ENGINEERING TRIPOS PART IB

Friday 4 June 2010 2.30 to 4.30

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer not more than **two** *questions from each section.*

All questions carry the same number of marks.

The **approximate** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Attachments: There are no attachments to this paper.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

VEM3

SECTION A

*Answer not more than **two** questions from this section.*

1 A two-dimensional fluid flow has velocity $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j}$, with $u_x = Sx$, $u_y = -Sy$, and S a positive constant. (In Fluid Mechanics, S is called the *strain rate* and this flow is called *Hiemenz flow*.)

(a) Derive a equation for the field lines of $\mathbf{u}$ (the streamlines) and sketch them. [5]

(b) Show that $\mathbf{u}$ is solenoidal and irrotational. [5]

(c) Find a scalar potential, ϕ, for $\mathbf{u}$ and evaluate the line integral

$$\int \mathbf{u} \cdot d\mathbf{l}$$

along the straight line between the points with coordinates $(0,0)$ and $(\sqrt{2/S}, 1/\sqrt{S})$. [5]

(d) Two particles at points (X_0, Y_0) and $(X_0 + r_0, Y_0)$ are released at $t = 0$ and follow the flow. Find the distance $r(t)$ between the particles after a time t. Hence show that $(1/r)dr/dt = S$. [5]

2 (a) If $x = f(u,v)$ and $y = g(u,v)$, where $u = \phi(r,s)$ and $v = \psi(r,s)$, it is known that

$$\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(r,s)}$$

Hence show that

$$\frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1}$$

[4]

(b) Sketch on the same graph the following curves: $x^2 - y^2 = 1$, $x^2 - y^2 = 2$, $2xy = \pi/2$, $2xy = \pi$ $(x > 0,\ y > 0)$ and mark the region $\Re$ bounded by these curves. [4]

(c) Using a suitable transformation, calculate

$$\int\int 4(x^4 - y^4)\sin(2xy)\ dxdy$$

over the region $\Re$ from part (b). [12]

3 The heat flux $\mathbf{q}$ in a material is given by $\mathbf{q} = -\lambda \nabla T$, where λ is the thermal conductivity and T the temperature. The energy conservation principle can be written as $(rate\ of\ internal\ energy\ accumulation) = (net\ heat\ flow\ in)$, with the internal energy per unit volume defined as $\rho c_p T$, with ρ the density and c_p the heat capacity of the material. Assume that λ, ρ and c_p are constant and uniform.

(a) Using Gauss's Theorem and the energy conservation principle applied to an infinitesimally small volume, derive the *heat equation*

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \ , \quad \alpha = \frac{\lambda}{\rho c_p}$$

[6]

(b) In one dimension, the heat equation becomes

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Consider the semi-infinite domain $0 \leq x < \infty$. The initial condition for $T(x,t)$ is given as $T(x,0) = 0$ at $t = 0$ and the boundary condition at $x = 0$ is $T(0,t) = T_0 \cos(\omega t)$ for $t > 0$. After many cycles of the oscillation ($t \gg 1/\omega$), $T(x,t)$ reaches a harmonic response to the time-varying boundary condition.

(i) Using separation of variables and paying attention to the nature of the separation constant, find an analytical expression for $T(x,t)$. [10]

(ii) Sketch $T(x,t)$. [4]

SECTION B

Answer not more than ***two*** *questions from this section.*

4 Requests arrive at a web server at an average rate of 3000 requests per hour. Assume that the request arrivals are independent. Internally, the server has 2 CPUs, A and B. CPU A has twice the capacity of CPU B. When requests arrive, they are randomly assigned to the processors, with probability $2/3$ for CPU A and $1/3$ for CPU B. The upper limit on the capacity of the CPU is 2100 requests per hour for CPU A and 1050 requests for CPU B. If the capacity is exceeded within an hour, the CPU fails. In parts (c), (d) and (e) you may use a similar Gaussian approximation to the one required in part (b). You may ignore the possibility that both processors fail within an hour.

(a) What is the distribution of the requests arriving at the web server? [2]

(b) Approximate the number of arrivals per hour for CPU B with a Gaussian distribution. What are the parameters of this Gaussian? [4]

(c) Using the Gaussian approximation from above, what is the probability that CPU B will fail in a given hour? [4]

(d) A user sends a request to the server. Unfortunately, the server fails due to overload. What is the probability that the failure was caused by CPU A? [6]

(e) What is the probability of failure per hour when assigning requests randomly as described? What would the probability of failure be if the requests were assigned optimally depending on the respective CPU loads (instead of randomly)? [4]

5 (a) By performing LU decomposition on the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & 4 \\ -1 & -1 & 0 & -3 \\ 2 & 4 & 1 & 7 \end{bmatrix}$$

find bases for each of the fundamental subspaces of the matrix $\mathbf{A}$. [12]

(b) Find the value of a for which the equation $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ has a solution, where

$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix}$$

[4]

(c) For this value of a, find the general solution of the equation $\mathbf{A}\,\mathbf{x} = \mathbf{b}$. Verify that your answer does indeed satisfy this equation. [4]

6 (a) Explain how a power method can be used to find the largest (in magnitude) eigenvalue and corresponding eigenvector of an $n \times n$ matrix **A**, and describe the factors that determine the rate of convergence of the iterative process. [5]

(b) The matrix **A** is believed to have an eigenvalue λ, not the largest, which is approximately 3. Explain how you would use the shifted inverse-power method to find this eigenvalue and discuss the convergence rate that you would expect to obtain. [5]

(c) Show that the shifted inverse-power method, with a shift value of 1, applied to the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}$$

furnishes the correct eigenvalues but does not determine the eigenvectors. Explain why this happens. [5]

(d) For large values of n, the computational cost of inverting a matrix is prohibitive. Explain how the shifted inverse-power method can be carried out without the need to perform any matrix inversion. [5]

END OF PAPER