$

b) Rough slope :



Before After

Conservation of momentum normal to plane :

$N = MV_2 + MV_0 \cos\theta$

Given : $V_2 = 0 \quad \Rightarrow \quad N = MV_0 \cos\theta$ ①

Tangential to plane:

$$F = MV_0 \sin\theta - MV_1 \quad (2)$$

Moments of momentum about G:

$$FR = J\omega = \frac{2}{5}MR^2\omega \quad (3) \text{ (data book)}$$

If ball rolls after impact:

$$\omega = \frac{V_1}{R}$$

From (3): $F = \frac{2}{5}MV_1$

Sub. into (2): $F\left(1 + \frac{5}{2}\right) = MV_0 \sin\theta$

$$\therefore F = \frac{2}{7}MV_0 \sin\theta$$

No sliding: $F \leq \mu N$

$$\frac{2}{7}MV_0 \sin\theta \leq \mu M V_0 \cos\theta$$

$$\therefore \boxed{\mu \geq \frac{2}{7}\tan\theta}$$

Comments

Question 2, a ball impacting a slope, was also a popular question, attempted by three quarters of candidates. Some candidates showed good intuition for impulse and momentum, and were able to solve part (a) almost by inspection after sketching the expected motion of the ball. A number struggled to make any progress. Common errors in this question:

(i) The majority incorrectly assumed in part (a) that the ball rolls down the slope after impact, even though nearly all could identify the correct direction of the impulse in the absence of friction (normal to the plane, through the centre of mass). Indeed, part (b) – where the ball does roll, nominally a more difficult analysis – was answered much better. The 'rolling versus sliding' type of question is perhaps more familiar from the examples paper and recent Tripos questions.

(ii) Many included the weight (and as a force, rather than an impulse) in the momentum conservation relations during the impact event.

(iii) Confusion between D'Alembert forces and momentum terms in balance equations was common. Some candidates attempted a 'D'Alembert impulse' type approach to the momentum balances, nearly always resulting in sign errors.

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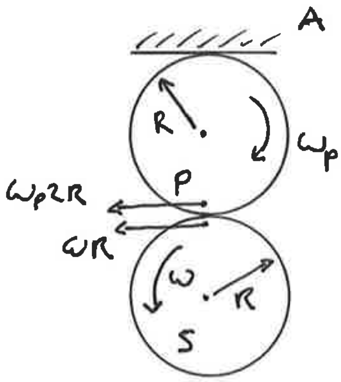
17/06/10

Q3

a)



$$\omega R = \omega_p 2R$$

Angular velocity of planets:

$$\boxed{\omega_p = \frac{\omega}{2}}$$

Let ω_c = angular vel. of C

Centre point of planet:

$$\omega_p R = \omega_c 2R$$

$$\therefore \boxed{\omega_c = \frac{\omega}{4}}$$

b) Kinetic energy of the gearbox:

$$U = \underbrace{\frac{1}{2} J \omega^2}_{\text{sun}} + \underbrace{4\left[\frac{1}{2} J \omega_p^2 + \frac{1}{2} M (\omega_p R)^2\right]}_{\text{planets}}$$

Polar moment of inertia of gears:

$$J = \frac{MR^2}{2} \quad \text{(data book)}$$

$$\therefore U = \frac{MR^2\omega^2}{4} + 4\left[\frac{MR^2\omega^2}{16} + \frac{MR^2\omega^2}{8}\right]$$

$$\therefore \boxed{U = MR^2\omega^2}$$ → energy dissipated in braking

c)(i) Moment of momentum:

$$\underline{H} = \underbrace{J\omega\,\underline{e}_3}_{\text{sun}} + \underbrace{4\left[-J\omega_p + M(\omega_p R)2R\right]\underline{e}_3}_{\text{planets}}$$

$$+ \underbrace{I_{GB}\,\Omega\,\underline{e}_1}_{\text{whole gearbox}}$$

$$\therefore\ \underline{H} = \frac{MR^2\omega}{2}\underline{e}_3 + 4\left[-\frac{MR^2\omega}{4} + M\omega R^2\right]\underline{e}_3$$

$$+ I_{GB}\,\Omega\,\underline{e}_1$$

$$\therefore\ \boxed{\underline{H} = \frac{7}{2}MR^2\omega\,\underline{e}_3 + I_{GB}\,\Omega\,\underline{e}_1}$$

no need to evaluate

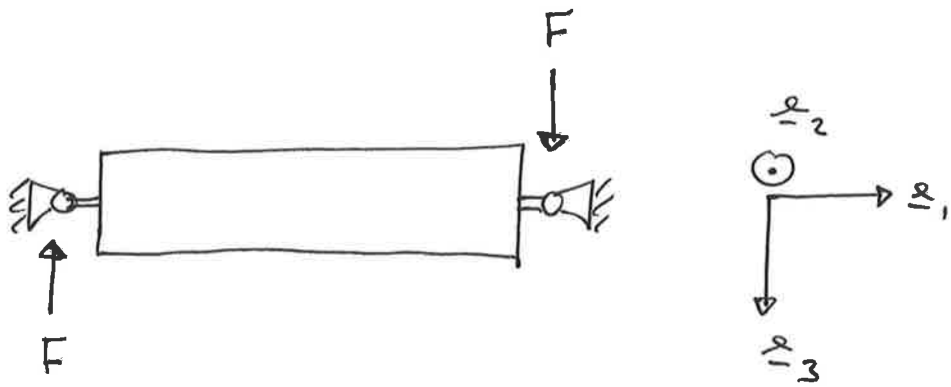
(ii) Torque:

$$\underline{T} = \dot{\underline{H}} = \frac{7}{2}MR^2\omega\,\dot{\underline{e}}_3$$

$$\dot{\underline{e}}_3 = (\Omega\,\underline{e}_1)\times\underline{e}_3 = -\Omega\,\underline{e}_2$$

$$\therefore\ \underline{T} = -\frac{7}{2}MR^2\omega\Omega\,\underline{e}_2$$

Support reactions: provide $\underline{T}$



$$\boxed{F = \frac{7}{2}\frac{MR^2\omega\Omega}{H}}$$

Comments

Question 3 was the least popular question in Section A, attempted by a third of candidates, but was mostly answered well. Most understood the principles of kinetic energy and moments of momentum for the rotating discs. But contributions were often missed, and errors crept in when summing them up. Common errors in this question:

(i) The kinematics caused difficulties for a significant number, the most common mistake being to take the angular velocity of the planets to be the same as the sun (because they have the same radius).

(ii) The kinetic energy and moment of momentum equations were frequently missing the contribution due to the motion of the centre of mass of the planets.

(iii) Using the wrong expression for the polar moment of inertia of the discs was common.

(iv) The relationship between gyroscopic torque and rate of change of moments of momentum was understood by many. However, errors crept in when differentiating the moment of momentum expression, either getting signs wrong in the cross product of unit vectors or including a time derivative for the unit vector parallel to the precession axis.

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17/06/10

Solutions to IB mechanics 2010 Section B

April 24, 2015

4) a) The moment of inertia of the skater is $2md^2$.

b) Moment of momentum of the two skaters before the collision is therefore $2(2md^2\omega)$. There are no external forces or torques on the system, so during the perfectly elastic collision, the moment of momentum is conserved, and so it remains the same.

After the collision the new angular velocity of the skaters is denoted by Ω and their speeds by v, the moment of momentum after the collision can be written as $2[(2m+M)vd+(2md^2)\Omega]$, so the conservation of moment of momentum implies

$$2(2md^2\omega) = 2[(2m+M)vd + (2md^2)\Omega] \tag{1}$$

c) Conservation of energy implies

$$2\frac{1}{2}(2md^2)\omega^2 = 2[\frac{1}{2}(2m+M)v^2 + \frac{1}{2}(2md^2)\Omega^2] \tag{2}$$

Solving (1) and (2) for v, we have

$$\begin{aligned}
2\frac{1}{2}(2md^2)\omega^2 - 2\frac{1}{2}(2m+M)v^2 &= (2md^2)\left[\frac{2(2md^2\omega) - 2(2m+M)vd}{(4md^2)}\right]^2 \\
\omega^2 - \frac{(2m+M)}{2md^2}v^2 &= \left[\omega - \frac{2m+M}{2md}v\right]^2 \\
0 &= \left[\left(\frac{2m+M}{2md}\right)^2 + \frac{(2m+M)}{2md^2}\right]v^2 - 2\omega\frac{2m+M}{2md}v \\
v &= \omega\frac{1}{md}\frac{(2md)^2}{2m+M+2m} \\
&= \omega d\frac{4m}{4m+M}
\end{aligned}$$

The directions of the equal and opposite velocity vectors are perpendicular to the line joining the two skaters before the collision, along the line of the impluse received by the skaters.

5) a) Let us denote the force in the BD struts by T. The angle between BD and the horizontal is also α. Resolving the forces on the slider vertically and equating their sum to zero in equilibrium gives

$$Mg = 2T \sin\alpha$$

Now consider one arm of the tachometer, and take moments around point A. The mass m is rotating at a distance $2l\cos\alpha$ from the shaft.

$$\begin{aligned} 0 &= mg2l\cos\alpha - m\omega^2(2l\cos\alpha)(2l\sin\alpha) + Tl\sin 2\alpha \\ 2mgl + Mgl &= 4ml^2\omega^2\sin\alpha \\ \sin\alpha &= \frac{(2m+M)g}{4ml\omega^2} \end{aligned}$$

b) The maximum value of $\sin\alpha$ is 1, so

$$\omega_0 = \left[\frac{2mg + Mg}{4ml}\right]^{1/2}$$

c) Now there is no force from the strut BD, so again considering just one arm, the moment of inertia of the mass around point A is $m(2l)^2$, and the torque is given by the first two terms from part a),

$$m(2l)^2\ddot{\alpha} = mg2l\cos\alpha - m\omega^2(2l\cos\alpha)(2l\sin\alpha)$$

d) Note the following relationship between $\ddot{\alpha}$ and $\dot{\alpha}$

$$\ddot{\alpha} = \frac{d\dot{\alpha}}{dt} = \frac{d\dot{\alpha}}{d\alpha}\frac{d\alpha}{dt} = \frac{d\dot{\alpha}}{d\alpha}\dot{\alpha}$$

This is now used to integrate the differential equation in part c),

$$\begin{aligned} 4ml^2\dot{\alpha}d\dot{\alpha} &= [mg2l\cos\alpha - m\omega^2(2l\cos\alpha)(2l\sin\alpha)]d\alpha \\ 2ml^2\dot{\alpha}^2 + C &= 2mgl\sin\alpha + ml^2\omega^2\cos 2\alpha \end{aligned}$$

where the constant comes from the indefinite integration. Both at the initial angle and at the maximum height, $\dot{\alpha}$ is zero, so the implicit equation

$$C = 2mgl\sin\alpha + ml^2\omega^2\cos 2\alpha$$

has two solutions, one of them is the initial angle (given in part a), this fixes C, the other corresponds to the maximum height.

6) a) If the rocker went all the way to the other edge of the well, it would undergo 1 and 3/4 revolutions, which would orient it with the heavy side on top, and thus its potential energy would need to be higher than at the starting point. But since it started from rest, this cannot happen, so the rocker will stop short of the other edge, at less than 1 and 3/4 revolution.

b) Cross sectional area of cylinder before drilling is πr^2. Area drilled out is $\pi r^2/4$. Area after drilling is $\pi r^2 - \pi r^2/4 = 3\pi r^2/4$. So if M is the mass remaining, the original cylinder had a mass of $4M/3$ and the mass that was removed is $M/3$. Therefore the centre of gravity position is $-r/2(M/3)/M = -r/6$ away from the geometric centre, $5r/6$ away from the edge of the cylinder along the symmetry axis. To calculate moment of inertia, note that for a uniform cylinder, it is mass $\times$ radius$^2/2$ around its centre, and we need to shift to the new centre of gravity. So for the rocker, it is

$$I_g = \frac{4}{3}Mr^2/2 + \left(\frac{r}{6}\right)^2 \frac{4}{3}M - \frac{M}{3}\left(\frac{r}{2}\right)^2/2 - \left(\frac{r}{2}+\frac{r}{6}\right)^2 \frac{M}{3} = \frac{37}{72}Mr^2$$

c) The angle of rotation of the rocker with respect to the vertical axis is $\phi - \theta$. The height of the centre of gravity is

$$y = R(1-\cos\theta) + r\cos\theta - \frac{r}{6}\cos(\phi-\theta)$$

and $r\phi = R\theta$. The horizontal displacement of the centre of gravity is

$$x = R\sin\theta - r\sin\theta - \frac{r}{6}\sin(\phi-\theta)$$

The kinetic energy is in two parts, the rotation around the centre of gravity gives

$$\frac{1}{2}I_g(\dot\phi - \dot\theta)^2$$

and the speed of the centre of gravity is

$$v = \begin{pmatrix} \dot x \\ \dot y \end{pmatrix}$$

And the corresponding kinetic term is

$$\frac{1}{2}Mv^2$$

.

d) For small angles, using $R = 7r/4$ gives $\phi - \theta = \frac{3}{4}\theta$, and the position vector $(x,y)^T$ is,

$$\begin{aligned} x &= \frac{7}{4}r\theta - r\theta - \frac{r}{6}\left(\frac{7}{4}\theta - \theta\right) = \left(\frac{3}{4} - \frac{1}{6}\frac{3}{4}\right)r\theta = \frac{5}{8}r\theta \\ y &= \frac{7}{4}r - \frac{3}{4}r(1-\theta^2/2) - \frac{r}{6}(1 - \frac{9}{16}\theta^2/2) = \frac{5}{6}r + \frac{27}{64}r\theta^2 \end{aligned}$$

so the potential energy (taking a factor of $\frac{1}{2}$ out) is

$$\frac{1}{2}Mg\frac{27}{32}r\theta^2$$

When working out the total kinetic energy, we will need the rotational part and the horizontal linear part. The vertical linear part has an extra factor of θ^2, which is small. The total kinetic energy is thus

$$\frac{1}{2}I_g\frac{9}{16}\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}Mr^2\frac{37}{72}\frac{9}{16}\dot{\theta}^2 + \frac{1}{2}Mr^2\frac{25}{64}\dot{\theta}^2 = \frac{1}{2}Mr^2\frac{87}{72}\frac{9}{16}\dot{\theta}^2$$

The frequency ω of small oscillations is given by $(k/m)^{1/2}$ for a simple harmonic oscillator with mass m and spring constant k. The oscillator has a potential energy of $\frac{1}{2}k\theta^2$ and a kinetic energy of $\frac{1}{2}m\dot{\theta}^2$, so here

$$\begin{aligned} k &= \frac{27}{32}Mgr \\ m &= \frac{87}{72}\frac{9}{16}Mr^2 \\ \omega^2 &= \frac{g}{r}\frac{27}{32}\frac{72}{87}\frac{16}{9} = \frac{36}{29}\frac{g}{r} \end{aligned}$$